SOME IDENTITIES CONNECTED WITH A CONTINUED FRACTION OF RAMANUJAN

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Dedicated to Professor H. M. Srivastava on his Seventieth Birthday

Abstract. We first prove two identities which are analogous to Entry 3.3.4 in Ramanujan's lost notebook. The identities in Entry 3.3.4 come out equal to a cubic theta function of Borwein and Borwein [**5**]. In our case they come out equal to $\frac{(q^4;q^4)^2}{(q^2;q^4)^2}C^2(q)$. We also express $C(q)$ in terms of theta functions $\phi(q)$ and $\psi(q)$. A series expansion of $\log C(q)$ is also given. One of the identities (9) is equivalent to a Theorem in partitions.

1. INTRODUCTION

In this paper we consider a continued fraction of Ramanujan $C(q)$ defined as (see Hirschhorn [**6**], Theorem 2)

$$(1) \qquad C(q) = \cfrac{1}{1+}\cfrac{1+q}{1+}\cfrac{q^2}{1+}\cfrac{q+q^3}{1+}\cfrac{q^4}{1+}\cdots$$

$$(2) \qquad = \frac{(q^2;q^4)_\infty^2}{(q;q^4)_\infty(q^3;q^4)_\infty},$$

and show that it is intimately connected with theta functions. This continued fraction $C(q)$ is analogous to the following famous Rogers-Ramanujan continued fraction $R(q)$:

$$(3) \qquad R(q) = \cfrac{1}{1+}\cfrac{q}{1+}\cfrac{q^2}{1+}\cfrac{q^3}{1+\ldots}$$

$$(4) \qquad = \frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}.$$

While studying this continued fraction $C(q)$, the author found identities for $C(q)$ which are analogous to the identities of Ramanujan, Entry 3.3.4 [2, p. 57]. These identities are equivalent to a Theorem in partitions and is stated in the respective section later. They become more interesting as they are also analogous to the Borwein and Borwein's cubic theta function. This makes the paper interesting. We also show that the continued fraction $C(q)$ can be expressed in terms of theta function $\phi(q)$, $\psi(q)$ and $\chi(q)$. Further study is done in the later part of the paper.

The paper is organized as follows. In Section 3 we prove two identities using Ramanujan's $_1\psi_1$ summation and show that each identity equals $\frac{(q^4;q^4)_\infty^2}{(q^2;q^4)_\infty^2}C^2(q)$. These identities are analogous to Entry 3.3.4 of Ramanujan [2, p. 57]. The identities in Entry 3.3.4 equal $\frac{1}{3}q^{-\frac{1}{3}}c(q)$ where $c(q)$ is one of the cubic theta functions of Borwein and Borwein [6] defined by

$$c(q) = \sum_{m,n}^{\infty} q^{(m+\frac{1}{3})^2+(m+\frac{1}{3})(n+\frac{1}{3})+(n+\frac{1}{3})^2},$$

see Berndt book [4, p. 109, Lemma 5.1 eq.(5.5)].

In Section 4 the continued fraction $C(q)$ is expressed in terms of theta functions $\phi(q)$ and $\psi(q)$ as $\chi(q)$-function. In Section 5 we give a series expansion for $\log C(q)$.

Lastly, in Section 6, we prove five q-identities The identities (25) and (26) may be seen as a factorization of the following identity:

$$\phi^2(-q) - \frac{\chi^2(q^2)}{\chi^2(-q^2)}\phi^2(-q^4) = -\frac{4q(q;q^2)_\infty}{(q^2;q^2)_\infty f^2(-q^4,-q^{12})}.$$

2. Basic Facts

We shall be using the customary q-product notation. Thus For $|q| < 1$,

$$(a)_0 = (a;q)_0 = 1, \quad \text{and for} \quad n \geq 1,$$
$$(a)_n = (a;q)_n = \prod_{k=0}^{n-1}(1 - aq^k).$$

Furthermore

$$(a)_\infty = (a;q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k), \quad |q| < 1.$$

Ramanujan defined general theta function by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, |ab| < 1.$$

The most important special cases of $f(a,b)$ are in Ramanujan's notation, $|q| < 1$

