TAIWANESE JOURNAL OF MATHEMATICS
Vol. 15, No. 6, pp. 2701-2719, December 2011
This paper is available online at http://tjm.math.ntu.edu.tw

MASCHKE-TYPE THEOREM AND DUALITY THEOREM FOR WEAK TWISTED SMASH PRODUCTS

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Abstract. Let H be a weak Hopf algebra in the sense of Böhm and Szlachányi [3] and A a weak H-bimodule algebra. Then in this paper we first introduce the notion of a weak twisted smash product $A \star H$ and then find some sufficient and necessary conditions making it into a weak bialgebra. Furthermore, we give a Maschke-type theorem for the weak twisted smash product over semisimple weak Hopf algebra H, which generalizes the well-known Maschke-type theorem in [5, 15, 17]. Finally, we obtain an analogue of the duality theorem for the weak twisted smash products.

1. Introduction

Let H be a Hopf algebra with a bijective antipode S over a fixed field and let A be an H-bimodule algebra. The twisted smash product $A \star H$ has been introduced by Wang and Li [16] and further studied by Wang and Kim [15]. It contains a usual smash product (Molnar [9]), a Drinfeld's double (Drinfeld [8]) and a Doi-Takeuchi's double algebra (Doi and Takeuchi [7]), so it plays an important role in quantum group theory.

In 1996, Böhm and Szlachányi [3] introduced and studied weak Hopf algebras (or quantum groupoids) as a generalization of ordinary Hopf algebras and groupoid algebras (see also Böhm et al. [2]). Shortly, the axioms of a weak Hopf algebra are the same as the ones for a Hopf algebra, except that the coproduct of the unit, the product of the counit and the antipode conditions are replaced by weaker properties. We refer the reader to [11] and [12] for the further study.

The main aim of this article is to study the weak twisted smash product $A \star H$ and to prove an analogue of the Maschke-type theorem (see [5]) and the duality theorem (see [1] and [6]) for the classical Hopf algebras in the setting of weak Hopf algebras.

Received July 22, 2008, accepted September 23, 2010.
Communicated by Ruibin Zhang.
2010 *Mathematics Subject Classification*: 16W30.
Key words and phrases: Weak Hopf algebras, Weak twisted smash products, Maschke-type theorem, Duality theorem.

This paper is organized as follows.

In Section 2, we recall some definitions and basic results related to weak Hopf algebras and weak module algebras.

In Section 3, we first introduce the notion of a weak twisted smash product $A\star H$ and give a weak Drinfeld's double as an example. Next we find some sufficient and necessary conditions making it into a weak bialgebra (see Theorem 3.7), generalizing the main result in [16]. Furthermore we give the sufficient conditions making $A\star H$ into a weak Hopf algebra.

In Section 4, we give a Maschke-type theorem for the weak twisted smash product $A\star H$ over a semisimple weak Hopf algebra H (see Theorem 4.5 and 4.6).

In Section 5, we prove an analogue of the duality theorem for the weak twisted smash products: Let H be a finite dimensional weak Hopf algebra and $A\star H$ be the weak twisted smash product. Then there is a canonical isomorphism between the algebras $(A\star H)\#H^*$ and End$(A\star H)_A$.

2. Basic Definitions and Results

In this section, we recall some basic definitions and results related to weak Hopf algebras introduced by Böhm et al. [2][3] and also about weak module algebras given by Caenepeel and Groot [4] that we will need later.

Throughout this paper, k denotes a fixed field, the tensor product $\otimes = \otimes_k$ and Hom are always assumed to be over k. If U and V are k-vector spaces, $T_{U,V} : U\otimes V \longrightarrow V\otimes U$ will denote the flip map defined by $T_{U,V}(u\otimes v) = v\otimes u$, for all $u\in U$ and $v\in V$. For an algebra A and a coalgebra C, we have the convolution algebra Conv(C,A) =Hom(C,A) as space, with the multiplication given by

$$(f*g)(c) = m_A(f\otimes g)\Delta_C(c) = f(c_1)g(c_2),$$

for all $f,g\in$ Hom$(C,A), c\in C$. Here we use the Sweedler's notation (see Sweedler [13]) for the comultiplication. Namely, $\Delta(c) = c_1\otimes c_2$.

2.1. Weak bialgebras

Recall from Böhm et al. [2] and Böhm and Szlachányi [3] that a weak k-bialgebra H is both a k-algebra (m,μ) and a k-coalgebra (Δ,ε) such that $\Delta(hk) = \Delta(h)\Delta(k)$, for all $h,k\in H$, and

$$\Delta^2(1) = 1_1\otimes 1_2 1'_1\otimes 1'_2 = 1_1\otimes 1'_1 1_2\otimes 1'_2, \tag{2.1}$$

$$\varepsilon(hkl) = \varepsilon(hk_1)\varepsilon(k_2 l) = \varepsilon(hk_2)\varepsilon(k_1 l), \tag{2.2}$$

for all $h,k,l\in H$, where $1'$ stands for another copy of 1. We summarize the elementary properties of weak bialgebras. The maps $\varepsilon_t,\varepsilon_s$: $H\longrightarrow H$ defined by

$$\varepsilon_t(h) = \varepsilon(1_1 h)1_2;\quad \varepsilon_s(h) = 1_1\varepsilon(h1_2)$$

are called the target map and source map, and their images H_t and H_s are called the target and source space. The source and target space can be described as follows:

$$H_t = \{h \in H \mid \varepsilon_t(h) = h\} = \{h \in H \mid \Delta(h) = 1_1 h \otimes 1_2 = h 1_1 \otimes 1_2\},$$
$$H_s = \{h \in H \mid \varepsilon_s(h) = h\} = \{h \in H \mid \Delta(h) = 1_1 \otimes h 1_2 = 1_1 \otimes 1_2 h\}.$$

For all $g, h \in H$, we also have

$$\varepsilon_t(h)\varepsilon_s(g) = \varepsilon_s(g)\varepsilon_t(h),$$

and its dual property

$$\varepsilon_s(h_1) \otimes \varepsilon_t(h_2) = \varepsilon_s(h_2) \otimes \varepsilon_t(h_1).$$

Finally $\varepsilon_t(1) = \varepsilon_s(1) = 1$ and

$$\varepsilon_t(h)\varepsilon_t(g) = \varepsilon_t(\varepsilon_t(h)g); \quad \varepsilon_s(h)\varepsilon_s(g) = \varepsilon_s(h\varepsilon_s(g)).$$

This implies that H_t and H_s are subalgebras of H.

2.2. Weak Hopf algebras

A weak Hopf algebra H is a weak bialgebra together with a k-linear map $S : H \longrightarrow H$ (called the antipode) satisfying

$$S * id_H = \varepsilon_s, \ id_H * S = \varepsilon_t, \ S * id_H * S = S,$$

where $*$ is the convolution product. It follows immediately that

$$S = \varepsilon_s * S = S * \varepsilon_t.$$

If the antipode exists, then it is unique. The antipode S is both an anti-algebra and an anti-coalgebra morphism. If H is a finite-dimensional weak Hopf algebra over k, then S is automatically bijective and the dual $H^* = \mathrm{Hom}(H, k)$ has a natural structure of a weak Hopf algebra with the structure operations dual to those of H. Now we recall some properties about S.

