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RELATIONS BETWEEN GENERALIZED VECTOR VARIATIONAL-LIKE INEQUALITIES AND VECTOR OPTIMIZATION PROBLEMS

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Abstract. In this paper, we consider generalized vector variational-like inequalities involving Dini subdifferential. Some relations among these inequalities and vector optimization problems are presented.

1. INTRODUCTION

It is well known that the vector variational inequality provides the necessary and sufficient conditions for a solution of a vector optimization problem if each component of the vector-valued function is convex and differentiable. In the last decade, several authors studied such kind of necessary and sufficient conditions when the vector-valued function is not necessarily convex or differentiable in some sense; See, for example [1-14] and the references therein. The vector optimization problem may have a nonsmooth objective function. Therefore, Crespi et al. [15] introduced the Minty vector variational inequality problem (in short, MVVIP) defined by means of lower Dini derivative. They established some relations between a MVVIP and the solutions of vector minimization problem (both ideal and weak efficient but not efficient) solutions. Further, Ansari and Lee [3], and Lalitha and Mehta [9] introduced both the Minty and the Stampacchia type vector variational inequalities (in short, MVVIs and SVVIs, respectively) defined by means of upper Dini derivative. They established some existence results and gave some relations along these inequalities and vector optimization problems. Their approach seems to be more direct than the one adopted in [15]. Mishra and Wang [11]

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considered Stampacchia type vector variational inequalities involving Clarke's subdifferential. They gave some relations between their vector variational inequalities and vector optimization. Recently, we [2] considered both the Minty and the Stampacchia type vector variational-like inequalities (in short, MVVLIs and SVVLIs, respectively) for a bifunction. Such bifunction can be taken as Dini upper directional derivative or any other kind of directional derivatives which satisfy certain properties. Some existence results for these kinds of inequalities have been established. We also studied some relationships among these inequalities and vector optimization problems.

In the present paper, we extend the MVVLIs and SVVLIs for Dini subdifferentials and present some relations among these inequalities and vector optimization problems.

2. Preliminaries

Throughout the paper, $\mathbf{0}$ will be considered as a zero vector in $\mathbb{R}^n$. We denote by $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{R}^n$.

Definition 2.1. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a function and $x \in \mathbb{R}^n$ be a point where f is finite. The *Dini upper directional derivative* at the point $x \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$ [16] is defined by

$$f^D(x; d) = \limsup_{t \to 0^+} \frac{f(x+td) - f(x)}{t} = \inf_{s>0} \sup_{0<t<s} \frac{f(x+td) - f(x)}{t}.$$

Definition 2.2. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a function and $x \in \mathbb{R}^n$ be a point where f is finite. The *Dini upper subdifferential* of f at x [17] is defined by

$$\partial^D f(x) = \left\{ \xi \in \mathbb{R}^n : \langle \xi, v \rangle \leq f^D(x, v) \text{ for all } v \in \mathbb{R}^n \right\}.$$

The following mean value theorem will be used to establish the main result of this paper.

Theorem 2.1. [18, Corollary 2.4]. *Let $K \subseteq \mathbb{R}^n$ and $f : K \to \mathbb{R} \cup \{\pm\infty\}$ be finite and upper semicontinuous on an open set containing the segment $[a, b]$ of K. If $\partial^D f(x)$ is an upper semicontinuous set-valued map with nonempty equicontinuous values on $[a, b]$, then there exist $c \in [a, b]$ and $\xi \in \partial^D f(c)$ such that*

$$f(b) - f(a) = \langle \xi, b - a \rangle.$$

The assumption that $\partial^D f(x)$ is nonempty and equicontinuous is satisfied when $f^D(x; \cdot)$ is convex and continuous.

We say that the map $\eta : K \times K \to \mathbb{R}^n$ is *skew* if for all $x, y \in K$,

$$\eta(y, x) + \eta(x, y) = \mathbf{0}.$$

Condition A. Let $K \subseteq \mathbb{R}^n$ be an invex set w. r. t. η and let $g : K \to \mathbb{R}$ be a function. Then

$$g(x + \eta(y, x)) \leq g(y), \quad \text{for all } x, y \in K.$$

Condition C. Let $K \subseteq \mathbb{R}^n$ be an invex set w. r. t. $\eta : K \times K \to \mathbb{R}^n$. Then, for all $x, y \in K$, $\lambda \in [0, 1]$,

(a) $\eta(x, x + \lambda\eta(y, x)) = -\lambda\eta(y, x)$

(b) $\eta(y, x + \lambda\eta(y, x)) = (1 - \lambda)\eta(y, x)$

Obviously, the map $\eta(y, x) = y - x$ satisfies Condition C.