(5) $$\phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty},$$

(6) $$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}.$$

Lastly, we define

(7) $$\chi(q) := (-q;q^2)_\infty.$$

Ramanujan's $_1\psi_1$-summation formula:

$$_1\psi_1(a;b;q,z)$$

(8) $$= \sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = \frac{(b/a)_\infty (az)_\infty (q/az)_\infty (q)_\infty}{(q/a)_\infty (b/az)_\infty (b)_\infty (z)_\infty}, |b/a| < |z| < 1.$$

3. ANALOGOUS IDENTITIES

3.1. Identities each equal to $\frac{(q^4;q^4)_\infty^2}{(q^2;q^4)_\infty^2} C^2(q)$

We shall prove the following two identities, using $_1\psi_1$ summation

(9) $$\psi(q)^2 = \frac{(q^2;q^2)_\infty^4}{(q;q)_\infty^2} = \sum_{n=0}^{\infty} \frac{q^n}{1-q^{4n+1}} - \sum_{n=0}^{\infty} \frac{q^{3n+2}}{1-q^{4n+3}}$$

(10) $$= \sum_{n=0}^{\infty} \frac{q^{4n^2+2n}(1+q^{4n+1})}{1-q^{4n+1}} - \sum_{n=0}^{\infty} \frac{q^{4n^2+6n+2}(1+q^{4n+3})}{1-q^{4n+3}}$$

The identity (9) is equivalent to the Theorem (see Hirschhorn [**8**]),

The number of representations of n as a sum of two triangular numbers is equal to the excess of the number of divisors of $4n+1$ that are 1 modulo 4 over the number of divisors of $4n+1$ that are congruent to 3 modulo n.

We shall show that these identities equals $\frac{(q^4;q^4)_\infty^2}{(q^2;q^4)_\infty^2} C^2(q)$. Moreover, we shall write the right-hand sides of (9) and (10) in a very symmetric form.

Proof of (9). We have

(11) $$\sum_{n=0}^{\infty} \frac{q^n}{1-q^{4n+1}} - \sum_{n=0}^{\infty} \frac{q^{3n+2}}{1-q^{4n+3}} = \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{4n+1}}.$$

Writing q^4 for q and setting $a = q, b = q^5$ and $z = q$ in (8), we get

$$_1\psi_1(q;q^5;q^4,q) = \sum_{n=-\infty}^{\infty} \frac{(q;q^4)_n}{(q^5;q^4)_n}q^n$$

$$= (1-q)\sum_{n=-\infty}^{\infty} \frac{(q^5;q^4)_{n-1}}{(q^5;q^4)_n}q^n$$

$$= (1-q)\sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{4n+1}}.$$

Hence

$$\sum_{n=0}^{\infty} \frac{q^n}{1-q^{4n+1}} - \sum_{n=0}^{\infty} \frac{q^{3n+2}}{1-q^{4n+3}} = \frac{1}{(1-q)}{}_1\psi_1(q;q^5;q^4,q)$$

$$= \frac{(q^4,q^4,q^2,q^2;q^4)_\infty}{(q,q^3,q,q^3;q^4)_\infty}$$

$$= \frac{(q^4;q^4)_\infty^2(q^2;q^4)_\infty^2}{(q;q^4)_\infty^2(q^3;q^4)_\infty^2}$$

$$= \frac{(q^4;q^4)_\infty^4(q^2;q^4)_\infty^4}{(q;q)_\infty^2}$$

$$= \frac{(q^2;q^2)_\infty^4}{(q;q)_\infty^2},$$

which proves (9).

Proof of (*10*). We recall the following result of Andrews [**2**, p. 58]:

$$(12) \qquad \sum_{n=0}^{\infty} \frac{q^{kn}}{1-q^{jn+k}} = \sum_{n=0}^{\infty} q^{jn^2+2kn}\frac{1+q^{jn+k}}{1-q^{jn-k}}.$$

Taking $j = 4$ and $k = 1$ and $k = 3$, respectively, in (12), we have

$$(13) \qquad \sum_{n=0}^{\infty} \frac{q^n}{1-q^{4n+1}} = \sum_{n=0}^{\infty} \frac{q^{4n^2+2n}(1+q^{4n+1})}{1-q^{4n+1}}$$

and

$$(14) \qquad \sum_{n=0}^{\infty} \frac{q^{3n+2}}{1-q^{4n+3}} = \sum_{n=0}^{\infty} \frac{q^{4n^2+6n+2}(1+q^{4n+3})}{1-q^{4n+3}}.$$

From (13) and (14), we have (10).