By Bӧhm et al. [2], let H be a weak Hopf algebra. Then we have the following conclusions:

(2.3) (1) $\varepsilon_t \circ S = \varepsilon_t \circ \varepsilon_s = S \circ \varepsilon_s, \quad \varepsilon_s \circ S = \varepsilon_s \circ \varepsilon_t = S \circ \varepsilon_t,$

(2.4) (2) $x_1 \otimes x_2 S(x_3) = x_1 \otimes \varepsilon_t(x_2) = 1_1 x \otimes 1_2,$

(2.5) (3) $S(x_1)x_2 \otimes x_3 = \varepsilon_s(x_1) \otimes x_2 = 1_1 \otimes x 1_2,$

(2.6) (4) $x_1 \otimes S(x_2)x_3 = x_1 \otimes \varepsilon_s(x_2) = x 1_1 \otimes S(1_2),$

(2.7) (5) $x_1 S(x_2) \otimes x_3 = \varepsilon_t(x_1) \otimes x_2 = S(1_1) \otimes 1_2 x,$

(2.8) (6) $x_1 y \otimes x_2 = x_1 \otimes x_2 S(y), \quad$ for all $\quad y \in H_s,$

(2.9) (7) $x_1 \otimes z x_2 = S(z) x_1 \otimes x_2, \quad$ for all $\quad z \in H_t.$

Let H be a weak Hopf algebra with a bijective antipode S_H, then H^{cop} is also a weak Hopf algebra with antipode S^{-1} (here S^{-1} is the composite-inverse of the antipode S_H). At this time

$$S^{-1}(h_2)h_1 = S^{-1}\varepsilon_s(h) = \varepsilon(h1_1)1_2 \triangleq \tilde{\varepsilon}_t(h),$$

$$h_2S^{-1}(h_1) = S^{-1}\varepsilon_t(h) = 1_1\varepsilon(1_2h) \triangleq \tilde{\varepsilon}_s(h).$$

2.3. Weak (bi)module algebras

Let H be a weak Hopf algebra.

(i) Recall from [4], an algebra A is called a left weak H-module algebra if A is left H-module via $h\otimes a \mapsto h \rightharpoonup a$ such that for all $h\in H,\ a,b\in A$,

$$h \rightharpoonup (ab) = (h_1 \rightharpoonup a)(h_2 \rightharpoonup b), \tag{2.10}$$

$$h \rightharpoonup 1_A = \varepsilon_t(h) \rightharpoonup 1_A. \tag{2.11}$$

Following [4, Proposition 4.15], the Eq.(2.11) is equivalent to

(2.12) $(a)\quad h \rightharpoonup 1_A = \tilde{\varepsilon}_s(h) \rightharpoonup 1_A;$

(2.13) $(b)\quad \varepsilon_t(h) \rightharpoonup a = (h \rightharpoonup 1_A)a;$

(2.14) $(c)\quad \tilde{\varepsilon}_s(h) \rightharpoonup a = a(h \rightharpoonup 1_A);$

(2.15) $(d)\quad \varepsilon_t(h) \rightharpoonup (ab) = (\varepsilon_t(h) \rightharpoonup a)b$

(2.16) $(e)\quad \tilde{\varepsilon}_s(h) \rightharpoonup (ab) = a(\tilde{\varepsilon}_s(h) \rightharpoonup b),$

for all $h\in H$ and $a,b\in A$.

(ii) Similarly, an algebra A is called a right weak H-module algebra if A is right H-module via $a\otimes h \mapsto a \leftharpoonup h$ such that for all $h\in H,\ a,b\in A$,

$$(ab) \leftharpoonup h = (a \leftharpoonup h_1)(b \leftharpoonup h_2), \tag{2.17}$$

$$1_A \leftharpoonup h = 1_A \leftharpoonup \varepsilon_s(h). \tag{2.18}$$

Following [4, Proposition 4.15], the Eq.(2.13) is equivalent to

(2.19) $(a)\quad 1_A \leftharpoonup h = 1_A \leftharpoonup \tilde{\varepsilon}_t(h);$

(2.20) $(b)\quad a \leftharpoonup \varepsilon_s(h) = a(1_A \leftharpoonup h);$

(2.21) $(c)\quad a \leftharpoonup \tilde{\varepsilon}_t(h) = (1_A \leftharpoonup h)a;$

(2.22) $(d)\quad (ab) \leftharpoonup \varepsilon_s(h) = a(b \leftharpoonup \varepsilon_s(h))\,;$

(2.23) $(e)\quad (ab) \leftharpoonup \tilde{\varepsilon}_t(h) = (a \leftharpoonup \tilde{\varepsilon}_t(h))b,$

for all $h\in H$ and $a,b\in A$.

(iii) Let A be an H-bimodule. If A is both a left weak H-module algebra and a right weak H-module algebra, then A is called *a weak H-bimodule algebra*.

3. The Weak Twisted Smash Product Bialgebra $A \star H$

In this section, we first introduce the notion of a weak twisted smash product $A \star H$ and give a weak Drinfeld's double as an example. Next we find some sufficient and necessary conditions making it into a weak bialgebra, generalizing the main constructions in [16] and [18]. Furthermore we give the sufficient conditions making $A \star H$ into a weak Hopf algebra.

Definition 3.1. Let H be a weak Hopf algebra and A a weak H-bimodule algebra with the left action $\rightharpoonup$ and the right action $\leftharpoonup$. A *weak twisted smash product* $A \star H$ of A and H is defined on the vector space $A\overline{\otimes}H = \{a\overline{\otimes}h \in A \otimes H \mid a\overline{\otimes}h = 1_1 \rightharpoonup a \leftharpoonup S(1_3) \otimes 1_2 h\}$ and the multiplication is given by

$$(a\overline{\otimes}h)(b\overline{\otimes}l) = a(h_1 \rightharpoonup b \leftharpoonup S(h_3))\overline{\otimes}h_2 l,$$

for all $a, b \in A, h, l \in H$. The element $a\overline{\otimes}h$ of $A \star H$ will usually be written as $a \star h$. It is not hard to show that the multiplication is well-defined and $A \star H$ is an associative algebra with the unit $1_A \star 1_H$.