Definition 2.3. Let x be arbitrary point of K. The set K is said to be *invex at* x w. r. t. η if for all $y \in K$,

$$x + \lambda\eta(y, x) \in K, \quad \text{for all } \lambda \in [0, 1].$$

K is said to be an *invex set* w. r. t. η if K is invex at every point $x \in K$ w. r. t. η.

The definition of an invex set essentially says that there is a path starting from x which is contained in K. It is not required that y should be one of the end points of the path.

Definition 2.4. Let $g : K \to \mathbb{R} \cup \{\pm\infty\}$ be a function such that for all $x \in K$, $f(x)$ is finite, and let $\eta : K \times K \to \mathbb{R}^n$ be a bifunction. The function g is said to be *generalized invex* w. r. t. η if

$$\langle \xi, \eta(y, x)\rangle \leq g(y) - g(x), \quad \text{for all } x, y \in K \text{ and all } \xi \in \partial^D g(x). \tag{2.1}$$

The function g is said to be *strictly generalized invex* w. r. t. η if strict inequality holds in (2.1) for all $x \neq y$.

Definition 2.5. Let $K \subseteq \mathbb{R}^n$ be an invex set w. r. t. $\eta : K \times K \to \mathbb{R}^n$. A function $g : K \to \mathbb{R}$ is said to be *preinvex* w. r. t. η if

$$g(x + \lambda\eta(y, x)) \leq \lambda g(y) + (1 - \lambda)g(x), \quad \text{for all } x, y \in K \text{ and all } \lambda \in [0, 1].$$

Lemma 2.1. *Let $K \subseteq \mathbb{R}^n$ be an invex set w. r. t. $\eta : K \times K \to \mathbb{R}^n$ and $g : K \to \mathbb{R}$ be a function such that the Conditions A and C hold. If g is generalized invex w. r. t. η, then it is preinvex w. r. t. the same η.*

Proof. Suppose that $x, y \in K$ and $\lambda \in (0, 1)$. Since K is invex, we have $\hat{x} = x + \lambda\eta(y, x) \in K$. By generalized invexity w. r. t. η of f, we have

$$\langle \xi, \eta(y, \hat{x})\rangle \leq g(y) - g(\hat{x}), \quad \text{for all } \zeta \in \partial^D g(\hat{x}). \tag{2.2}$$

Similarly, the condition of generalized invexity applied to the pair $x, \hat{x}$ yields

$$\langle \zeta, \eta(x, \hat{x}) \rangle \leq g(x) - g(\hat{x}), \quad \text{for all } \zeta \in \partial^D g(\hat{x}). \tag{2.3}$$

Multiplying inequality (2.2) by λ and inequality (2.3) by $(1-\lambda)$ and then adding the resultants, we obtain

$$\lambda\langle \zeta, \eta(y, \hat{x}) \rangle + (1-\lambda)\langle \zeta, \eta(x, \hat{x}) \rangle \leq \lambda g(y) + (1-\lambda) g(x) - g(\hat{x}). \tag{2.4}$$

By Condition C, we have

$$\lambda\eta(y, \hat{x}) + (1-\lambda)\eta(x, \hat{x}) = \lambda(1-\lambda)\eta(y, x) - \lambda(1-\lambda)\eta(y, x) = \mathbf{0}.$$