3.2. Symmetric form of right-hand sides of (9) and (10)

We now write the right-hand sides of (9) and (10) in a symmetric form so that we can see that they are analogous to Borwein and Borwein's cubic $c(q)$ function.

We write the right-hand side of (9) in the following more symmetric form:

$$\sum_{n=0}^{\infty} \frac{q^n}{1-q^{4n+1}} - \sum_{n=0}^{\infty} \frac{q^{3n+2}}{1-q^{4n+3}}$$

(15)
$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} [q^n q^{m(4n+1)} - q^{3n+2} q^{m(4n+3)}],$$

$$= q^{-\frac{1}{4}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} [q^{(4n+1)(4m+1)/4} - q^{(4n+3)(4m+3)/4}].$$

We now write the right-hand side of (10) in the following symmetric form:

$$\sum_{n=0}^{\infty} \frac{q^{4n^2+2n}(1+q^{4n+1})}{1-q^{4n+1}} - \sum_{n=0}^{\infty} \frac{q^{4n^2+6n+2}(1+q^{4n+3})}{1-q^{4n+3}}$$

$$= \sum_{n=0}^{\infty} \frac{q^{4n^2+2n}(1+q^{4n+1})}{1-q^{4n+1}} - \sum_{n=1}^{\infty} \frac{q^{4n^2-2n}(1+q^{4n-1})}{1-q^{4n-1}}$$

(16)
$$= \sum_{n=0}^{\infty} \frac{q^{4n^2+2n}(1+q^{4n+1})}{1-q^{4n+1}} + \sum_{n=-1}^{-\infty} \frac{q^{4n^2+2n}(1+q^{4n+1})}{1-q^{4n+1}}$$

$$= \sum_{n=-\infty}^{\infty} \frac{q^{4n^2+2n}(1+q^{4n+1})}{1-q^{4n+1}}$$

$$= q^{-1/4} \sum_{n=-\infty}^{\infty} \frac{q^{\frac{(4n+1)^2}{4}}(1+q^{4n+1})}{1-q^{4n+1}}.$$

Now, finally, we express the right-hand side of (9) in terms of $C^2(q)$.
Since

$$\sum_{n=0}^{\infty} \frac{q^n}{1-q^{4n+1}} - \sum_{n=0}^{\infty} \frac{q^{3n+2}}{1-q^{4n+3}}$$

$$= \frac{(q^4;q^4)_{\infty}^2 (q^2;q^4)_{\infty}^2}{(q;q^4)_{\infty}^2 (q^3;q^4)_{\infty}^2} = \frac{(q^4;q^4)_{\infty}^2}{(q^2;q^4)_{\infty}^2} C^2(q),$$

by using (11), we can write $C(q)$ as follows:

(17)
$$\frac{(q^4;q^4)_{\infty}^2}{(q^2;q^4)_{\infty}^2} C^2(q) = \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{4n+1}}.$$

Here we have used the definition of $C(q)$ given in (2). Thus (9), (10), (15) and (16) can be expressed in terms of $\frac{(q^4;q^4)_{\infty}^2}{(q^2;q^4)_{\infty}^2} C^2(q)$.

4. Relation of $C(q)$ with Theta Functions $\phi(q)$ and $\psi(q)$

In this section we give representation of $C(q)$ in terms of the theta functions $\phi(q)$ and $\psi(q)$:

$$(18) \qquad\qquad C(q) = \frac{\phi(-q^2)}{\psi(-q)}$$

$$(19) \qquad\qquad C(q) = \frac{\psi(q)}{\psi(q^2)}$$

$$(20) \qquad\qquad C(q) = \chi(q)\chi(-q^2)$$

and

$$(21) \qquad\qquad C(q) = \frac{\phi(q)}{\psi(q)}$$

By (2), we have

$$C(q) = \frac{(q^2; q^4)_\infty^2}{(q; q^4)_\infty (q^3; q^4)_\infty} = \frac{(q^2; q^4)_\infty^2 (q^4; q^4)_\infty}{(q; q^4)_\infty (q^3; q^4)_\infty (q^4; q^4)_\infty}$$

$$= \frac{f(-q^2, -q^2)}{(q; q^2)_\infty (q^4; q^4)_\infty} = \frac{\phi(-q^2)}{(q; q^2)_\infty (q^4; q^4)_\infty}, \quad \text{by (5)}.$$