Example 3.2. Let H be a finite dimensional weak Hopf algebra with a bijective antipode S. We define actions: $h \rightharpoonup f = f_1\langle f_2, h\rangle$, $f \leftharpoonup h = f_2\langle f_1, S^{-2}(h_1)\rangle$, for all $h \in H$, $f \in H^*$. Then it is easy to check $(H^*, \rightharpoonup, \leftharpoonup)$ is a weak H-bimodule algebra. Now we define the weak Drinfeld's double $\overline{D(H)} = H^*\overline{\otimes}H = \{f\overline{\otimes}h \in H^* \otimes H \mid f\overline{\otimes}h = f_2\langle f_1, S^{-1}(1_3)\rangle\langle f_3, 1_1\rangle \otimes 1_2 h\}$ as vector space. The multiplication is given by

$$(f \star h)(g \star l) = fg_2\langle g_1, S^{-1}(h_3)\rangle\langle g_3, h_1\rangle \star h_2 l.$$

In Nikshych [12], there is another definition of weak Drinfeld's double as follows: $\widehat{D(H)} = H^* \otimes H/kerJ$ as vector space, here $J : H^* \otimes H \to H^* \otimes H$, $J(f \otimes h) = (\varepsilon \otimes 1_H)(f \otimes h) = (f \otimes h)(\varepsilon \otimes 1_H)$. We denote $[f \otimes h]$ the class of $f \otimes h$ in $\widehat{D(H)}$. The multiplication is same as $\overline{D(H)}$. We show that $\overline{D(H)} = \widehat{D(H)}$ as: $\forall\ f\overline{\otimes}h \in \overline{D(H)}$, $(\varepsilon\overline{\otimes}1_H)(f\overline{\otimes}h) = (1_1 \rightharpoonup f \leftharpoonup S(1_3))\overline{\otimes}1_2 h = f\overline{\otimes}h$, $(f\overline{\otimes}h)(\varepsilon\overline{\otimes}1_H) = f(h_1 \rightharpoonup \varepsilon \leftharpoonup S(h_3))\overline{\otimes}h_2 = f(1_1 \rightharpoonup \varepsilon \leftharpoonup S(1_3))\overline{\otimes}1_2 h = (1_1 \rightharpoonup f \leftharpoonup S(1_3))\overline{\otimes}1_2 h = f\overline{\otimes}h$, and we get $f\overline{\otimes}h \in \widehat{D(H)}$. Conversely, if $[f \otimes h] \in \widehat{D(H)}$, then $[f \otimes h] = [\varepsilon \otimes 1_H][f \otimes h] = [(1_1 \rightharpoonup f \leftharpoonup S(1_3)) \otimes 1_2 h]$ and $[f \otimes h] \in \overline{D(H)}$. We obtain that the weak Drinfeld's double is one kind of weak twisted smash products.

The following lemma is straightforward.

Lemma 3.3. *Let $A \star H$ be a weak twisted smash product algebra. If A is a weak bialgebra, then $A \star H$ is a coalgebra, whose comultiplication is given by*

$$\Delta_{A\star H}(a \star h) = a_1 \star h_1 \otimes a_2 \star h_2,$$

and counit is given by

$$\varepsilon(a \star h) = \varepsilon_A(a)\varepsilon_H(h),$$

for all $a \in A$ and $h \in H$.

Lemma 3.4. *Let $A \star H$ be a weak twisted smash product algebra. If A is a weak bialgebra, then the comultiplication $\Delta_{A\star H}$ is a multiplicative map if and only if for all $h \in H, b \in A$,*

$$\begin{aligned} &(h_1 \rightharpoonup b \leftharpoonup S(h_4))_1 \star h_2 \otimes (h_1 \rightharpoonup b \leftharpoonup S(h_4))_2 \star h_3 \\ &= 1_1(h_1 \rightharpoonup b_1 \leftharpoonup S(h_3)) \star h_2 1_{H1} \otimes 1_2(h_4 \rightharpoonup b_2 \leftharpoonup S(h_6)) \star h_5 1_{H2}. \end{aligned} \tag{3.1}$$

Proof. As a matter of fact, for all $a \star h, b \star g \in A \star H$, we have

$$\begin{aligned} &\Delta_{A\star H}((a \star h)(b \star g)) = \Delta_{A\star H}(a(h_1 \rightharpoonup b \leftharpoonup S(h_3)) \star h_2 g) \\ &= a_1(h_1 \rightharpoonup b \leftharpoonup S(h_4))_1 \star h_2 g_1 \otimes a_2(h_1 \rightharpoonup b \leftharpoonup S(h_4))_2 \star h_3 g_2, \end{aligned} \tag{A}$$

$$\begin{aligned} &\Delta_{A\star H}(a \star h)\Delta_{A\star H}(b \star g) = (a_1 \star h_1)(b_1 \star g_1) \otimes (a_2 \star h_2)(b_2 \star g_2) \\ &= a_1(h_1 \rightharpoonup b_1 \leftharpoonup S(h_3)) \star h_2 g_1 \otimes a_2(h_4 \rightharpoonup b_2 \leftharpoonup S(h_6)) \star h_5 g_2. \end{aligned} \tag{B}$$

If (3.1) holds, then we obtain $(A) \overset{(3.1)}{=} a_1 1_1(h_1 \rightharpoonup b_1 \leftharpoonup S(h_3)) \star h_2 1_{H1} g_1 \otimes a_2 1_2(h_4 \rightharpoonup b_2 \leftharpoonup S(h_6)) \star h_5 1_{H2} g_2 = a_1(h_1 \rightharpoonup b_1 \leftharpoonup S(h_3)) \star h_2 g_1 \otimes a_2(h_4 \rightharpoonup b_2 \leftharpoonup S(h_6)) \star h_5 g_2 = (B)$. So $\Delta_{A\star H}$ is a multiplicative map.

Conversely, if $\Delta_{A\star H}((a \star h)(b \star g)) = \Delta_{A\star H}(a \star h)\Delta_{A\star H}(b \star g)$, that is,

$$\begin{aligned} &a_1(h_1 \rightharpoonup b \leftharpoonup S(h_4))_1 \star h_2 g_1 \otimes a_2(h_1 \rightharpoonup b \leftharpoonup S(h_4))_2 \star h_3 g_2 \\ &= a_1(h_1 \rightharpoonup b_1 \leftharpoonup S(h_3)) \star h_2 g_1 \otimes a_2(h_4 \rightharpoonup b_2 \leftharpoonup S(h_6)) \star h_5 g_2. \end{aligned}$$

In the above equality, taking $a = 1_A$ and $g = 1_H$, then we get

$$\begin{aligned} &1_1(h_1 \rightharpoonup b \leftharpoonup S(h_4))_1 \star h_2 1_{H1} \otimes 1_2(h_1 \rightharpoonup b \leftharpoonup S(h_4))_2 \star h_3 1_{H2} \\ &= (h_1 \rightharpoonup b \leftharpoonup S(h_4))_1 \star h_2 \otimes (h_1 \rightharpoonup b \leftharpoonup S(h_4))_2 \star h_3 \\ &= 1_1(h_1 \rightharpoonup b_1 \leftharpoonup S(h_3)) \star h_2 1_{H1} \otimes 1_2(h_4 \rightharpoonup b_2 \leftharpoonup S(h_6)) \star h_5 1_{H2}. \end{aligned}$$