Since

$$\lambda\langle \zeta, \eta(y, \hat{x}) \rangle + (1-\lambda)\langle \zeta, \eta(x, \hat{x}) \rangle = \langle \zeta, \lambda\eta(y, \hat{x}) + (1-\lambda)\eta(x, \hat{x}) \rangle = 0,$$

the inequality (2.4) yields the conclusion. ■

Lemma 2.2. *Let $K \subseteq \mathbb{R}^n$ be a nonempty set, $\eta : K \times K \to \mathbb{R}^n$ be skew and $g : K \to \mathbb{R}$ be a function. If g is generalized invex w. r. t. η, then $\partial^D g$ is generalized monotone w. r. t. the same η, that is, for all $x, y \in K$ and all $\zeta \in \partial^D g(x)$, $\xi \in \partial^D g(y)$,*

$$\langle \xi - \zeta, \eta(y, x) \rangle \geq 0.$$

Proof. Since g is generalized invex w. r. t. η, for all for all $x, y \in K$ and all $\zeta \in \partial^D g(x)$, $\xi \in \partial^D g(y)$, we have

$$\langle \zeta, \eta(y, x) \rangle \leq f(y) - f(x) \tag{2.5}$$

and

$$\langle \xi, \eta(x, y) \rangle \leq f(x) - f(y). \tag{2.6}$$

Adding inequalities (2.5) and (2.6), we obatin

$$\langle \xi, \eta(x, y) \rangle + \langle \zeta, \eta(y, x) \rangle \leq 0.$$

Since η is skew, we have

$$\langle \xi - \zeta, \eta(y, x) \geq 0,$$

and hence, $\partial^D g$ is generalized monotone w. r. t. η. ■

3. Generalized Vector Variational-like Inequalities and Vector Optimization Problems

Throughout the paper, unless otherwise specified, we assume that K is a nonempty subset of $\mathbb{R}^n$ and $\eta : K \times K \to \mathbb{R}^n$ is a given map. The interior of K is denoted by $\text{int } K$.

Let $f = (f_1, \ldots, f_\ell) : \mathbb{R}^n \to \mathbb{R}^\ell$ be a vector-valued function. We consider the following *vector optimization problem*:

(VOP) $\quad$ Minimize $f(x) = \big(f_1(x), \ldots, f_\ell(x)\big)$ subject to $x \in K$.

A point $\bar{x} \in K$ is said to be an *efficient* (or *Pareto*) *solution* (respectively, *weak efficient solution*) of (VOP) if

$$f(y) - f(\bar{x}) = \big(f_1(y) - f_1(\bar{x}), \ldots, f_\ell(y) - f_\ell(\bar{x})\big) \notin -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}, \text{ for all } y \in K$$
$$\Big(\text{respectively, } f(y) - f(\bar{x}) = \big(f_1(y) - f_1(\bar{x}), \ldots, f_\ell(y) - f_\ell(\bar{x})\big) \notin -\text{int}\, \mathbb{R}^\ell_+, \text{ for all } y \in K\Big),$$

where $\mathbb{R}^\ell_+$ is the nonnegative orthant of $\mathbb{R}^\ell$ and $\mathbf{0}$ is the origin of the nonnegative orthant, namely $\mathbb{R}^\ell_+$.

It is clear that every efficient solution is a weak efficient solution. However, Proposition 4.1 says that every weak efficient solution is an efficient solution if each f_i ($i \in \mathfrak{I}$) is strictly generalized invex w. r. t. η.

For each $i \in \mathfrak{I}$, let $f_i : K \to \mathbb{R}$ be a function such that for all $x \in K$, $f^D(x; \cdot)$ is convex and continuous. We consider the following *generalized Minty vector variational-like inequality problem*:

(GMVVLIP) Find $\bar{x} \in K$ such that for all $y \in K$ and all $\xi_i \in \partial^D f_i(y)$, $i \in \mathfrak{I} = \{1, \ldots, \ell\}$,

$$\langle \xi, \eta(y, \bar{x}) \rangle_\ell = \Big(\langle \xi_1, \eta(y, \bar{x}) \rangle, \ldots, \langle \xi_\ell, \eta(y, \bar{x}) \rangle\Big) \notin -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}.$$

When $\eta(y, x) = y - x$, then (GMVVLIP) is called *generalized Minty vector variational inequality problem* (in short, GMVVIP).

We also consider the following *generalized Stampacchia vector variational-like inequality problem*:

(GSVVLIP) Find $\bar{x} \in K$ such that for all $y \in K$, there exists $\zeta_i \in \partial^D f_i(\bar{x})$, $i \in \mathfrak{I} = \{1, \ldots, \ell\}$ such that

$$\langle \zeta, \eta(y, \bar{x}) \rangle_\ell = \Big(\langle \zeta_1, \eta(y, \bar{x}) \rangle, \ldots, \langle \zeta_\ell, \eta(y, \bar{x}) \rangle\Big) \notin -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}.$$

When $\eta(y, x) = y - x$, then (GSVVLIP) is called *generalized Stampacchia vector variational inequality problem* (in short, GSVVIP).

