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TWO GENERALIZED STRONG CONVERGENCE THEOREMS OF HALPERN'S TYPE IN HILBERT SPACES AND APPLICATIONS

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Abstract. Let C be a closed convex subset of a real Hilbert space H. Let A be an inverse-strongly monotone mapping of C into H and let B be a maximal monotone operator on H such that the domain of B is included in C. We introduce two iteration schemes of finding a point of $(A+B)^{-1}0$, where $(A+B)^{-1}0$ is the set of zero points of $A+B$. Then, we prove two strong convergence theorems of Halpern's type in a Hilbert space. Using these results, we get new and well-known strong convergence theorems in a Hilbert space.

1. INTRODUCTION

Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively. Let $f : C \times C \to \mathbb{R}$ be a bifunction and let A be a nonlinear mapping of C into H. Then, a generalized equilibrium problem (with respect to C) is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) + \langle A\hat{x}, y - \hat{x}\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of such solutions $\hat{x}$ is denoted by $EP(f, A)$, i.e.,

$$EP(f, A) = \{\hat{x} \in C : f(\hat{x}, y) + \langle A\hat{x}, y - \hat{x}\rangle \geq 0,\ \forall y \in C\}.$$

In the case of $A = 0$, $EP(f, A)$ is denoted by $EP(f)$. In the case of $f = 0$, $EP(f, A)$ is also denoted by $VI(C, A)$. This is the set of solutions of the variational inequality for A; see [15] and [19]. Let T be a mapping of C into H. We denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to H$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

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For a nonexpansive mapping $T : C \to C$, the iteration procedure of Halpern's type is as follows: $u \in C$, $x_1 \in C$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$; see [10]. Let $\alpha > 0$ be a given constant. A mapping $A\colon C \to H$ is said to be α-inverse-strongly monotone if

$$\langle x - y, Ax - Ay\rangle \geq \alpha \left\| Ax - Ay \right\|^2, \quad \forall x, y \in C.$$

A multi-valued mapping B on H is said to be monotone if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in \mathrm{dom}(B)$, $u \in Bx$, and $v \in By$, where $\mathrm{dom}(B)$ is the domain of B. A monotone operator B on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1}\colon H \to \mathrm{dom}(B)$, which is called the resolvent of B for $r > 0$. The resolvent of B for $r > 0$ is nonexpansive, see [23]. A mapping $U : C \to H$ is a strict pseudo-contraction [7] if there is $k \in \mathbb{R}$ with $0 \leq k < 1$ such that

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + k\|(I - U)x - (I - U)y\|^2, \quad \forall x, y \in C.$$

We call such U a k-strict pseudo-contraction. A k-strict pseudo-contraction $U : C \to H$ is nonexpansive if $k = 0$. A mapping $T : C \to H$ is quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tu - v\| \leq \|u - v\|, \quad \forall u \in C,\ v \in F(T).$$

If $S : C \to H$ is a nonexpansive mapping, then $I - S$ is $\frac{1}{2}$-inverse-strongly monotone, where I is the identity mapping. A nonexpansive mapping $S : C \to H$ with $F(S) \neq \emptyset$ is quasi-nonexpansive; see [23]. We also know that if $U : C \to H$ is a k-strict pseudo-contraction with $0 \leq k < 1$, then $A = I - U$ is a $\frac{1-k}{2}$-inverse-strongly monotone mapping; see, for instance, Marino and Xu [14]. Zhou [29] proved the following strong convergence theorem of Halpern's type for strict pseudo-contractions in a Hilbert space.

Theorem 1. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let k be a real number with $0 \leq k < 1$ and let $U : C \to H$ be a k-strict pseudo-contraction such that $F(U) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $u \in C$, $x_1 = x \in C$ and*

$$\begin{cases} y_n = P_C[\beta_n x_n + (1 - \beta_n)Ux_n], \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy

$$\alpha_n \to 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty,$$

$$k \leq \beta_n \leq b < 1, \quad and \quad \sum_{n=1}^{\infty} |\beta_n - \beta_{n+1}| < \infty.$$

Then, $\{x_n\}$ converges strongly to $z_0 = P_{F(U)}u$, where $P_{F(U)}$ is the metric projection of H onto $F(U)$.

In this paper, motivated by the generalized equilibrium problem and Zhou's theorem (Theorem 1), we first pove a strong convergence theorem for finding a zero point of $A+B$, where A is an inverse-strongly monotone mapping of C into H and B is a maximal monotone operator on H such that the domain of B is included in C. For eample, if $A = I - U$, where U is a strict pseodo-contraction, and B is the indicator function of C, then this result generalizes Zhou's one. Furthermore, we prove another strong convergence theorem which is different from the above form in a Hilbert space. Using these results, we get new and well-known strong convergence theorems in a Hilbert space.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot\,, \cdot \rangle$ and norm $\|\cdot\|$, respectively. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have from [23] that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.1}$$

All Hilbert spaces satisfy Opial's condition, that is,

$$\liminf_{n\to\infty} \|x_n - u\| < \liminf_{n\to\infty} \|x_n - v\|$$

if $x_n \rightharpoonup u$ and $u \neq v$; see [16]. Let C be a nonempty closed convex subset of a Hilbert space H. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle \tag{2.2}$$

for all $x, y \in H$. Furthermore $\langle x - P_C x, y - P_C x\rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [21]. Let $\alpha > 0$ be a given constant. A mapping $A\colon C \to H$ is said to be α-inverse-strongly monotone if $\langle x - y, Ax - Ay\rangle \geq \alpha \|Ax - Ay\|^2$ for all $x, y \in C$. It is known that $\|Ax - Ay\| \leq (1/\alpha)\|x - y\|$ for all $x, y \in C$ if A is α-inverse-strongly monotone; see, for example, [25]. Let B be a mapping of H into 2^H. The effective domain of B is denoted by $\mathrm{dom}(B)$, that is, $\mathrm{dom}(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping B on H is said to be monotone if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in \mathrm{dom}(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1}: H \to \mathrm{dom}(B)$, which is called the resolvent of B for r. Let B be a maximal monotone operator on H and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$. It is known that the resolvent J_r is firmly nonexpansive and $B^{-1}0 = F(J_r)$ for all $r > 0$. It is also known that $\|J_\lambda x - J_\mu x\| \leq (|\lambda - \mu| / \lambda) \|x - J_\lambda x\|$ holds for all $\lambda, \mu > 0$ and $x \in H$; see [9, 21] for more details. As a matter of fact, we know the following lemma [20].

Lemma 2. *Let H be a real Hilbert space and let B be a maximal monotone operator on H. For $r > 0$ and $x \in H$, define the resolvent $J_r x$. Then the following holds:*

$$\frac{s-t}{s}\langle J_s x - J_t x, J_s x - x\rangle \geq \|J_s x - J_t x\|^2$$

for all $s, t > 0$ and $x \in H$.