So (3.1) holds. The proof is completed. □

Lemma 3.5. *Let $A \star H$ be a weak twisted smash product algebra. If A is a weak bialgebra, then*

$$\begin{aligned}
(1) \qquad & (\Delta_{A\star H} \otimes id_{A\star H})\Delta_{A\star H}(1_A \star 1_H) \\
&= (\Delta_{A\star H}(1_A \star 1_H) \otimes 1_A \star 1_H)(1_A \star 1_H \otimes \Delta_{A\star H}(1_A \star 1_H)), \\
(2) \qquad & (\Delta_{A\star H} \otimes id_{A\star H})\Delta_{A\star H}(1_A \star 1_H) \\
&= (1_A \star 1_H \otimes \Delta_{A\star H}(1_A \star 1_H))(\Delta_{A\star H}(1_A \star 1_H) \otimes 1_A \star 1_H).
\end{aligned}$$

Proof. We check (1) as follows:

$$\begin{aligned}
& (\Delta_{A\star H} \otimes id_{A\star H})\Delta_{A\star H}(1_A \star 1_H) \\
&= 1_1 \star \widetilde{1}_1 \otimes 1_2 \star \widetilde{1}_2 \otimes 1_3 \star \widetilde{1}_3, \\
& (\Delta_{A\star H}(1_A \star 1_H) \otimes 1_A \star 1_H)(1_A \star 1_H \otimes \Delta_{A\star H}(1_A \star 1_H)) \\
&= 1_1 \star \widetilde{1}_1 \otimes (1_2 \star \widetilde{1}_2)(1'_1 \star \widehat{1}_1) \otimes 1'_2 \star \widehat{1}_2 \\
&= 1_1 \star \widetilde{1}_1 \otimes 1_2(\widetilde{1}_2 \rightharpoonup 1'_1 \leftharpoonup S(\widetilde{1}_4)) \star \widetilde{1}_3\widehat{1}_1 \otimes 1'_2 \star \widehat{1}_2 \\
&= 1_1 \star \widetilde{1}_2 \otimes 1_2(\widetilde{1}_1 \rightharpoonup 1'_1 \leftharpoonup S(\widetilde{1}_4)) \star \widetilde{1}_3\widehat{1}_1 \otimes 1'_2 \star \widehat{1}_2 \\
&= 1_1 \star \widetilde{1}_1 \otimes (\widetilde{1}_2 \rightharpoonup 1_2 1'_1 \leftharpoonup S(\widetilde{1}_4)) \star \widetilde{1}_3\widehat{1}_1 \otimes 1'_2 \star \widehat{1}_2 \\
&= 1_1 \star \widetilde{1}_1 \otimes (\widetilde{1}_3 \rightharpoonup 1_2 1'_1 \leftharpoonup S(\widetilde{1}_4)) \star \widetilde{1}_2\widehat{1}_1 \otimes 1'_2 \star \widehat{1}_2 \\
&= 1_1 \star \widetilde{1}_1 \otimes (\overline{1}_2 \rightharpoonup 1_2 1'_1 \leftharpoonup S(\overline{1}_3)) \star \overline{1}_1\widetilde{1}_2\widehat{1}_1 \otimes 1'_2 \star \widehat{1}_2 \\
&= 1_1 \star \widetilde{1}_1 \otimes (\overline{1}_1 \rightharpoonup 1_2 1'_1 \leftharpoonup S(\overline{1}_3)) \star \overline{1}_2\widetilde{1}_2\widehat{1}_1 \otimes 1'_2 \star \widehat{1}_2 \\
&= 1_1 \star \widetilde{1}_1 \otimes 1_2 1'_1 \star \widetilde{1}_2\widehat{1}_1 \otimes 1'_2 \star \widehat{1}_2 \\
&= 1_1 \star \widetilde{1}_1 \otimes 1_2 \star \widetilde{1}_2 \otimes 1_3 \star \widetilde{1}_3.
\end{aligned}$$

In a similar way, we can prove (2). ■

Lemma 3.6. *Let $A \star H$ be a weak twisted smash product algebra. If A is a weak bialgebra, then we have the following conclusions*

(1) $\varepsilon((a\star x)(b\star g)(c\star p)) = \varepsilon((a\star x)(b\star g)_1)\varepsilon((b\star g)_2(c\star p))$ *if and only if*

$$\begin{aligned}
& \varepsilon(a(x_1 \rightharpoonup b \leftharpoonup S(x_5))(x_2g_1 \rightharpoonup c \leftharpoonup S(x_3g_2)) \\
(3.2) \qquad = \; & \varepsilon(a(x_1 \rightharpoonup b_1 \leftharpoonup S(x_3)))\varepsilon(x_2g_1)\varepsilon(b_2(g_2 \rightharpoonup c \leftharpoonup S(g_3))).
\end{aligned}$$

(2) $\varepsilon((a\star x)(b\star g)(c\star p)) = \varepsilon((a\star x)(b\star g)_2)\varepsilon((b\star g)_1(c\star p))$ *if and only if*

$$\begin{aligned}
& \varepsilon(a(x_1 \rightharpoonup b \leftharpoonup S(x_5))(x_2g_1 \rightharpoonup c \leftharpoonup S(x_3g_2)) \\
(3.3) \qquad = \; & \varepsilon(a(x_1 \rightharpoonup b_2 \leftharpoonup S(x_3)))\varepsilon(x_2g_3)\varepsilon(b_1(g_1 \rightharpoonup c \leftharpoonup S(g_2))).
\end{aligned}$$

Proof. (1) For all $a \star x, b \star g, c \star p \in A \star H$, we compute

$$\begin{aligned}
&\varepsilon((a\star x)(b\star g)(c\star p))\\
=\ &\varepsilon((a(x_1\rightharpoonup b\leftharpoonup S(x_3))\star x_2g)(c\star p))\\
=\ &\varepsilon(a(x_1\rightharpoonup b\leftharpoonup S(x_5))(x_2g_1\rightharpoonup c\leftharpoonup S(x_4g_3)))\varepsilon(x_3g_2p),\\
&\varepsilon((a\star x)(b\star g)_1)\varepsilon((b\star g)_2(c\star p))\\
=\ &\varepsilon((a\star x)(b_1\star g_1))\varepsilon((b_2\star g_2)(c\star p))\\
=\ &\varepsilon(a(x_1\rightharpoonup b_1\leftharpoonup S(x_3))\star x_2g_1)\varepsilon(b_2(g_2\rightharpoonup c\leftharpoonup S(g_4))\star g_3p)\\
=\ &\varepsilon(a(x_1\rightharpoonup b_1\leftharpoonup S(x_3)))\varepsilon(x_2g_1)\varepsilon(b_2(g_2\rightharpoonup c\leftharpoonup S(g_4)))\varepsilon(g_3p).
\end{aligned}$$

If $\varepsilon((a\star x)(b\star g)(c\star p)) = \varepsilon((a\star x)(b\star g)_1)\varepsilon((b\star g)_2(c\star p))$, then by the above discussion we obtain the following equality: $\varepsilon(a(x_1\rightharpoonup b\leftharpoonup S(x_5))(x_2g_1\rightharpoonup c\leftharpoonup S(x_4g_3)))\varepsilon(x_3g_2p) = \varepsilon(a(x_1\rightharpoonup b_1\leftharpoonup S(x_3)))\varepsilon(x_2g_1)\varepsilon(b_2(g_2\rightharpoonup c\leftharpoonup S(g_4)))\varepsilon(g_3p)$. Taking $p=1_H$ in the equality, we obtain (3.2) holds.