Now, by [**3**, p. 37, Eq. (22.2)], we get

$$(-q; q^2)_\infty = \frac{1}{(q; q^2)_\infty (-q^2; q^2)_\infty} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty (q^4; q^4)_\infty}$$

$$(q; q^2)_\infty (q^4; q^4)_\infty = \frac{(q^2; q^2)_\infty}{(-q; q^2)_\infty} = \psi(-q). \quad \text{by (6)}$$

Hence

$$C(q) = \frac{\phi(-q^2)}{\psi(-q)},$$

which is (18).

$$= \frac{\psi(q)}{\psi(q^2)}, \quad \text{by [**3**, p. 40, Entry 25 (iii)]},$$

which is (19).

Lastly, we have

$$C(q) = \frac{(q^2; q^4)_\infty^2}{(q; q^4)_\infty (q^3; q^4)_\infty} = \frac{(q^2; q^4)_\infty^2}{(q; q^2)_\infty}$$

$$= (q^2; q^4)_\infty (-q; q^2)_\infty$$

$$= \chi(q)\chi(-q^2),$$

which is (20).

By means of a result given by Berndt [**3**, p. 40, Entry 25(iv)], we obtain

$$\frac{\phi(q)}{\psi(q)} = \frac{\psi(q)}{\psi(q^2)}.$$

Also, by (19), we get

$$C(q) = \frac{\phi(q)}{\psi(q)},$$

which is (21).

5. APPLICATION OF THE DEFINITION

Using another known result [3, p. 38, Entry 23(ii)]:

(22)
$$\log \psi(q) = \sum_{k=1}^{\infty} \frac{q^k}{k(1+q^k)},$$

we give a series expansion of $\log C(q)$.

Taking logarithmic of both sides of (19), we have

$$\log C(q) = \log \psi(q) - \log \psi(q^2)$$

$$= \sum_{k=1}^{\infty} \frac{q^k}{k(1+q^k)} - 2\sum_{k=1}^{\infty} \frac{q^{2k}}{2k(1+q^{2k})}, \quad \text{by (22)}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}q^k}{k(1+q^k)},$$

which provides a series expansion for $\log C(q)$.

6. FURTHER q-IDENTITIES

(23)
$$C(q)C(q^2)C(q^4)C(q^8)\ldots = \psi(q),$$

(24)
$$C(q)^2 C(q^2)C(q^4)C(q^8)\ldots = \phi(q),$$

(25)
$$\phi(-q) + \frac{\chi(q^2)}{\chi(-q^2)}\phi(-q^4)$$
$$= \frac{2(q;q^2)_\infty(-q^4;q^4)_\infty(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty} \frac{f^2(-q^6,-q^{10})f^2(-q^8,-q^8)}{f^2(-q^3,-q^{13})f^2(-q^5,-q^{11})}C^{-1}(q^4),$$

(26)
$$\phi(-q) - \frac{\chi(q^2)}{\chi(-q^2)}\phi(-q^4)$$
$$= -2q\frac{(q;q^2)_\infty(q^{16};q^{16})_\infty^4}{(q^2;q^2)_\infty(-q^4;q^4)_\infty}\frac{f^2(-q^2,-q^{14})}{f^2(-q,-q^{15})f^2(-q^7,-q^9)}C(q^4),$$

and

(27) $$\phi^2(-q) - \frac{\chi^2(q^2)}{\chi^2(-q^2)}\phi^2(-q^4) = -\frac{4q(q,q^2)_\infty}{(q^2;q^2)_\infty f^2(-q^4,-q^{12})}.$$

The identities (25) and (26) are a factorization of the identity (27).

6.1. Proofs of the identities

6.1.1. Proof of (23) and (24)

The identity (23) follows directly from (19). The identity (24) follows from (23) and (21).