Furthermore, for a mapping A of C into H, we know that $F(J_\lambda(I - \lambda A)) = (A + B)^{-1}0$ for all $\lambda > 0$; see [4]. We also know the following lemmas:

Lemma 3. ([18]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space and let $\{\beta_n\}$ be a sequence in $[0, 1]$ such that $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all $n \in \mathbb{N}$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 4. ([2, 28]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence in $[0, 1]$ with $\sum_{n=1}^\infty \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^\infty \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\gamma_n + \beta_n$$

for all $n = 1, 2, \ldots$. Then $\lim_{n\to\infty} s_n = 0$.

3. Inverse-strongly Monotone Mappings

Let H be a Hilbert space and let C be a nonempty closed convex subset of H. A mapping $U : C \to H$ is called a widely strict pseudo-contraction if there is a real number $k \in \mathbb{R}$ with $k < 1$ such that

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + k\|(I - U)x - (I - U)y\|^2 \tag{3.1}$$

for all $x, y \in C$. Such a mapping U is called a widely k-strict pseudo-contraction. We know that a widely k-strict pseudo-contraction is a strict pseudo-contraction [7] if $0 \leq k < 1$. A widely k-strict pseudo-contraction is also a nonexpansive mapping if

$k = 0$. Conversely, we have that if $T : C \to H$ is a nonexpansive mapping, then for any $n \in \mathbb{N}$, $U = \frac{1}{1+n}T + \frac{n}{1+n}I$ is a widely $(-n)$-strict pseudo-contraction. As in Zhou [29], we obtain the following result.

Lemma 5. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $k < 1$ and let $U : C \to H$ be a widely k-strict pseudo-contraction such that $F(U) \neq \emptyset$ and let P_C be the metric projection of H onto C. Then, $F(P_C U) = F(U)$.*

Proof. Take $z, v \in C$ with $P_C Uz = z$ and $Uv = v$. Then we obtain from (2.1) and (2.2) that

$$\begin{aligned} 2\|z - v\|^2 &= 2\|P_C Uz - P_C Uv\|^2 \\ &\leq 2\langle Uz - Uv, P_C Uz - P_C Uv\rangle \\ &= 2\langle Uz - v, z - v\rangle \\ &= \|Uz - v\|^2 + \|v - z\|^2 - \|Uz - z\|^2 - \|v - v\|^2 \end{aligned}$$

and hence

$$\|z - v\|^2 + \|Uz - z\|^2 \leq \|Uz - v\|^2.$$

Since U is a widely strict pseudo-contraction, we have that

$$\|z - v\|^2 + \|Uz - z\|^2 \leq \|Uz - v\|^2 \leq \|z - v\|^2 + k\|z - Uz\|^2$$

and hence $(1 - k)\|Uz - z\|^2 \leq 0$. From $1 - k > 0$, we have $\|Uz - z\|^2 \leq 0$ and then $Uz = z$. This completes the proof. ∎

We also know that a mapping $A : C \to H$ is called inverse-strongly monotone if there exisis $\alpha > 0$ such that

$$\alpha\|Ax - Ay\|^2 \leq \langle x - y, Ax - Ay\rangle \tag{3.2}$$

for all $x, y \in C$. Such a mapping A is called α-inverse strongly monotone. Recently, Hojo, Takahashi and Yao [11] also introduced a class of nonlinear mappings in a Hilbert space which contains the class of generalized hybrid mappings: A mapping $U : C \to H$ is called extended hybrid if there are $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned} \alpha(1+\gamma)\ &\|Ux - Uy\|^2 + (1 - \alpha(1+\gamma))\|x - Uy\|^2 \\ &\leq (\beta + \alpha\gamma)\|Ux - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2 \\ &\quad -(\alpha - \beta)\gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2 \end{aligned} \tag{3.3}$$

for all $x \in C$. Such a mapping U is called (α, β, γ)-extended hybrid. In [11], they proved the following theorem which represents a relation between the class of generalized hybrid mappings and the class of extended hybrid mappings in a Hilbert space; see also [12] and [26].

Theorem 6. *Let C be a nonempty closed convex subset of a Hilbert space H and let α, β and γ be real numbers with $\gamma \neq -1$. Let T and U be mappings of C into H such that $U = \frac{1}{1+\gamma}T + \frac{\gamma}{1+\gamma}I$. Then, for $1+\gamma > 0$, $T : C \to H$ is an (α, β)-generalized hybrid mapping if and only if $U : C \to H$ is an (α, β, γ)-extended hybrid mapping. In this case, $F(T) = F(U)$.*

Now, we deal with some properties for inverse-strongly monotone mappings in a Hilbert space.

Lemma 7. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\alpha > 0$ and let A, U and T be mappings of C into H such that $U = I - A$ and $T = 2\alpha\, U + (1-2\alpha)I$. Then, the following are equivalent:*

(a) *A is an α-inverse-strongly monotone mapping, i.e.,*

$$\alpha\|Ax - Ay\|^2 \leq \langle x - y, Ax - Ay\rangle, \quad \forall x, y \in C;$$

(b) *U is a widely $(1-2\alpha)$-strict pseudo-contraction, i.e.,*

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + (1-2\alpha)\|(I-U)x - (I-U)y\|^2, \quad \forall x, y \in C;$$

(c) *U is a $(1, 0, 2\alpha - 1)$-extended hybrid mapping, i.e.,*

$$\begin{aligned} 2\alpha\|Ux - Uy\|^2 &+ (1-2\alpha)\|x - Uy\|^2 \\ &\leq (2\alpha - 1)\|Ux - y\|^2 + 2(1-\alpha)\|x - y\|^2 \\ &\quad - (2\alpha - 1)\|x - Ux\|^2 - (2\alpha - 1)\|y - Uy\|^2, \quad \forall x, y \in C; \end{aligned}$$

(d) *T is a nonexpansive mapping.*

In this case, $Z(A) = F(U) = F(T)$, where $Z(A) = \{u \in C : Au = 0\}$.

Proof. Let us show (a) $\Longleftrightarrow$ (b). We have that for all $x, y \in C$,

$$\begin{aligned} &\alpha\|Ax - Ay\|^2 \leq \langle x - y, Ax - Ay\rangle \\ &\quad\Longleftrightarrow 2\alpha\|Ax - Ay\|^2 \leq 2\langle x - y, Ax - Ay\rangle \\ &\quad\Longleftrightarrow 2\alpha\|Ax - Ay\|^2 \leq \|x - y\|^2 + \|Ax - Ay\|^2 - \|x - Ax - (y - Ay)\|^2 \\ &\quad\Longleftrightarrow \|x - Ax - (y - Ay)\|^2 \leq \|x - y\|^2 + (1-2\alpha)\|Ax - Ay\|^2 \\ &\quad\Longleftrightarrow \|Ux - Uy\|^2 \leq \|x - y\|^2 + (1-2\alpha)\|(I-U)x - (I-U)y\|^2. \end{aligned}$$