Conversely, if (3.2) holds, we get

$$\begin{aligned}
&\varepsilon((a\star x)(b\star g)(c\star p))\\
=\ &\varepsilon(a(x_1\rightharpoonup b\leftharpoonup S(x_5))(x_2g_1\rightharpoonup c\leftharpoonup S(x_4g_3)))\varepsilon(x_3g_2p)\\
=\ &\varepsilon(a(x_1\rightharpoonup b\leftharpoonup S(x_5))(x_2g_1\rightharpoonup c\leftharpoonup S(x_4g_3)))\varepsilon(x_3g_21_2)\varepsilon(1_1p)\\
=\ &\varepsilon(a(x_1\rightharpoonup b\leftharpoonup S(x_4))(x_2g_1\rightharpoonup c\leftharpoonup S(x_3g_2\varepsilon_t(p))))\\
=\ &\varepsilon(a(x_1\rightharpoonup b\leftharpoonup S(x_4))(x_2g_1\rightharpoonup (c\leftharpoonup S(\varepsilon_t(p)))\leftharpoonup S(x_3g_2)))\\
\overset{(3.2)}{=}\ &\varepsilon(a(x_1\rightharpoonup b_1\leftharpoonup S(x_3)))\varepsilon(x_2g_1)\varepsilon(b_2(g_2\rightharpoonup (c\leftharpoonup S(\varepsilon_t(p)))\leftharpoonup S(g_3)))\\
=\ &\varepsilon(a(x_1\rightharpoonup b_1\leftharpoonup S(x_3)))\varepsilon(x_2g_1)\varepsilon(b_2(g_2\rightharpoonup c\leftharpoonup S(g_3\varepsilon_t(p))))\\
=\ &\varepsilon(a(x_1\rightharpoonup b_1\leftharpoonup S(x_3)))\varepsilon(x_2g_1)\varepsilon(b_2(g_2\rightharpoonup c\leftharpoonup S(g_4)))\varepsilon(g_3p)\\
=\ &\varepsilon((a\star x)(b\star g)_1)\varepsilon((b\star g)_2(c\star p)).
\end{aligned}$$

In a similar way, we can prove (2). The proof is completed. ■

The following is the main result in this section.

Theorem 3.7. *Let $A\star H$ be a weak twisted smash product algebra. If A is a weak bialgebra, then $A\star H$ is a weak bialgebra if and only if (3.1)-(3.3) are satisfied.*

In this case, if A and H are two weak Hopf algebras, and for all $a\in A, x\in H$,

$$a_1(\varepsilon_t(x)\rightharpoonup S(a_2))\star 1_H = \varepsilon(x)(\varepsilon_t(a\leftharpoonup S(1_{H1})))\star 1_{H2}, \tag{3.4}$$

$$\begin{aligned}
&S(x_3)\rightharpoonup \varepsilon_s(a)\leftharpoonup S^2(x_1)\star S(x_2)x_4\\
&= 1_1\star S(1_{H2})\varepsilon(a(x_11_{H1}\rightharpoonup 1_2\leftharpoonup S(x_2)))
\end{aligned} \tag{3.5}$$

hold, then $A\star H$ is a weak Hopf algebra with an antipode

$$S_{A\star H}(a\star x)=(1\star S(x))(S(a)\star 1)=S(x_3)\rightharpoonup S(a)\leftharpoonup S(S(x_1))\star S(x_2).$$

Proof. By Lemma 3.3−3.6, $A\star H$ is a weak bialgebra if and only if Eqs. (3.1)-(3.3) are satisfied.

Next, we will show that $A\star H$ is a weak Hopf algebra with an antipode $S_{A\star H}$. In fact, for all $a\in A, x\in H$, we have

$$\begin{aligned}
&S(a_1\star x_1)(a_2\star x_2)\\
&=(S(x_3)\rightharpoonup S(a_1)\leftharpoonup S(S(x_1))\star S(x_2))(a_2\star x_4)\\
&=[(S(x_5)\rightharpoonup S(a_1)\leftharpoonup S(S(x_1)))(S(x_4)\rightharpoonup a_2\leftharpoonup S(S(x_2)))]\star S(x_3)x_6\\
&=S(x_3)\rightharpoonup S(a_1)a_2\leftharpoonup S(S(x_1))\star S(x_2)x_4,
\end{aligned}$$

and while

$$\begin{aligned}
&(1_1\star 1_{H1})\varepsilon((a\star x)(1_2\star 1_{H2}))\\
&=(1_1\star 1_{H1})\varepsilon(a(x_1\rightharpoonup 1_2\leftharpoonup S(x_3)))\varepsilon(x_2 1_{H2})\\
&=1_1\star\varepsilon_s(x_2)\varepsilon(a(x_1\rightharpoonup 1_2\leftharpoonup S(x_3)))\\
&=1_1\star S(1_{H2})\varepsilon(a(x_1 1_{H1}\rightharpoonup 1_2\leftharpoonup S(x_2))).
\end{aligned}$$

So $S(a_1\star x_1)(a_2\star x_2)=(1_1\star 1_{H1})\varepsilon((a\star x)(1_2\star 1_{H2}))$ if (3.5) holds.

$$\begin{aligned}
&(a_1\star x_1)S(a_2\star x_2)\\
&=(a_1\star x_1)(S(x_4)\rightharpoonup S(a_2)\leftharpoonup S(S(x_2))\star S(x_3))\\
&=a_1(x_1S(x_6)\rightharpoonup S(a_2)\leftharpoonup S(S(x_4))S(x_3))\star x_2S(x_5)\\
&=a_1(x_1S(x_5)\rightharpoonup S(a_2)\leftharpoonup S(\varepsilon_t(x_3)))\star x_2S(x_4)\\
&=a_1(x_1S(x_4)\rightharpoonup S(a_2)\leftharpoonup S(1_2))\star 1_1x_2S(x_3)\\
&=a_1(1'_1x_1S(x_2)\rightharpoonup S(a_2)\leftharpoonup S(1_2))\star 1_11'_2\\
&=a_1(1_1\varepsilon_t(x)\rightharpoonup S(a_2)\leftharpoonup S(1_3))\star 1_2\\
&=a_1(1_1\rightharpoonup(\varepsilon_t(x)\rightharpoonup S(a_2))\leftharpoonup S(1_3))\star 1_2\\
&=(1_1\rightharpoonup a_1(\varepsilon_t(x)\rightharpoonup S(a_2))\leftharpoonup S(1_3))\star 1_2\\
&=a_1(\varepsilon_t(x)\rightharpoonup S(a_2))\star 1_H,
\end{aligned}$$

and while

$$\begin{aligned}
&\varepsilon((1_1\star 1_{H1})(a\star x))(1_2\star 1_{H2})\\
&=\varepsilon(1_1(1_{H1}\rightharpoonup a\leftharpoonup S(1_{H3}))\star 1_{H2}x)(1_2\star 1_{H4})\\
&=\varepsilon(1_1(1_{H1}\rightharpoonup(a\leftharpoonup S(1'_{H1}))\leftharpoonup S(1_{H3}))\star 1_{H2}x)(1_2\star 1'_{H2})\\
&=\varepsilon(1_{H1}\rightharpoonup 1_1(a\leftharpoonup S(1'_{H1}))\leftharpoonup S(1_{H3})\star 1_{H2}x)(1_2\star 1'_{H2})\\
&=\varepsilon(1_1(a\leftharpoonup S(1'_{H1}))\star x)(1_2\star 1'_{H2})\\
&=\varepsilon(x)(\varepsilon_t(a\leftharpoonup S(1'_{H1}))\star 1'_{H2}).
\end{aligned}$$