The following result provides the necessary and sufficient conditions for an efficient solution of (VOP).

Theorem 3.1. *Let $K \subseteq \mathbb{R}^n$ be an invex set w. r. t. $\eta : K \times K \to \mathbb{R}^n$ which is skew such that the Conditions A and C hold. For each $i \in \mathfrak{I} = \{1, \ldots, \ell\}$, let $f_i : K \to \mathbb{R}$ be generalized invex w. r. t. η and upper semicontinuous on any segment S of K such that for each $x \in K$, $f_i^D(x; \cdot)$ is convex and continuous and $\partial^D f_i$ is upper semicontinuous on S. Then, $\bar{x} \in K$ is an efficient solution of (VOP) if and only if it is a solution of (GMVVLIP).*

Proof. Let $\bar{x} \in K$ be a solution of (GMVVLIP) but not an efficient solution of (VOP). Then, there exists $z \in K$ such that

$$\big(f_1(z) - f_1(\bar{x}), \ldots, f_\ell(z) - f_\ell(\bar{x})\big) \in -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}. \tag{3.1}$$

Set $z(\lambda) := \bar{x} + \lambda\eta(z, \bar{x})$ for all $\lambda \in [0, 1]$. Since K is invex w. r. t. η, $z(\lambda) \in K$ for all $\lambda \in [0, 1]$. By Lemma 2.1, each f_i is preinvex w. r. t. η, and therefore,

$$f_i(z(\lambda)) = f_i(\bar{x} + \lambda\eta(z, \bar{x})) \leq \lambda f_i(z) + (1 - \lambda)f_i(\bar{x}), \quad \text{for each } i = 1, 2, \ldots, \ell,$$

that is,

$$f_i(\bar{x} + \lambda\eta(z, \bar{x})) - f_i(\bar{x}) \leq \lambda[f_i(z) - f_i(\bar{x})],$$

for all $\lambda \in [0, 1]$ and for each $i = 1, \ldots, \ell$. In particular, for $\lambda = 1$, we have

$$f_i(\bar{x} + \eta(z, \bar{x})) - f_i(\bar{x}) \leq f_i(z) - f_i(\bar{x}), \quad \text{for each } i = 1, \ldots, \ell. \tag{3.2}$$

By Theorem 2.1, there exist $\lambda_i \in (0, 1)$ and $\xi_i \in \partial^D f_i(z(\lambda_i))$, $i \in \mathfrak{I}$, where $z(\lambda_i) := \bar{x} + \lambda_i\eta(z, \bar{x})$, such that

$$\langle \xi_i, \eta(z, \bar{x})\rangle = f_i(\bar{x} + \eta(z, \bar{x})) - f_i(\bar{x}), \quad \text{for each } i = 1, \ldots, \ell. \tag{3.3}$$

By combining (3.2)–(3.3), we obtain

$$\langle \xi_i, \eta(z, \bar{x})\rangle \leq f_i(z) - f_i(\bar{x}), \quad \text{for each } i = 1, \ldots, \ell. \tag{3.4}$$

From (3.1)–(3.4), we get

$$\langle \xi_i, \eta(z, \bar{x})\rangle \leq 0, \quad \text{for each } i = 1, \ldots, \ell \tag{3.5}$$

with strict inequality holds for some k, where $1 \leq k \leq \ell$. Choose $\lambda_0 \in (0, 1)$ such that $\lambda_0 < \min\{\lambda_1, \ldots, \lambda_\ell\}$. By Condition C, for all $i = 1, 2, \ldots, \ell$, we have