Let us show (b) $\Longleftrightarrow$ (c). Since

$$\begin{aligned} \|(I-U)x - (I-U)y\|^2 &= \|x - y - (Ux - Uy)\|^2 \\ &= \|x - y\|^2 + \|Ux - Uy\|^2 - 2\langle x - y, Ux - Uy\rangle \\ &= \|x - y\|^2 + \|Ux - Uy\|^2 \\ &\quad - \|x - Uy\|^2 - \|y - Ux\|^2 + \|x - Ux\|^2 + \|y - Uy\|^2, \end{aligned}$$

for all $x, y \in C$, we have that

$$\begin{aligned}
&\|Ux - Uy\|^2 \le \|x - y\|^2 + (1 - 2\alpha)\|(I - U)x - (I - U)y\|^2 \\
&\quad\Longleftrightarrow \|Ux - Uy\|^2 \le \|x - y\|^2 + (1 - 2\alpha)(\|x - y\|^2 + \|Ux - Uy\|^2 \\
&\qquad\qquad - \|x - Uy\|^2 - \|y - Ux\|^2 + \|x - Ux\|^2 + \|y - Uy\|^2) \\
&\quad\Longleftrightarrow 2\alpha\|Ux - Uy\|^2 + (1 - 2\alpha)\|x - Uy\|^2 \\
&\qquad\quad \le (2\alpha - 1)\|Ux - y\|^2 + 2(1 - \alpha)\|x - y\|^2 \\
&\qquad\qquad - (2\alpha - 1)\|x - Ux\|^2 - (2\alpha - 1)\|y - Uy\|^2.
\end{aligned}$$

Let us show (b) $\Longleftrightarrow$ (d). We have that for all $x, y \in C$,

$$\begin{aligned}
&\|Tx - Ty\|^2 \le \|x - y\|^2 \\
&\quad\Longleftrightarrow \|2\alpha Ux + (1 - 2\alpha)x - 2\alpha Uy - (1 - 2\alpha)y\|^2 \le \|x - y\|^2 \\
&\quad\Longleftrightarrow 2\alpha\|Ux - Uy\|^2 + (1 - 2\alpha)\|x - y\|^2 \\
&\qquad\qquad - 2\alpha(1 - 2\alpha)\|(I - U)x - (I - U)y\|^2 - \|x - y\|^2 \le 0 \\
&\quad\Longleftrightarrow 2\alpha\|Ux - Uy\|^2 - 2\alpha\|x - y\|^2 \\
&\qquad\qquad - 2\alpha(1 - 2\alpha)\|(I - U)x - (I - U)y\|^2 \le 0 \\
&\quad\Longleftrightarrow \|Ux - Uy\|^2 - \|x - y\|^2 - (1 - 2\alpha)\|(I - U)x - (I - U)y\|^2 \le 0 \\
&\quad\Longleftrightarrow \|Ux - Uy\|^2 \le \|x - y\|^2 + (1 - 2\alpha)\|(I - U)x - (I - U)y\|^2.
\end{aligned}$$

Finally, let us show $Z(A) = F(U) = F(T)$. In fact, we have that for $u \in C$,

$$Au = 0 \Longrightarrow Uu = u - Au = u \Longrightarrow Tu = 2\alpha Uu + (1 - 2\alpha)u = u.$$

We can also show the reverse implication. This completes the proof. ∎

Remark 1. Let $\alpha > 0$ and let $A \colon C \to H$ be α-inverse-strongly monotone. Then, it is obvious that for any $\beta \in \mathbb{R}$ with $0 < \beta \le 2\alpha$, A is $\frac{\beta}{2}$-inverse-strongly monotone. So, we have from Lemma 3.1 that

$$T = I - \beta A = I - \beta(I - U) = \beta U + (1 - \beta)I$$

is nonexpansive.

Using Lemma 7, we can get the following important result.

Lemma 8. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let k be a real number with $k < 1$ and let A, U and T be mappings of C into H such that $U = I - A$ and $T = (1 - k)U + kI$. Then, the following are equivalent:*

(a) *A is a $\frac{1-k}{2}$-inverse-strongly monotone mapping;*

(b) *U is a widely k-strict pseudo-contraction;*

(c) *U is a $(1, 0, -k)$-extended hybrid mapping;*

(d) *T is a nonexpansive mapping.*

In this case, $Z(A) = F(U) = F(T)$.

Proof. Putting $\alpha = \frac{1-k}{2}$ for $k < 1$, we have $\alpha > 0$. Furthermore, we have

$$1 - 2\alpha = 1 - (1 - k) = k.$$

This means (a) $\Longleftrightarrow$ (b). Similarly, we obtain (b) $\Longleftrightarrow$ (c) $\Longleftrightarrow$ (d). ∎

Remark 2. Let k be a real number with $k < 1$. If U is a widely k-strict pseudo-contraction, then for any $t \in \mathbb{R}$ with $k \le t < 1$, U is a widely t-strict pseudo-contraction. So, we have from Lemma 8 that

$$T = (1-t)U + tI$$

is nonexpansive.

4. Main Results

In this section, we first prove a strong convergence theorem which generalizes Zhou's theorem (Theorem 1) in a Hilbert space.

Theorem 9. *Let H be a real Hilbert space and let C be a closed convex subset of H. Let $\alpha > 0$. Let A be an α-inverse-strongly monotone mapping of C into H and let B be a maximal monotone operator on H such that the domain of B is included in C. Let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Suppose that $(A+B)^{-1}0 \neq \emptyset$. Let $u \in C,\ x_1 = x \in C$ and let $\{x_n\} \subset C$ be a sequence generated by*

$$x_{n+1} = \alpha_n u + (1-\alpha_n) J_{\lambda_n}(x_n - \lambda_n A x_n)$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0,\infty)$ and $\{\alpha_n\} \subset (0,1)$ satisfy

$$0 < a \le \lambda_n \le 2\alpha, \quad \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty,$$

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \textit{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Then $\{x_n\}$ converges strongly to a point z_0 of $(A+B)^{-1}0$, where $z_0 = P_{(A+B)^{-1}0}u$.

Proof. Put $y_n = J_{\lambda_n}(x_n - \lambda_n A x_n)$ and let $z \in (A+B)^{-1}0$. Then, we have from $z = J_{\lambda_n}(z - \lambda_n A z)$ that

$$\begin{aligned}
\|y_n - z\|^2 &= \|J_{\lambda_n}(x_n - \lambda_n A x_n) - z\|^2 \\
&= \|J_{\lambda_n}(x_n - \lambda_n A x_n) - J_{\lambda_n}(z - \lambda_n A z)\|^2 \\
&\le \|(x_n - \lambda_n A x_n) - (z - \lambda_n A z)\|^2 \\
&= \|(x_n - z) - \lambda_n (A x_n - A z)\|^2 \\
&= \|x_n - z\|^2 - 2\lambda_n \langle x_n - z, A x_n - A z\rangle + \lambda_n^2 \|A x_n - A z\|^2 \\
&\le \|x_n - z\|^2 - 2\lambda_n \alpha \|A x_n - A z\|^2 + \lambda_n^2 \|A x_n - A z\|^2 \\
&= \|x_n - z\|^2 + \lambda_n(\lambda_n - 2\alpha) \|A x_n - A z\|^2 \\
&\le \|x_n - z\|^2 .
\end{aligned} \tag{4.1}$$