So $(a_1\star x_1)S(a_2\star x_2)=\varepsilon((1_1\star 1_{H1})(a\star x))(1_2\star 1_{H2})$ if (3.4) holds.

Moreover, for all $a\in A, x\in H$, we have

$$
\begin{aligned}
& S(a_1\star x_1)(a_2\star x_2)S(a_3\star x_3)\\
=\ & S(a_1\star x_1)\varepsilon((1_1\star 1_{H1})(a_2\star x_2))(1_2\star 1_{H2})\\
=\ & S(a_1\star x_1)(a_2(\varepsilon_t(x_2)\rightharpoonup S(a_3))\star 1_H)\\
=\ & (S(x_3)\rightharpoonup S(a_1)\leftharpoonup S^2(x_1)\star S(x_2))(a_2(\varepsilon_t(x_4)\rightharpoonup S(a_3))\star 1_H)\\
=\ & (S(x_5)\rightharpoonup S(a_1)\leftharpoonup S^2(x_1))(S(x_4)\rightharpoonup a_2(\varepsilon_t(x_6)\rightharpoonup S(a_3))\leftharpoonup S^2(x_2))\star S(x_3)\\
=\ & S(x_3)\rightharpoonup S(a_1)a_2(\varepsilon_t(x_4)\rightharpoonup S(a_3))\leftharpoonup S(S(x_1))\star S(x_2)\\
=\ & S(1_1x_3)\rightharpoonup S(a_1)a_2(1_2\rightharpoonup S(a_3))\leftharpoonup S(S(x_1))\star S(x_2)\\
=\ & S(1_2x_3)\rightharpoonup S(a_1)a_2(1_1\rightharpoonup S(a_3))\leftharpoonup S(S(x_1))\star S(x_2)\\
=\ & S(1_1x_3)\rightharpoonup (1_2\rightharpoonup S(a_1)a_2S(a_3))\leftharpoonup S(S(x_1))\star S(x_2)\\
=\ & S(x_3)S(1_1)1_2\rightharpoonup S(a)\leftharpoonup S(S(x_1))\star S(x_2)\\
=\ & S(x_3)\rightharpoonup S(a)\leftharpoonup S(S(x_1))\star S(x_2)=S(a\star x).
\end{aligned}
$$

Thus $A\star H$ is a weak Hopf algebra. ∎

Corollary 3.8. (*1*) *If H is an ordinary Hopf algebra, then $A\star H$ is a twisted smash product constructed by Wang and Li* [*16*]. *If A and H are two Hopf algebras, then we get Theorem 3.6 is exactly the Theorem 1.7 in* [*16*].

(*2*) *If A is a left weak H-module algebra and the right action is trivial, then we denote $A\star H=A\#H$. The multiplication is turned into $(a\#h)(b\#l)=a(h_1\cdot b)\#h_2l$. So $A\star H$ is the weak smash product constructed in* [*10*] *and we get the results in Zhang and Zhu* [*18*].

The following proposition is obvious.

Proposition 3.9. *Let $A\star H$ be a weak twisted smash product, then A and H are subalgebras of $A\star H$ with inclusion maps $i: A\to A\star H,\ a\mapsto a\star 1_H$, and $j: H\to A\star H,\ h\mapsto 1_A\star h$ respectively. Furthermore, i and j are algebra maps.*

Theorem 3.10. *Let $A\star H$ be a weak twisted smash product and M a vector space over k. Then M is a left $A\star H$-module if and only if M is a left A-module and a left H-module such that*

$$h\cdot(a\cdot m)=(h_1\rightharpoonup a\leftharpoonup S(h_3))\cdot(h_2\cdot m), \tag{3.6}$$

for all $a\in A, h\in H$ and $m\in M$.

Proof. Let $(M,\rightharpoonup)$ be a left $A\star H$-module. We define

$$a\cdot m=(a\star 1_H)\rightharpoonup m,\ \ h\cdot m=(1_A\star h)\rightharpoonup m.$$

Then M is a left A-module and left H-module by Proposition 3.9. Moreover,

$$\begin{aligned} h\cdot(a\cdot m) &= (1_A\star h)\rightharpoonup((a\star 1_H)\rightharpoonup m)\\ &= (h_1\rightharpoonup a\leftharpoonup S(h_3)\star h_2)\rightharpoonup m\\ &= ((h_1\rightharpoonup a\leftharpoonup S(h_3)\star 1_H)(1_A\star h_2))\rightharpoonup m\\ &= (h_1\rightharpoonup a\leftharpoonup S(h_3))\cdot(h_2\cdot m). \end{aligned}$$

Conversely, we define $(a\star h)\rightharpoonup m = a\cdot(h\cdot m)$. It is easy to check $(1_A\star 1_H)\rightharpoonup m = m$. Now we prove

$$\begin{aligned} & [(a\star h)(b\star g)]\rightharpoonup m\\ = \;& [a(h_1\rightharpoonup b\leftharpoonup S(h_3))\star h_2 g]\rightharpoonup m\\ = \;& a(h_1\rightharpoonup b\leftharpoonup S(h_3))\cdot(h_2 g\cdot m)\\ = \;& a\cdot[(h_1\rightharpoonup b\leftharpoonup S(h_3))\cdot(h_2\cdot(g\cdot m))]\\ \overset{(3.6)}{=} \;& a\cdot(h\cdot(b\cdot(g\cdot m)))\\ = \;& (a\star h)\rightharpoonup(b\cdot(g\cdot m))\\ = \;& (a\star h)\rightharpoonup((b\star g)\rightharpoonup m). \end{aligned}$$

This shows M is a left $A\star H$-module. ■

4. The Maschke-type Theorem for $A\star H$

In this section, we will give a Maschke-type theorem for the weak twisted smash product $A\star H$ over a semisimple weak Hopf algebra H, which extends the Maschke-type theorem in [5, 15, 17].

The following lemma given in Bӧhm et al. [2] is needed in the sequel.