$$\begin{aligned}
\eta\left(z(\lambda_0), z(\lambda_i)\right) &= \eta\left(\bar{x} + \lambda_0\eta(z, \bar{x}), \bar{x} + \lambda_i\eta(z, \bar{x})\right) \\
&= \eta\left(\bar{x} + \lambda_0\eta(z, \bar{x}), \bar{x} + \lambda_0\eta(z, \bar{x}) + (\lambda_i - \lambda_0)\eta(z, \bar{x})\right) \\
&= \eta\left(\bar{x}+\lambda_0\eta(z, \bar{x}), \bar{x}+\lambda_0\eta(z, \bar{x})+\frac{(\lambda_i-\lambda_0)}{(1-\lambda_0)}\eta(z, \bar{x} + \lambda_0\eta(z, \bar{x}))\right) \\
&= \frac{(\lambda_0 - \lambda_i)}{(1 - \lambda_0)}\eta\left(z, \bar{x} + \lambda_0\eta(z, \bar{x})\right) \\
&= (\lambda_0 - \lambda_i)\eta(z, \bar{x}),
\end{aligned}$$

that is,

$$\eta\left(z(\lambda_0), z(\lambda_i)\right) = (\lambda_0 - \lambda_i)\, \eta(z, \bar{x}). \tag{3.6}$$

By the skewness of η, we obtain

$$\eta\left(z(\lambda_i), z(\lambda_0)\right) = (\lambda_i - \lambda_0)\, \eta(z, \bar{x}). \tag{3.7}$$

Combining (3.5)–(3.6), we get

$$\langle \xi_i, \eta(z(\lambda_0), z(\lambda_i))\rangle \geq 0, \quad \text{for each } i = 1, \ldots, \ell$$

with strict inequality holds for some k, where $1 \leq k \leq \ell$. By Lemma 2.2, we have

$$\langle \xi_{i_0}, \eta(z(\lambda_i), z(\lambda_0))\rangle \leq 0, \quad \text{for all } \xi_{i_0} \in \partial^D f_i(z(\lambda_0)) \text{ and all } i = 1, \ldots, \ell$$

with strict inequality holds for some k, where $1 \leq k \leq \ell$. Therefore, by (3.7), we deduce

$$\langle \xi_{i_0}, \eta(z, \bar{x})\rangle \leq 0, \quad \text{for all } \xi_{i_0} \in \partial^D f_i(z(\lambda_0)) \text{ and all } i = 1, \ldots, \ell \tag{3.8}$$

with strict inequality holds for some k, where $1 \leq k \leq \ell$. By Condition C, $\eta(z, z(\lambda_0)) = (1 - \lambda_0)\eta(z, \bar{x})$, and thus, we have

$$\langle \xi_{i_0}, \eta(z, z(\lambda_0))\rangle \leq 0, \quad \text{for all } \xi_{i_0} \in \partial^D f_i(z(\lambda_0)) \text{ and all } i = 1, \ldots, \ell$$

with strict inequality holds for some k, where $1 \leq k \leq \ell$. This implies that

$$\Big(\langle \xi_{1_0}, \eta(z, z(\lambda_0))\rangle, \ldots, \langle \xi_{\ell_0}, \eta(x, z(\lambda_0))\rangle\Big) \in -\mathbb{R}^{\ell}_{+} \setminus \{\mathbf{0}\}$$

which contradicts to our supposition that $\bar{x}$ is a solution of (GMVVLIP).

Conversely, suppose that $\bar{x} \in K$ is an efficient solution of (VOP). Then, we have

$$\big(f_1(y) - f_1(\bar{x}), \ldots, f_\ell(y) - f_\ell(\bar{x})\big) \notin -\mathbb{R}^{\ell}_{+} \setminus \{\mathbf{0}\}, \quad \text{for all } y \in K. \tag{3.9}$$

Since each f_i is generalized invex w. r. t η, we deduce that

$$\langle \xi_i, \eta(\bar{x}, y)\rangle \leq f_i(\bar{x}) - f_i(y), \quad \text{for all } y \in K,\ \xi_i \in \partial^D f_i(y) \text{ and all } i \in \mathfrak{I}.$$

Also, since η is skew, we obtain

$$\langle \xi_i, \eta(y, \bar{x})\rangle \geq f_i(y) - f_i(\bar{x}), \quad \text{for all } y \in K,\ \xi_i \in \partial^D f_i(y) \text{ and all } i \in \mathfrak{I}. \tag{3.10}$$

From (3.9) and (3.10), it follows that $\bar{x}$ is a solution of (GMVVLIP). ■

Remark 3.1. Theorem 3.1 extends [6, Proposition 1], [10, Theorem 2.1], [13, Theorem 3.1] and [1, Theorem 3.1].