From $x_{n+1} = \alpha_n u + (1-\alpha_n)y_n$, we have

$$\begin{aligned}\|x_{n+1} - z\| &= \|\alpha_n(u-z) + (1-\alpha_n)(y_n - z)\| \\ &\le \alpha_n \|u-z\| + (1-\alpha_n)\|x_n - z\|.\end{aligned}$$

Putting $K = \max\{\|u-z\|, \|x_1 - z\|\}$, we have that $\|x_n - z\| \le K$ for all $n \in \mathbb{N}$. In fact, it is obvious that $\|x_1 - z\| \le K$. Suppose that $\|x_k - z\| \le K$ for some $k \in \mathbb{N}$. Then, we have that

$$\begin{aligned}\|x_{k+1} - z\| &\le \alpha_k\|u-z\| + (1-\alpha_k)\|x_k - z\| \\ &\le \alpha_k K + (1-\alpha_k)K \\ &= K.\end{aligned}$$

By induction, we obtain that $\|x_n - z\| \le K$ for all $n \in \mathbb{N}$. Then, $\{x_n\}$ is bounded. Furthermore, $\{Ax_n\}$ and $\{y_n\}$ are bounded. Putting $u_n = x_n - \lambda_n Ax_n$, we have

$$\begin{aligned}x_{n+2} - x_{n+1} &= (\alpha_{n+1} - \alpha_n)u + (1-\alpha_{n+1})J_{\lambda_{n+1}}(x_{n+1} - \lambda_{n+1}Ax_{n+1}) \\ &\quad - (1-\alpha_n)J_{\lambda_n}(x_n - \lambda_n Ax_n) \\ &= (\alpha_{n+1} - \alpha_n)u + (1-\alpha_{n+1})\{J_{\lambda_{n+1}}(x_{n+1} - \lambda_{n+1}Ax_{n+1}) \\ &\quad - J_{\lambda_{n+1}}u_n + J_{\lambda_{n+1}}u_n - J_{\lambda_n}u_n + J_{\lambda_n}u_n\} - (1-\alpha_n)J_{\lambda_n}u_n.\end{aligned}$$

So, we have from Lemma 2 that

$$\begin{aligned}&\|x_{n+2} - x_{n+1}\| \\ &\le |\alpha_{n+1} - \alpha_n|\ \|u\| + (1-\alpha_{n+1})\|x_{n+1} - \lambda_{n+1}Ax_{n+1} - (x_n - \lambda_n Ax_n)\| \\ &\quad + (1-\alpha_{n+1})\|J_{\lambda_{n+1}}u_n - J_{\lambda_n}u_n\| + |\alpha_{n+1} - \alpha_n|\|J_{\lambda_n}u_n\| \\ &\le |\alpha_{n+1} - \alpha_n|\ \|u\| + (1-\alpha_{n+1})\|(I - \lambda_{n+1}A)x_{n+1} - (I - \lambda_{n+1}A)x_n \\ &\quad + (I - \lambda_{n+1}A)x_n - (x_n - \lambda_n Ax_n)\| \\ &\quad + (1-\alpha_{n+1})\|J_{\lambda_{n+1}}u_n - J_{\lambda_n}u_n\| + |\alpha_{n+1} - \alpha_n|\|J_{\lambda_n}u_n\| \\ &\le |\alpha_{n+1} - \alpha_n|\ \|u\| + (1-\alpha_n)\|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\| \\ &\quad + |\alpha_{n+1} - \alpha_n|\|J_{\lambda_n}u_n\| + \|J_{\lambda_{n+1}}u_n - J_{\lambda_n}u_n\| \\ &\le |\alpha_{n+1} - \alpha_n|\ \|u\| + (1-\alpha_n)\|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\| \\ &\quad + |\alpha_{n+1} - \alpha_n|\|J_{\lambda_n}u_n\| + \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}}\|J_{\lambda_{n+1}}u_n - u_n\| \\ &\le |\alpha_{n+1} - \alpha_n|\ \|u\| + (1-\alpha_n)\|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\| \\ &\quad + |\alpha_{n+1} - \alpha_n|\|J_{\lambda_n}u_n\| + \frac{|\lambda_{n+1} - \lambda_n|}{a}\|J_{\lambda_{n+1}}u_n - u_n\|.\end{aligned}$$

Using Lemma 4, we obtain that

$$\|x_{n+2} - x_{n+1}\| \to 0. \tag{4.2}$$

We also have from (2.1) that

$$\begin{aligned}\|x_{n+1}-x_n\|^2 &= \|\alpha_n(u-x_n)+(1-\alpha_n)(y_n-x_n)\|^2\\ &= \alpha_n\|u-x_n\|^2+(1-\alpha_n)\|y_n-x_n\|^2-\alpha_n(1-\alpha_n)\|u-y_n\|^2\end{aligned}$$

and hence

$$\begin{aligned}(1-\alpha_n)\|y_n-x_n\|^2 = \|x_{n+1}-x_n\|^2 \\ -\alpha_n\|u-x_n\|^2+\alpha_n(1-\alpha_n)\|u-y_n\|^2\end{aligned}$$

From $\alpha_n \to 0$, we get

$$y_n - x_n \to 0. \tag{4.3}$$

From $\sum_{n=1}^{\infty}|\lambda_n-\lambda_{n+1}|<\infty$, we have that $\{\lambda_n\}$ is a Cauchy sequence. So, we have $\lambda_n \to \lambda_0 \in [a, 2\alpha]$. Putting $u_n = x_n - \lambda_n A x_n$ and $y_n = J_{\lambda_n}(I-\lambda_n A)x_n$, we have from Lemma 2 that

$$\begin{aligned}&\|J_{\lambda_0}(I-\lambda_0 A)x_n - y_n\| = \|J_{\lambda_0}(I-\lambda_0 A)x_n - J_{\lambda_n}(I-\lambda_n A)x_n\|\\ &= \|J_{\lambda_0}(I-\lambda_0 A)x_n - J_{\lambda_0}(I-\lambda_n A)x_n\\ &\quad + J_{\lambda_0}(I-\lambda_n A)x_n - J_{\lambda_n}(I-\lambda_n A)x_n\|\\ &\le \|(I-\lambda_0 A)x_n-(I-\lambda_n A)x_n\| + \|J_{\lambda_0}u_n - J_{\lambda_n}u_n\|\\ &\le |\lambda_0-\lambda_n|\|Ax_n\| + \frac{|\lambda_0-\lambda_n|}{\lambda_0}\|J_{\lambda_0}u_n-u_n\| \to 0.\end{aligned} \tag{4.4}$$

We also have from (4.3) and (4.4) that

$$\|x_n - J_{\lambda_0}(I-\lambda_0 A)x_n\| \le \|x_n-y_n\| + \|y_n - J_{\lambda_0}(I-\lambda_0 A)x_n\| \to 0. \tag{4.5}$$

We will use (4.4) and (4.5) later.