Lemma 4.1. *The following conclusions on the weak Hopf algebra H are equivalent:*

(1) *H is semisimple;*

(2) *There exists a normalized right integral $x\in H$, that is, for all $h\in H$, $xh = x\varepsilon_s(h)$, and $\varepsilon_s(x)=1$.*

Lemma 4.2. *Let H be a weak Hopf algebra with invertible antipode S and $A\star H$ a weak twisted smash product. Then*

$$ha = (h_1\rightharpoonup a\leftharpoonup S(h_3))h_2, \tag{4.1}$$

$$ah = h_2(S^{-1}(h_1)\rightharpoonup a\leftharpoonup S^2(h_3)). \tag{4.2}$$

Here ha we denote $(1\star h)(a\star 1)$ and $ah=(a\star 1)(1\star h)$ as in Proposition 3.9.

Proof. We get (4.1) by straightforward computation.

Next we prove (4.2) holds.

$$\begin{aligned} & h_2(S^{-1}(h_1)\rightharpoonup a\leftharpoonup S^2(h_3)) \\ \overset{(4.1)}{=} & (h_2S^{-1}(h_1)\rightharpoonup a\leftharpoonup S^2(h_5)S(h_4))h_3 \\ = & (S^{-1}(\varepsilon_t(h_1))\rightharpoonup a\leftharpoonup S^2(h_4)S(h_3))\star h_2 \\ = & (1_1\rightharpoonup a\leftharpoonup S(h_2S(h_3)))\star 1_2h_1 \\ = & (1_1\rightharpoonup a\leftharpoonup S(\varepsilon_t(h_2)))\star 1_2h_1 \\ = & (1_1\rightharpoonup a\leftharpoonup S(1'_2))\star 1_21'_1h \\ = & (1_1\rightharpoonup a\leftharpoonup S(1_3))\star 1_2h \\ = & a\star h=ah, \end{aligned}$$

for all $a\in A$ and $h\in H$. ∎

Let x be a right integral of H. In the following proposition, we assume that the following formula holds in $A\star H$, for all $a\in A$,

$$S(x_1)\otimes(x_2\rightharpoonup a\leftharpoonup S(x_4))x_3=S(x_2)\otimes(x_3\rightharpoonup a\leftharpoonup S^3(x_1))x_4. \tag{4.3}$$

Proposition 4.3. *Let H be a finite dimensional weak Hopf algebra, and $A\star H$ a weak twisted smash product, and x a right integral in H. Assume that W and V are (left) $A\star H$-modules and $\lambda: V\to W$ is a left A-module map. If the right integral x satisfying the Eq. (4.3), then $\widetilde{\lambda}: V\to W$, $v\mapsto S(x_1)\cdot\lambda(x_2\cdot v)$ is a left $A\star H$-module map.*

Proof. We only need to prove $\widetilde{\lambda}$ is both a left H-module map and a left A-module map.

Because x is a right integral, we get $x_1h_1\otimes x_2h_2=\Delta(xh)=\Delta(x\varepsilon_s(h))=x_1\otimes x_2\varepsilon_s(h)$ and

$$x_1h_1\otimes x_2h_2\otimes h_3=x_1\otimes x_2\varepsilon_s(h_1)\otimes h_2. \tag{4.4}$$

For all $g\in H, v\in V$, since S is bijective, there exists $h\in H$ such that $S(h)=g$. Now we have

$$\begin{aligned} g\cdot(\widetilde{\lambda}(v)) &= S(h)\cdot(\widetilde{\lambda}(v)) \\ &= S(h_1)h_2S(h_3)\cdot(\widetilde{\lambda}\cdot v) \\ &= S(h_1)\varepsilon_t(h_2)S(x_1)\cdot\lambda(x_2\cdot v) \\ &\overset{(2.8)}{=} S(h_1)S(x_1)\cdot\lambda(x_2\varepsilon_t(h_2)\cdot v) \\ &= S(x_1h_1)\cdot\lambda(x_2h_2S(h_3)\cdot v) \end{aligned}$$

$$\begin{aligned}
&\overset{(4.4)}{=} S(x_1)\cdot\lambda(x_2\varepsilon_s(h_1)S(h_2)\cdot v)\\
&= S(x_1)\cdot\lambda(x_2S(h_1)h_2S(h_3)\cdot v)\\
&= S(x_1)\cdot\lambda(x_2S(h)\cdot v)\\
&= S(x_1)\cdot\lambda(x_2g\cdot v)=\widetilde{\lambda}(g\cdot v).
\end{aligned}$$

So $\widetilde{\lambda}$ is a left H-module map.

On the other hand, for all $a\in A, v\in V$, we compute

$$\begin{aligned}
a\cdot(\widetilde{\lambda}(v)) &= aS(x_1)\cdot\lambda(x_2\cdot v)\\
&\overset{(4.2)}{=} S(x_2)(x_3\rightharpoonup a\leftharpoonup S^3(x_1))\cdot\lambda(x_4\cdot v)\\
&= S(x_2)\cdot\lambda((x_3\rightharpoonup a\leftharpoonup S^3(x_1))x_4\cdot v)\ (\lambda \text{ is a left } A\text{-linear map})\\
&\overset{(4.3)}{=} S(x_1)\cdot\lambda((x_2\rightharpoonup a\leftharpoonup S(x_4))x_3\cdot v)\\
&\overset{(4.1)}{=} S(x_1)\cdot\lambda(x_2a\cdot v)=\widetilde{\lambda}(a\cdot v),
\end{aligned}$$

that is, $\widetilde{\lambda}$ is a left A-module map.

By the above discussion, we know that $\widetilde{\lambda}$ is a left $A\star H$-module map. ■

Lemma 4.4. *If H is a finite dimensional weak Hopf algebra and H^* is unimodular, then the equation (4.3) holds.*

Proof. Following from [14, Corollary 6.5], we have $l_2\otimes l_1=l_1\otimes S^2(l_2a^{-1})$, where l is a left integral and a is the distinguished group-like element of H; our hypothesis that H^* is unimodular implies $a=1$, hence $l_2\otimes l_1=l_1\otimes S^2(l_2)$. Replacing the left integral l with a right integral x, where $l=S(x)$, we get $x_1\otimes x_2=x_2\otimes S^2(x_1)$. This follows $x_1\otimes x_2\otimes x_3\otimes x_4=x_2\otimes x_3\otimes x_4\otimes S^2(x_1)$ and we immediately get the equation (4.3) holds.

We can now obtain our version of Maschke's Theorem.

Theorem 4.5. *Let H be a finite dimensional weak Hopf algebra such that H is semisimple and H^* is unimodular and let $A\star H$ be a weak twisted smash product. Assume that V is a left $A\star H$-module and W an $A\star H$-submodule of V. If W is a summand of V as A-module, then W is a summand of V as $A\star H$-module.*

Proof. Let $\lambda: V\to W$ be an A-module projection map. Define

$$\widetilde{\lambda}: V\to W,\ v\mapsto S(x_1)\cdot\lambda(x_2\cdot v).$$

By Proposition 4.3 and Lemma 4.4, $\widetilde{\lambda}$ is a left $A\star H$-module map, where x is a normalized right integral of H in Lemma 4.1.