Theorem 3.2. *Let $K \subseteq \mathbb{R}^n$ be a nonempty set and $\eta : K \times K \to \mathbb{R}^n$ be a map. For each $i \in \mathfrak{I} = \{1, \ldots, \ell\}$, let $f_i : K \to \mathbb{R}$ be generalized invex w. r. t. η such that for all $x \in K$, $f_i^D(x; \cdot)$ is convex and continuous. If $\bar{x} \in K$ is a solution (GSVVLIP), then it is an efficient solution of (VOP).*

Proof. Since $\bar{x} \in X$ is a solution of (GSVVLIP), for any $y \in K$, there exist $\zeta_i \in \partial^D f_i(\bar{x})$, $i = 1, \ldots, \ell$, such that

$$\Big(\langle \zeta_1, \eta(y, \bar{x})\rangle, \ldots, \langle \zeta_\ell, \eta(y, \bar{x})\rangle\Big) \notin -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}. \tag{3.11}$$

Since each f_i is generalized invex w. r. t. η, we have

$$\langle \zeta_i, \eta(y, \bar{x})\rangle \leq f_i(y) - f_i(\bar{x}) \quad \text{for any } y \in K \text{ and for all } i \in \mathfrak{I}. \tag{3.12}$$

By combining (3.11) and (3.12), we obtain

$$\big(f_1(y) - f_1(\bar{x}), \ldots, f_\ell(y) - f_\ell(\bar{x})\big) \notin -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}, \quad \text{for all } y \in K.$$

Thus, $\bar{x} \in K$ is an efficient solution of (VOP). ∎

Remark 3.2. Theorem 3.2 extends [10, Theorem 2.2] to the nonconvex setting and generalizes [1, Theorem 3.2]. Theorem 3.2 is different from [11, Theorem 3.1] in the following aspects. We used Dini upper subdifferential but Mishra and Wang [11] used Clarke's subdifferential. In Theorem 3.2, f_i is generalized invex but not necessarily locally Lipschitz. However, in Theorem 3.1 in [11], f_i is Locally Lipschtiz and invex.

4. Generalized Weak Vector Variational-like Inequalities

Throughout this section, we assume that for each $i \in \mathfrak{I}$, $f_i : K \to \mathbb{R}$ is generalized invex w. r. t. $\eta : K \times K \to \mathbb{R}^n$ such that for all $x \in K$, $f_i^D(x; \cdot)$ is convex and continuous.

We consider the following weak forms of generalized Minty vector variational-like inequality problem and generalized Stampaccia vector variational-like inequality problem.

(WGMVVLIP) Find $\bar{x} \in K$ such that for all $y \in K$ and all $\xi_i \in \partial^D f_i(y)$, $i \in \mathfrak{I} = \{1, \ldots, \ell\}$

$$\langle \xi, \eta(y, \bar{x})\rangle_\ell = \Big(\langle \xi_1, \eta(y, \bar{x})\rangle, \ldots, \langle \xi_\ell, \eta(y, \bar{x})\rangle\Big) \notin -\text{int } \mathbb{R}^\ell_+.$$

(WGSVVLIP) Find $\bar{x} \in K$ such that for all $y \in K$, there exists $\zeta_i \in \partial^D f_i(\bar{x})$, $i \in \mathcal{I} = \{1, \ldots, \ell\}$ such that

$$\langle \zeta, \eta(y, \bar{x}) \rangle_\ell = \Big(\langle \zeta_1, \eta(y, \bar{x}) \rangle, \ldots, \langle \zeta_\ell, \eta(y, \bar{x}) \rangle \Big) \notin -\mathrm{int}\ \mathbb{R}^\ell_+.$$

Of course, when $\eta(y, x) = y = x$, then (WGMVVLIP) and (WGSVVLIP) are called weak forms of generalized Minty vector variational inequality problem and generalized Stampacchia vector variational inequality problem, respectively.

The following result says that every solution of (WGSVVLIP) is a solution of (WGMVVLIP) if each f_i is generalized invex.