Put $z_0 = P_{(A+B)^{-1}0}u$. Let us show that $\limsup_{n\to\infty}\langle u-z_0, y_n-z_0\rangle \le 0$. Put $A = \limsup_{n\to\infty}\langle u-p_0, y_n-p_0\rangle$. Then without loss of generality, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $A = \lim_{i\to\infty}\langle u-p_0, y_{n_i}-p_0\rangle$ and $\{y_{n_i}\}$ converges weakly some point $w \in C$. From $\|x_n - y_n\| \to 0$, we also have that $\{x_{n_i}\}$ converges weakly to $w \in C$. On the other hand, from $\lambda_n \to \lambda_0 \in [a, 2\alpha]$, we have $\lambda_{n_i} \to \lambda_0 \in [a, 2\alpha]$. Using (4.4), we have that

$$\|J_{\lambda_0}(I-\lambda_0 A)x_{n_i} - y_{n_i}\| \to 0.$$

Furthermore, using (4.5), we have that

$$\|x_{n_i} - J_{\lambda_0}(I-\lambda_0 A)x_{n_i}\| \to 0.$$

Since $J_{\lambda_0}(I-\lambda_0 A)$ is nonexpansive, we have $w = J_{\lambda_0}(I-\lambda_0 A)w$. This means that $0 \in Aw + Bw$. So, we have

$$A = \lim_{i\to\infty}\langle u - z_0, y_{n_i} - z_0\rangle = \langle u - z_0, w - z_0\rangle \leq 0.$$

Since $x_{n+1} - z_0 = \alpha_n(u - z_0) + (1-\alpha_n)(y_n - z_0)$, we have

$$\begin{aligned}\|x_{n+1}-z_0\|^2 &\leq (1-\alpha_n)^2\|y_n - z_0\|^2 + 2\langle\alpha_n(u-z_0), x_{n+1}-z_0\rangle \\ &\leq (1-\alpha_n)\|y_n-z_0\|^2 \\ &\quad + 2\alpha_n\langle u - z_0, x_{n+1}-x_n+x_n-y_n+y_n-z_0\rangle.\end{aligned} \tag{4.6}$$

Putting $s_n = \|x_n - z_0\|^2$, $\gamma_n = 2\langle u-z_0, x_{n+1}-x_n+x_n-y_n+y_n-z_0\rangle$ and $\beta_n = 0$ in Lemma 4, from $\sum_{n=1}^{\infty}\alpha_n = \infty$ and (4.6) we have that $x_n \to z_0$. This completes the proof. ■

Next, we prove another strong convergence theorem which is related to [19].

Theorem 10. *Let C be a closed convex subset of a real Hilbert space H. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H and let B be a maximal monotone operator on H such that the domain of B is included in C. Let $J_\lambda = (I+\lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Suppose that $(A+B)^{-1}0 \neq \emptyset$. Let $u \in C$, $x_1 = x \in C$ and let $\{x_n\} \subset C$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)(\alpha_n u + (1-\alpha_n)J_{\lambda_n}(x_n - \lambda_n A x_n))$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0,\infty)$, $\{\beta_n\} \subset (0,1)$ and $\{\alpha_n\} \subset (0,1)$ satisfy

$$0 < a \leq \lambda_n \leq 2\alpha, \quad \lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0,$$

$$0 < c \leq \beta_n \leq d < 1, \quad \lim_{n\to\infty}\alpha_n = 0 \quad \textit{and} \quad \sum_{n=1}^{\infty}\alpha_n = \infty.$$

Then $\{x_n\}$ converges strongly to a point z_0 of $(A+B)^{-1}0$, where $z_0 = P_{(A+B)^{-1}0}u$.

Proof. Let $z \in (A+B)^{-1}0$. From $z = J_{\lambda_n}(z - \lambda_n Az)$, we obtain that

$$\begin{aligned}&\|J_{\lambda_n}(x_n - \lambda_n Ax_n) - z\|^2 \\ &= \|J_{\lambda_n}(x_n-\lambda_n Ax_n) - J_{\lambda_n}(z-\lambda_n Az)\|^2 \\ &\leq \|(x_n - \lambda_n Ax_n) - (z - \lambda_n Az)\|^2 \\ &= \|(x_n - z) - \lambda_n(Ax_n - Az)\|^2 \\ &= \|x_n - z\|^2 - 2\lambda_n\langle x_n - z, Ax_n - Az\rangle + \lambda_n^2\|Ax_n - Az\|^2 \\ &\leq \|x_n - z\|^2 - 2\lambda_n\alpha\|Ax_n - Az\|^2 + \lambda_n^2\|Ax_n - Az\|^2 \\ &= \|x_n - z\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Az\|^2 \\ &\quad \leq \|x_n - z\|^2.\end{aligned} \tag{4.7}$$

Let $y_n = \alpha_n u + (1-\alpha_n)J_{\lambda_n}(x_n - \lambda_n A x_n)$. Then we have

$$\begin{aligned}\|y_n - z\| &= \|\alpha_n(u-z) + (1-\alpha_n)(J_{\lambda_n}(x_n - \lambda_n A x_n) - z)\| \\ &\leq \alpha_n \|u - z\| + (1-\alpha_n)\|x_n - z\|.\end{aligned}$$

Using this, we get

$$\begin{aligned}&\|x_{n+1} - z\| \\ &= \|\beta_n(x_n - z) + (1-\beta_n)(y_n - z)\| \\ &\leq \beta_n \|x_n - z\| + (1-\beta_n)\|y_n - z\| \\ &\leq \beta_n \|x_n - z\| + (1-\beta_n)(\alpha_n \|u-z\| + (1-\alpha_n)\|x_n - z\|) \\ &= (1-\alpha_n(1-\beta_n))\|x_n - z\| + \alpha_n(1-\beta_n)\|u - z\|.\end{aligned}$$

Putting $K = \max\{\|x_1 - z\|, \|u - z\|\}$, we have that $\|x_n - z\| \leq K$ for all $n \in \mathbb{N}$. In fact, it is obvious that $\|x_1 - z\| \leq K$. Suppose that $\|x_k - z\| \leq K$ for some $k \in \mathbb{N}$. Then, we have that

$$\begin{aligned}\|x_{k+1} - z\| &\leq (1-\alpha_k(1-\beta_k))\|x_k - z\| + \alpha_k(1-\beta_k)\|u - z\| \\ &\leq (1-\alpha_k(1-\beta_k))K + \alpha_k(1-\beta_k)K = K.\end{aligned}$$