6.1.2. Proof of (25)

Now, by using the identity:

$$\phi(-q) + \frac{\chi(q^2)}{\chi(-q^2)}\phi(-q^4) = \frac{(q;q)_\infty}{(-q;q)_\infty} + \frac{(-q^2;q^4)_\infty}{(q^2;q^4)_\infty}\frac{(q^4;q^4)_\infty}{(-q^4;q^4)_\infty}$$
$$= \frac{(q;q)_\infty}{(-q;q)_\infty}\left[1 + \frac{(-q;q)_\infty}{(q;q)_\infty}\frac{(q^4;q^4)_\infty}{(-q^4;q^4)_\infty}\frac{(-q^2;q^4)_\infty}{(q^2;q^4)_\infty}\right],$$

we have

$$\frac{(q;q)_\infty}{(-q;q)_\infty}\left[1 + \frac{(-q;q^4)_\infty}{(q;q^4)_\infty}\frac{(-q^2;q^4)_\infty^2}{(q^2;q^4)_\infty^2}\frac{(-q^3;q^4)_\infty}{(q^3;q^4)_\infty}\right]$$
$$= \frac{(q;q)_\infty}{(-q;q)_\infty}\left[1 + \frac{f(q,q^3)f(q^2,q^2)}{f(-q,-q^3)f(-q^2,-q^2)}\right]$$
$$= \frac{(q;q)_\infty}{(-q;q)_\infty}\left[\frac{f(-q,-q^3)f(-q^2,-q^2) + f(q,q^3)f(q^2,q^2)}{f(-q,-q^3)f(-q^2,-q^2)}\right].$$

Applying a known result [**3**, p. 45, Entry 29(i)] on the right-hand side of the above identity, we find that

$$\phi(-q) + \frac{\chi(q^2)}{\chi(-q^2)}\phi(-q^4) = \frac{2(q;q)_\infty}{(-q;q)_\infty}\frac{f(q^3,q^5)f(q^3,q^5)}{f(-q,-q^3)f(-q^2,-q^2)}$$
$$= \frac{2(q;q^2)_\infty(-q^4;q^4)_\infty(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty}\frac{f^2(-q^6,-q^{-10})f^2(-q^8,-q^8)}{f^2(-q^3,-q^{13})f^2(-q^5,-q^{11})}C^{-1}(q^4),$$

which proves (25).

6.1.3. Proof of (26)

Now we prove the companion identity of (25):

$$\phi(-q) - \frac{\chi(q^2)}{\chi(-q^2)}\phi(-q^4)$$

$$=\frac{(q;q)_\infty}{(-q;q)_\infty}\left[1 - \frac{(-q;q^4)_\infty}{(q;q^4)_\infty}\frac{(-q^2;q^4)_\infty^2}{(q^2;q^4)_\infty^2}\frac{(-q^3;q^4)_\infty}{(q^3;q^4)_\infty}\right]$$

$$= \frac{(q;q)_\infty}{(-q;q)_\infty}\left[\frac{f(-q,-q^3)f(-q^2,-q^2) - f(q,q^3)f(q^2,q^2)}{f(-q,-q^3)f(-q^2,-q^2)}\right]$$

$$= -\frac{2q(q;q)_\infty}{(-q;q)_\infty}\left[\frac{f^2(q,q^7)}{f(-q,-q^3)f(-q^2,-q^2)}\right],$$

where we have used a known identity [**3**, p. 45, Entry 29(ii)].
Finally, we have

$$\phi(-q) - \frac{\chi(q^2)}{\chi(-q^2)}\phi(-q^4)$$

$$= -2q\frac{(q;q^2)_\infty(q^{16};q^{16})_\infty^4}{(q^2;q^2)_\infty(-q^4;q^4)_\infty}\frac{f^2(-q^2,-q^{14})}{f^2(-q,-q^{15})f^2(-q^7,-q^9)}C(q^4),$$

which proves (26).

6.1.4. Proof of (27)

Multiplying (25) and (26), we have

$$\phi^2(-q) - \frac{\chi^2(q^2)}{\chi^2(-q^2)}\phi^2(-q^4) = -\frac{4q(q;q^2)_\infty}{(q^2;q^2)_\infty f^2(-q^4,-q^{12})},$$

which proves (27).

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