Theorem 4.1. *Let $K \subseteq \mathbb{R}^n$ be a nonempty set and $\eta : K \times K \to \mathbb{R}^n$ be skew. For each $i \in \mathcal{I} = \{1, \ldots, \ell\}$, let $f_i : K \to \mathbb{R}$ be generalized invex w. r. t. η. If $\bar{x} \in K$ is a solution (WGSVVLIP), then it is a solution of (WGMVVLIP).*

Proof. Let $\bar{x} \in K$ be a solution of (WGSVVLIP). Then, for any $y \in K$, there exist $\zeta_i \in \partial^D f_i(\bar{x})$, $i = 1, \ldots, \ell$, such that

$$\Big(\langle \zeta_1, \eta(y, \bar{x}) \rangle, \ldots, \langle \zeta_\ell, \eta(y, \bar{x}) \rangle \Big) \notin -\mathrm{int}\ \mathbb{R}^\ell_+. \tag{4.1}$$

Since each f_i is invex w. r. t. η, by Lemma 2.2, each $\partial^D f_i$ ($i \in \mathcal{I}$) is generalized monotone, and therefore, we have

$$\langle \xi_i - \zeta_i, \eta(y, \bar{x}) \rangle \geq 0 \quad \text{for all } y \in K,\ \xi_i \in \partial^D f_i(y) \text{ and for all } i \in \mathcal{I}. \tag{4.2}$$

From (4.1) and (4.2), it follows that for any $y \in K$ and any $\xi_i \in \partial^D f_i(y)$, $i \in \mathcal{I}$,

$$\Big(\langle \xi_1, \eta(y, \bar{x}) \rangle, \ldots, \langle \xi_\ell, \eta(y, \bar{x}) \rangle \Big) \notin -\mathrm{int}\ \mathbb{R}^\ell_+.$$

Thus, $\bar{x} \in K$ is a solution of (WGMVVLIP). ■

Now we present some relationship between the solution of (WGSVVLIP) and a weak efficient solution of (VOP).

Theorem 4.2. *Let $K \subseteq \mathbb{R}^n$ be a nonempty set and $\eta : K \times K \to \mathbb{R}^n$ be a map. For each $i \in \mathcal{I} = \{1, \ldots, \ell\}$, let $f_i : K \to \mathbb{R}$ be generalized invex w. r. t. η. If $\bar{x} \in K$ is a solution of (WGSVVLIP), then it is a weak efficient solution of (VOP).*

Proof. Suppose that $\bar{x}$ is a solution of (WGSVVLIP) but not a weak efficient solution of (VOP). Then, there exists $y \in K$ such that

$$\big(f_1(y) - f_1(\bar{x}), \ldots, f_\ell(y) - f_\ell(\bar{x}) \big) \in -\mathrm{int}\ \mathbb{R}^\ell_+. \tag{4.3}$$

Since each f_i, $i \in \mathcal{I}$, is generalized invex w. r. t. η, we have

$$\langle \zeta_i, \eta(y, \bar{x}) \rangle \leq f_i(y) - f_i(\bar{x}), \quad \text{for all } \zeta_i \in \partial^D f_i(\bar{x}). \tag{4.4}$$

Combining (4.3) and (4.4), we obtain

$$\Big(\langle\zeta_1,\eta(y,\bar{x})\rangle,\cdots,\langle\zeta_\ell,\eta(y,\bar{x})\rangle\Big)\in -\mathrm{int}\ \mathbb{R}^\ell_+,\quad \text{for all } \zeta_i\in\partial^D f_i(\bar{x})$$

which contradicts to our supposition that $\bar{x}$ is a solution of (WGSVVLIP). This completes the proof. ■

Theorem 4.3. *Let $K\subseteq\mathbb{R}^n$ be a nonempty set and $\eta: K\times K\to\mathbb{R}^n$ be a map. For each $i\in\mathfrak{I}=\{1,\ldots,\ell\}$, let $f_i: K\to\mathbb{R}$ be a function such that $-f_i$ is strictly generalized invex w. r. t. η. If $\bar{x}\in K$ is a weak efficient solution of (VOP), then it is a solution of (GSVVLIP).*