By induction, we obtain that $\|x_n - z\| \leq K$ for all $n \in \mathbb{N}$. Then, $\{x_n\}$ is bounded. Furthermore, $\{Ax_n\}$, $\{y_n\}$ and $\{J_{\lambda_n}(x_n - \lambda_n A x_n)\}$ are bounded. Putting $u_n = x_n - \lambda_n A x_n$, we have

$$\begin{aligned}y_{n+1} - y_n &= (\alpha_{n+1} - \alpha_n)u + (1-\alpha_{n+1})J_{\lambda_{n+1}}(x_{n+1} - \lambda_{n+1} A x_{n+1}) \\ &\quad - (1-\alpha_n)J_{\lambda_n}(x_n - \lambda_n A x_n) \\ &= (\alpha_{n+1} - \alpha_n)u + (1-\alpha_{n+1})\{J_{\lambda_{n+1}}(x_{n+1} - \lambda_{n+1} A x_{n+1}) \\ &\quad - J_{\lambda_{n+1}}u_n + J_{\lambda_{n+1}}u_n - J_{\lambda_n}u_n + J_{\lambda_n}u_n\} - (1-\alpha_n)J_{\lambda_n}u_n.\end{aligned}$$

So, we have from Lemma 2 that

$$\begin{aligned}\|y_{n+1} - y_n\| &\leq |\alpha_{n+1} - \alpha_n|\|u\| \\ &\quad + (1-\alpha_{n+1})\|x_{n+1} - \lambda_{n+1} A x_{n+1} - (x_n - \lambda_n A x_n)\| \\ &\quad + (1-\alpha_{n+1})\|J_{\lambda_{n+1}}u_n - J_{\lambda_n}u_n\| + |\alpha_{n+1} - \alpha_n|\|J_{\lambda_n}u_n\| \\ &\leq |\alpha_{n+1} - \alpha_n|\|u\| + \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\| \\ &\quad + |\alpha_{n+1} - \alpha_n|\|J_{\lambda_n}u_n\| + \|J_{\lambda_{n+1}}u_n - J_{\lambda_n}u_n\| \\ &\leq |\alpha_{n+1} - \alpha_n|\|u\| + \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\| \\ &\quad + |\alpha_{n+1} - \alpha_n|\|J_{\lambda_n}u_n\| + \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}}\|J_{\lambda_{n+1}}u_n - u_n\|.\end{aligned}$$

It follows that

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\le 0.$$

From Lemma 3, we get

$$y_n - x_n \to 0. \tag{4.8}$$

Consequently, we obtain

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|y_n-x_n\| = 0.$$

Take $\lambda_0 \in [a, 2\alpha]$. Putting $u_n = x_n - \lambda_n A x_n$ and $y_n = \alpha_n u + (1-\alpha_n)J_{\lambda_n}(I-\lambda_n A)x_n$, we have from Lemma 2 that

$$\begin{aligned}
&\|\alpha_n u + (1-\alpha_n)J_{\lambda_0}(I-\lambda_0 A)x_n - y_n\| \\
&= (1-\alpha_n)\|J_{\lambda_0}(I-\lambda_0 A)x_n - J_{\lambda_n}(I-\lambda_n A)x_n\| \\
&= (1-\alpha_n)\|J_{\lambda_0}(I-\lambda_0 A)x_n - J_{\lambda_0}(I-\lambda_n A)x_n \\
&\quad + J_{\lambda_0}(I-\lambda_n A)x_n - J_{\lambda_n}(I-\lambda_n A)x_n\| \\
&\le (1-\alpha_n)\{\|(I-\lambda_0 A)x_n - (I-\lambda_n A)x_n\| + \|J_{\lambda_0}u_n - J_{\lambda_n}u_n\|\} \\
&\le (1-\alpha_n)\{|\lambda_0-\lambda_n|\|Ax_n\| + \frac{|\lambda_0-\lambda_n|}{\lambda_0}\|J_{\lambda_0}u_n - u_n\|\}.
\end{aligned} \tag{4.9}$$

We also have

$$\begin{aligned}
&\|x_n - J_{\lambda_0}(I-\lambda_0 A)x_n\| \\
&\le \|x_n - y_n\| + \|y_n - (\alpha_n u + (1-\alpha_n)J_{\lambda_0}(I-\lambda_0 A)x_n)\| \\
&\quad + \|\alpha_n u + (1-\alpha_n)J_{\lambda_0}(I-\lambda_0 A)x_n - J_{\lambda_0}(I-\lambda_0 A)x_n\| \\
&= \|x_n - y_n\| + \|y_n - (\alpha_n u + (1-\alpha_n)J_{\lambda_0}(I-\lambda_0 A)x_n)\| \\
&\quad + \alpha_n\|u - J_{\lambda_0}(I-\lambda_0 A)x_n\|.
\end{aligned} \tag{4.10}$$

We will use (4.9) and (4.10) later.

Put $z_0 = P_{(A+B)^{-1}0}u$. Let us show that $\limsup_{n\to\infty}\langle u - z_0, y_n - z_0\rangle \le 0$. Put $A = \limsup_{n\to\infty}\langle u-p_0, y_n - p_0\rangle$. Then without loss of generality, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $A = \lim_{i\to\infty}\langle u - p_0, y_{n_i} - p_0\rangle$ and $\{y_{n_i}\}$ converges weakly some point $w \in C$. From $\|x_n - y_n\| \to 0$, we also have that $\{x_{n_i}\}$ converges weakly to $w \in C$. On the other hand, since $\{\lambda_n\} \subset (0,\infty)$ satisfies $0 < a \le \lambda_n \le 2\alpha$, there exists a subsequence $\{\lambda_{n_{i_j}}\}$ of $\{\lambda_{n_i}\}$ such that $\{\lambda_{n_{i_j}}\}$ converges to a number $\lambda_0 \in [a, 2\alpha]$. Using (4.9), we have that

$$\|\alpha_{n_{i_j}}u + (1-\alpha_{n_{i_j}})J_{\lambda_0}(I-\lambda_0 A)x_{n_{i_j}} - y_{n_{i_j}}\| \to 0.$$

Furthermore, using (4.10), we have that

$$\begin{aligned}
&\|x_{n_{i_j}} - J_{\lambda_0}(I-\lambda_0 A)x_{n_{i_j}}\| \\
&\le \|x_{n_{i_j}} - y_{n_{i_j}}\| + \|y_{n_{i_j}} - \{\alpha_{n_{i_j}} u + (1-\alpha_{n_{i_j}})J_{\lambda_0}(I-\lambda_0 A)x_{n_{i_j}}\}\| \\
&\quad + \alpha_{n_{i_j}} \|u - J_{\lambda_0}(I-\lambda_0 A)x_{n_{i_j}}\| \to 0.
\end{aligned}$$

Since $J_{\lambda_0}(I-\lambda_0 A)$ is nonexpansive, we have $w = J_{\lambda_0}(I-\lambda_0 A)w$. This means that $0 \in Aw + Bw$. So, we have

$$A = \lim_{j\to\infty} \langle u - z_0, y_{n_{i_j}} - z_0\rangle = \langle u - z_0, w - z_0\rangle \le 0.$$

Since $y_n - p_0 = \alpha_n(u-p_0) + (1-\alpha_n)(J_{\lambda_n}(x_n - \lambda_n Ax_n) - p_0)$, we have

$$\begin{aligned}
&\|y_n - p_0\|^2 - 2\alpha_n \langle u - p_0, y_n - p_0\rangle \\
&= (1-\alpha_n)^2 \|J_{\lambda_n}(x_n - \lambda_n Ax_n) - p_0\|^2 - \alpha_n^2 \|u - p_0\|^2 \\
&\le (1-\alpha_n)^2 \|J_{\lambda_n}(x_n - \lambda_n Ax_n) - p_0\|^2
\end{aligned}$$

and hence

$$\|y_n - p_0\|^2 \le (1-\alpha_n)^2 \|J_{\lambda_n}(x_n - \lambda_n Ax_n) - p_0\|^2 + 2\alpha_n \langle x - p_0, y_n - p_0\rangle.$$

From (4.7), we have

$$\|y_n - p_0\|^2 \le (1-\alpha_n)^2 \|x_n - p_0\|^2 + 2\alpha_n \langle x - p_0, y_n - p_0\rangle.$$

This implies that

$$\begin{aligned}
&\|x_{n+1} - p_0\|^2 \\
&\le \beta_n \|x_n - p_0\|^2 + (1-\beta_n)\|y_n - p_0\|^2 \\
&\le \beta_n \|x_n - p_0\|^2 + (1-\beta_n)\Big((1-\alpha_n)^2 \|x_n - p_0\|^2 + 2\alpha_n \langle x - p_0, y_n - p_0\rangle\Big) \\
&= \big(\beta_n + (1-\beta_n)(1-\alpha_n)^2\big)\|x_n - p_0\|^2 + 2(1-\beta_n)\alpha_n \langle x - p_0, y_n - p_0\rangle \\
&\le (1-(1-\beta_n)\alpha_n)\|x_n - p_0\|^2 + 2(1-\beta_n)\alpha_n \langle x - p_0, y_n - p_0\rangle.
\end{aligned}$$

By Lemma 4, we obtain that $x_n \to p_0$. This completes the proof. ∎

5. Applications

Let H be a Hilbert space and let f be a proper lower semicontinuous convex function of H into $(-\infty, \infty]$. Then, the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{z \in H : f(x) + \langle z, y - x\rangle \le f(y),\ y \in H\}$$

for all $x \in H$; see, for instance, [23]. From Rockafellar [17], we know that ∂f is maximal monotone. Let C be a nonempty closed convex subset of H and let i_C be the indicator function of C, i.e.,

$$i_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}$$

Then, i_C is a proper lower semicontinuous convex function of H into $(-\infty, \infty]$ and then the subdifferential ∂i_C of i_C is a maximal monotone operator. So, we can define the resolvent J_λ of ∂i_C for $\lambda > 0$, i.e.,

$$J_\lambda x = (I + \lambda \partial i_C)^{-1} x$$

for all $x \in H$. We have that for any $x \in H$ and $z \in C$,

$$\begin{aligned} z = J_\lambda x &\iff x \in z + \lambda \partial i_C z \\ &\iff x \in z + \lambda N_C z \\ &\iff x - z \in \lambda N_C z \\ &\iff \frac{1}{\lambda}\langle x - z, v - z\rangle \le 0, \ \forall v \in C \\ &\iff \langle x - z, v - z\rangle \le 0, \ \forall v \in C \\ &\iff z = P_C x, \end{aligned}$$

where $N_C z$ is the normal cone to C at z, i.e.,

$$N_C z = \{x \in H : \langle x, v - z\rangle \le 0, \ \forall v \in C\}.$$

Now, using Theorems 9 and 10, we can obtain strong convergence theorems for finding a solution of the variational inequality in a Hilbert space.

Theorem 11. *Let C be a closed convex subset of a real Hilbert space H. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H such that $VI(C, A) \neq \emptyset$. Let $u \in C$, $x_1 = x \in C$ and let $\{x_n\}$ be a sequence in C generated by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) P_C(x_n - \lambda_n A x_n)$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy

$$0 < a \le \lambda_n \le 2\alpha, \quad \lim_{n\to\infty} \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty,$$

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \textit{and} \quad \lim_{n\to\infty} \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty.$$

Then $\{x_n\}$ converges strongly to a point z_0 of $VI(C, A)$, where $z_0 = P_{VI(C,A)} u$.

Proof. Setting $B = \partial i_C$ in Theorem 9, we know that $J_{\lambda_n} = P_C$ for all $\lambda_n > 0$. Furthermore, we have

$$\begin{aligned} z \in (A + \partial i_C)^{-1}0 &\Longleftrightarrow 0 \in Az + \partial i_C z \\ &\Longleftrightarrow 0 \in Az + N_C z \\ &\Longleftrightarrow -Az \in N_C z \\ &\Longleftrightarrow \langle -Az, v - z\rangle \le 0,\ \forall v \in C \\ &\Longleftrightarrow \langle Az, v - z\rangle \ge 0,\ \forall v \in C \\ &\Longleftrightarrow z \in VI(C, A). \end{aligned}$$

So we obtain the desired result by Theorem 10. ∎

As in the proof of Theorem 11, we get the following theorem.

Theorem 12. *Let C be a closed convex subset of a real Hilbert space H. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H such that $VI(C, A) \neq \emptyset$. Let $u \in C$, $x_1 = x \in C$ and let $\{x_n\}$ be a sequence in C generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)\{\alpha_n u + (1 - \alpha_n)P_C(x_n - \lambda_n A x_n)\}$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, \infty)$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy

$$\begin{aligned} &0 < a \le \lambda_n \le 2\alpha, \quad \lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0, \\ &0 < c \le \beta_n \le d < 1, \quad \lim_{n\to\infty} \alpha_n = 0 \quad \textit{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty. \end{aligned}$$

Then $\{x_n\}$ converges strongly to a point z_0 of $VI(C, A)$, where $z_0 = P_{VI(C,A)}u$.

Using Theorems 11 and 12, we can obtain strong convergence theorems for widely strict pseudo-contractions in a Hilbert space.

Theorem 13. *Let H be a real Hilbert space and let C be a closed convex subset of H. Let $k < 1$. Let U be a widely k-strict pseudo-contraction of C into H such that $F(U) \neq \emptyset$. Let $u \in C$, $x_1 = x \in C$ and let $\{x_n\} \subset C$ be a sequence generated by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)P_C\{(1 - t_n)Ux_n + t_n x_n\}$$

for all $n \in \mathbb{N}$, where $\{t_n\} \subset (-\infty, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy

$$\begin{aligned} &k \le t_n \le b < 1, \quad \sum_{n=1}^{\infty} |t_n - t_{n+1}| < \infty, \\ &\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \textit{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty. \end{aligned}$$

Then $\{x_n\}$ converges strongly to a point z_0 of $F(U)$, where $z_0 = P_{F(U)}u$.