Proof. Suppose that $\bar{x}$ is not a solution of (GSVVLIP). Then, there exists $y\in K$ such that

$$\begin{aligned}&\Big(\langle\zeta_1,\eta(y,\bar{x})\rangle,\cdots,\langle\zeta_\ell,\eta(y,\bar{x})\rangle\Big)\in -\mathbb{R}^\ell_+\setminus\{\mathbf{0}\},\\ &\text{for all } \zeta_i\in\partial^D f_i(\bar{x}) \text{ and all } i\in\mathfrak{I}.\end{aligned}\tag{4.5}$$

Since each f_i, $i\in\mathfrak{I}$, is strictly generalized invex w. r. t. η, we have

$$\langle\zeta_i,\eta(y,\bar{x})\rangle > f_i(y)-f_i(\bar{x}),\quad \text{for all } \zeta_i\in\partial^D f_i(\bar{x}) \text{ and all } i\in\mathfrak{I}.\tag{4.6}$$

Combining (4.5) and (4.6), we obtain

$$\big(f_1(y)-f_1(\bar{x}),\ldots,f_\ell(y)-f_\ell(\bar{x})\big)\in -\mathrm{int}\ \mathbb{R}^\ell_+.$$

which contradicts to our supposition that $\bar{x}$ is a weak efficient solution of (VOP). This completes the proof. ■

Corollary 4.1. *Let $K\subseteq\mathbb{R}^n$ be a nonempty set and $\eta: K\times K\to\mathbb{R}^n$ be a map. For each $i\in\mathfrak{I}=\{1,\ldots,\ell\}$, let $f_i: K\to\mathbb{R}$ be a function such that $-f_i$ is strictly generalized invex w. r. t. η. If $\bar{x}\in K$ is an efficient solution of (VOP), then it is a solution of (GSVVLIP).*

Remark 4.3. In Theorem 4.3 and Corollary 4.1, we used Dini upper subdiiferential. However, Mishra and Wang [11] used Clarke's sundifferential in which locally Lipschitz condition is required.

Proposition 4.1. *Let $K\subseteq\mathbb{R}^n$ be a nonempty set and $\eta: K\times K\to\mathbb{R}^n$ be a map. For each $i\in\mathfrak{I}=\{1,\ldots,\ell\}$, let $f_i: K\to\mathbb{R}$ be strictly generalized invex w. r. t. η. If $\bar{x}\in K$ is a weak efficient solution of (VOP), then it is an efficient solution of (VOP).*

Proof. Suppose that $\bar{x}$ is a weak efficient solution of (VOP) but not an efficient solution of (VOP). Then, there exists $y\in K$ such that

$$\big(f_1(y)-f_1(\bar{x}),\ldots,f_\ell(y)-f_\ell(\bar{x})\big)\in -\mathbb{R}^\ell_+\setminus\{\mathbf{0}\}.\tag{4.7}$$

Since each f_i, $i \in \mathfrak{I}$, is strictly generalized invex w. r. t. η, we have

$$\langle \zeta_i, \eta(y, \bar{x}) \rangle \leq f_i(y) - f_i(\bar{x}), \quad \text{for all } \zeta_i \in \partial^D f_i(\bar{x}) \text{ and all } i \in \mathfrak{I}. \tag{4.8}$$

Combining (4.7) and (4.8), we obtain

$$\Big(\langle \zeta_1, \eta(y, \bar{x}) \rangle, \cdots, \langle \zeta_\ell, \eta(y, \bar{x}) \rangle \Big) \in -\text{int } \mathbb{R}^\ell_+, \quad \text{for all } \zeta_i \in \partial^D f_i(\bar{x}).$$

Hence $\bar{x}$ is not a solution of (WGSVVLIP). Then by Theorem 4.2, $\bar{x}$ is not a weak efficient solution of (VOP), a contradiction of our supposition. This completes the proof. ■

Theorem 4.4. *Let $K \subseteq \mathbb{R}^n$ be an invex set w. r. t. $\eta : K \times K \to \mathbb{R}^n$ such that η is skew. For each $i \in \mathfrak{I} = \{1, \ldots, \ell\}$, let $f_i : K \to \mathbb{R}$ be generalized invex w. r. t. η. If $\bar{x} \in K$ is a weak efficient solution of (VOP), then it is a solution of (WGMVVLIP).*

Proof. It is similar to the proof of the second part on Theorem 3.1. ■

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