Proof. We know from Lemma 8 that $I - U$ is $\frac{1-k}{2}$-inverse-strongly monotone. Setting $A = I - U$, $a = 1 - b, \lambda_n = 1 - t_n$ and $2\alpha = 1 - k$ in Theorem 11, we get from $k \leq t_n \leq b < 1$ that $0 < a \leq \lambda_n \leq 2\alpha$,

$$\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| = \sum_{n=1}^{\infty} |t_{n+1} - t_n| < \infty$$

and

$$I - \lambda_n A = I - (1 - t_n)(I - U) = (1 - t_n)U + t_n I.$$

Furthermore, putting $B = \partial i_C$, we have from Lemma 5 that

$$\begin{aligned} z \in (A + \partial i_C)^{-1} &\Longleftrightarrow 0 \in Az + \partial i_C z \\ &\Longleftrightarrow 0 \in z - Uz + N_C z \\ &\Longleftrightarrow Uz - z \in N_C z \\ &\Longleftrightarrow \langle Uz - z, v - z \rangle \leq 0, \ \forall v \in C \\ &\Longleftrightarrow P_C U z = z \\ &\Longleftrightarrow Uz = z. \end{aligned}$$

So, we obtain $(A + \partial i_C)^{-1} 0 = \mathrm{F}(U)$. Thus, we obtain the desired result by using Theorem 11. ∎

We obtain Zhou's theorem (Theorem 1) by assumming $0 \leq k < 1$ in Theorem 13. As in the proof of Theorem 13, we also get the following theorem.

Theorem 14. *Let H be a real Hilbert space and let C be a closed convex subset of H. Let $k < 1$. Let U be a widely k-strict pseudo-contraction of C into H such that $F(U) \neq \emptyset$. Let $u \in C, x_1 = x \in C$ and let $\{x_n\} \subset C$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u + (1 - \alpha_n) P_C\{(1 - t_n)Ux_n + t_n x_n\}$$

for all $n \in \mathbb{N}$, where $\{t_n\} \subset (-\infty, 1)$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy

$$\begin{aligned} &k \leq t_n \leq b, \quad \lim_{n\to\infty} (t_n - t_{n+1}) = 0, \\ &0 < c \leq \beta_n \leq d < 1, \quad \lim_{n\to\infty} \alpha_n = 0 \quad \textit{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty. \end{aligned}$$

Then $\{x_n\}$ converges strongly to a point z_0 of $F(U)$, where $z_0 = P_{F(U)}u$.

Next, using Theorems 9 and 10, we consider the problem for finding a solution of the generalized equilibrium problem in a Hilbert space. For solving the equilibrium problem, let us assume that the bifunction $f : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $f(x,x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x,y) + f(y,x) \le 0$ for all $x, y \in C$;
(A3) for all $x, y, z \in C$,

$$\limsup_{t \downarrow 0} f(tz + (1-t)x, y) \le f(x,y);$$

(A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

The following lemma appears implicitly in Blum and Oettli [5].

Lemma 15. ([Blum and Oettli]). *Let C be a nonempty closed convex subset of H and let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \ \forall y \in C.$$

The following lemma was also given in Combettes and Hirstoaga [8].

Lemma 16. *Assume that $f : C \times C \to \mathbb{R}$ satisfies $(A1)-(A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r x = \left\{ z \in C : f(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \ \forall y \in C \right\}$$

for all $x \in H$. Then, the following hold:

(1) *T_r is single-valued;*

(2) *T_r is a firmly nonexpansive mapping, i.e., for all $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y\rangle;$$

(3) *$F(T_r) = EP(f)$;*

(4) *$EP(f)$ is closed and convex.*

We call such T_r the resolvent of f for $r > 0$. Using Lemmas 15 and 16, we know the following lemma [20]. See [1] for a more general result.

Lemma 17. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $f : C \times C \to \mathbb{R}$ satisfy $(A1)-(A4)$. Let A_f be a set-valued mapping of H into itself defined by*

$$A_f x = \begin{cases} \{z \in H : f(x,y) \ge \langle y - x, z\rangle, \ \forall y \in C\}, \ x \in C, \\ \emptyset, \ x \notin C. \end{cases}$$

Then, $EP(f) = A_f^{-1}0$ and A_f is a maximal monotone operator with $\mathrm{dom}(A_f) \subset C$. Furthermore, for any $x \in H$ and $r > 0$, the resolvent T_r of f coincides with the resolvent of A_f, i.e.,

$$T_r x = (I + rA_f)^{-1} x.$$

Using Lemma 17, we obtain the following result.

Theorem 18. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\alpha > 0$ and let A be an α-inverse- strongly monotone mapping of C into H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$ and let T_λ be the resolvent of f for $\lambda > 0$. Suppose that $EP(f, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by $u \in C$, $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T_{\lambda_n}(I - \lambda_n A)x_n, \ \forall n \in \mathbb{N},$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset (0, \infty)$ satisfy

$$0 < a \leq \lambda_n \leq 2\alpha, \ \lim_{n\to\infty} \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty,$$
$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty, \ \textit{and} \ \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty.$$

Then, $\{x_n\}$ converges strongly to $P_{EP(f,A)}u$.

Proof. For the bifunction f, we can define A_f in Lemma 17. Putting $B = A_f$ in Theorem 9, we obtain from Lemma 17 that $J_{\lambda_n} = T_{\lambda_n}$ for all $n \in \mathbb{N}$. Furthermore, we have that for $\lambda > 0$,

$$\begin{aligned} z \in (A + A_f)^{-1}0 &\Longleftrightarrow 0 \in Az + A_f z \\ &\Longleftrightarrow 0 \in \lambda Az + \lambda A_f z \\ &\Longleftrightarrow z - \lambda Az \in z + \lambda A_f z \\ &\Longleftrightarrow z = T_\lambda(z - \lambda Az) \\ &\Longleftrightarrow f(z, y) + \frac{1}{\lambda}\langle y - z, z - (z - \lambda Az)\rangle \geq 0, \ \forall y \in C \\ &\Longleftrightarrow f(z, y) + \langle y - z, Az\rangle \geq 0, \ \forall y \in C \\ &\Longleftrightarrow z \in EP(f, A). \end{aligned}$$

So, we obtain the desired result by Theorem 9. ∎

As in the proof of Theorem 18, we get the following theorem which is related to [19, Theorem 4.1].

Theorem 19. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\alpha > 0$ and let A be an α-inverse- strongly monotone mapping of C into H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$ and let T_λ be the resolvent of f for $\lambda > 0$. Suppose that $EP(f, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by $u \in C$, $x_1 = x \in C$ and*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)\{\alpha_n u + (1 - \alpha) T_{\lambda_n}(I - \lambda_n A)x_n\}, \ \forall n \in \mathbb{N},$$

where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$ *and* $\{\lambda_n\} \subset (0,\infty)$ *satisfy*

$$0 < a \le \lambda_n \le 2\alpha,\ \lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0,$$
$$0 < c \le \beta_n \le d < 1, \quad \lim_{n\to\infty} \alpha_n = 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then, $\{x_n\}$ *converges strongly to* $P_{EP(f,A)}u$.

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