Next, we need only to show that $\widetilde{\lambda}$ is a projection, that is, for any $w \in W$, $\widetilde{\lambda}(w) = w$.

In fact, for any $w \in W$,

$$\begin{aligned}\widetilde{\lambda}(w) &= S(x_1) \cdot \lambda(x_2 \cdot w) \\ &= S(x_1) \cdot (x_2 \cdot w) \text{ (since } \lambda \mid_W = id) \\ &= (S(x_1)x_2) \cdot w = \varepsilon_s(x) \cdot w \\ &= 1 \cdot w = w.\end{aligned}$$

The proof is completed. ∎

Theorem 4.6. *Let H be a finite dimensional weak Hopf algebra such that H is semisimple and H^* is unimodular and let $A \star H$ be a weak twisted smash product.*

(1) Let V be an $A \star H$-module. If V is completely reducible as an A-module, then V is completely reducible as an $A \star H$-module.

(2) If A is semisimple Artinian, then so is $A \star H$.

Proof. (1) is immediately from Theorem 4.5.

(2) follows from (1), using the fact that an algebra is semisimple Artinian if and only if every module is completely reducible. ∎

5. The Duality Theorem for $A \star H$

In this section, we will prove an analogue of the Blattner-Cohen-Montgomery's duality theorem for weak twisted smash products, which extends the main result given by Nikshych [10].

Throughout this section, we will always assume H is a finite dimensional weak Hopf algebra, A a weak H-bimodule algebra and the following equation holds:

$$a \leftharpoonup h_1 \otimes h_2 = a \leftharpoonup h_2 \otimes h_1, \ \forall\ a \in A, h \in H. \tag{5.1}$$

First, we define a left H^*-module algebra $A \star H$ via the formula

$$\varphi \cdot (a \star h) = a \star (\varphi \rightharpoonup h) = \langle \varphi, h_2 \rangle a \star h_1,$$

for all $\varphi \in H^*, h \in H$ and $a \in A$.

Moreover, we can also define a right H_t^*-module on $A \star H$ by

$$(a \star h) \cdot \varphi' = \langle S_{H^*}^{-1}(\varphi'), h_2 \rangle a \star h_1 = \langle \varphi', S^{-1}(h_2) \rangle a \star h_1,$$

for all $\varphi' \in H_t^*$.

Now we will construct a canonical isomorphism between the weak smash product algebra $(A \star H)\#H^*$ and the endomorphism algebra $\mathrm{End}(A \star H)_A$, where the right A-module on $A \star H$ is the multiplication, i.e., $(a \star h) \cdot b = (a \star h)(b \star 1_H)$.

Lemma 5.1. *The map* $\alpha : (A \star H)\#H^* \to$ *End*$(A \star H)_A$ *defined by*

$$\alpha((x \star h)\#\varphi)(y \star g) = (x \star h)(y \star (\varphi \rightharpoonup g)) = (x \star h)(y \star \langle \varphi, g_2 \rangle g_1),$$

is a homomorphism of algebras, for any $x, y \in A, h, g \in H, \varphi \in H^*$.

Proof. Firstly, we need to check that α is well-defined.
In fact, for any $\xi \in H_t^*$, we need to compute that

$$\begin{aligned}
&\alpha((x \star h)\#\xi\varphi)(y \star g) \\
=\ & x(h_1 \rightharpoonup y \leftharpoonup S(h_3)) \star h_2(\xi\varphi \rightharpoonup g) \\
=\ & x(h_1 \rightharpoonup y \leftharpoonup S(h_3)) \star h_2(\xi \rightharpoonup 1_H)(\varphi \rightharpoonup g) \\
=\ & x(h_1 \rightharpoonup y \leftharpoonup S(h_2)) \star h_3(\xi \rightharpoonup 1_H)(\varphi \rightharpoonup g) \\
=\ & x(h_1(\xi \rightharpoonup 1_H)_1 \rightharpoonup y \leftharpoonup S(h_2(\xi \rightharpoonup 1_H)_2)) \star h_3(\xi \rightharpoonup 1_H)_3(\varphi \rightharpoonup g) \\
=\ & x(h_1(\xi \rightharpoonup 1_H)_1 \rightharpoonup y \leftharpoonup S(h_3(\xi \rightharpoonup 1_H)_3)) \star h_2(\xi \rightharpoonup 1_H)_2(\varphi \rightharpoonup g) \\
=\ & x((h(\xi \rightharpoonup 1_H))_1 \rightharpoonup y \leftharpoonup S((h(\xi \rightharpoonup 1_H))_3)) \star (h(\xi \rightharpoonup 1_H))_2(\varphi \rightharpoonup g) \\
=\ & (x \star h(\xi \rightharpoonup 1_H))(y \star \varphi \rightharpoonup g) \\
=\ & \alpha((x \star h(\xi \rightharpoonup 1_H))\#\varphi)(y \star g) \\
=\ & \alpha((x \star S_{H^*}^{-1}(\xi) \rightharpoonup h)\#\varphi)(y \star g) \\
=\ & \alpha((x \star h) \leftharpoonup \xi\#\varphi)(y \star g).
\end{aligned}$$

Secondly, we know that Im$\alpha \subseteq$ End$(A \star H)_A$:

$$\begin{aligned}
&\alpha((x \star h)\#\varphi)((y \star g) \cdot w) \\
=\ & \alpha((x \star h)\#\varphi)(y(g_1 \rightharpoonup w \leftharpoonup S(g_3)) \star g_2) \\
=\ & (x \star h)(y(g_1 \rightharpoonup w \leftharpoonup S(g_3)) \star \varphi \rightharpoonup g_2) \\
=\ & (x \star h)(y(g_1 \rightharpoonup w \leftharpoonup S(g_3)) \star \langle \varphi, g_3 \rangle g_2) \\
=\ & (x \star h)(y \star (\varphi \rightharpoonup g))(w \star 1_H) \\
=\ & (\alpha((x \star h)\#\varphi)(y \star g)) \cdot w.
\end{aligned}$$

Finally, for any $x, x', y \in A, h, h', g \in H, \varphi, \varphi' \in H^*$,

$$\begin{aligned}
&\alpha[((x \star h)\#\varphi)((x' \star h')\#\varphi')](y \star g) \\
=\ & \alpha((x \star h)(x' \star (\varphi_1 \rightharpoonup h'))\#\varphi_2\varphi')(y \star g) \\
=\ & (x \star h)(x' \star (\varphi_1 \rightharpoonup h'))(y \star (\varphi_2\varphi' \rightharpoonup g)) \\
=\ & (x \star h)(\varphi \cdot ((x' \star h')(y \star (\varphi' \rightharpoonup g)))) \\
=\ & \alpha((x \star h)\#\varphi) \circ \alpha((x' \star h')\#\varphi')(y \star g),
\end{aligned}$$

so, α is a homomorphism of algebras. ■

Let $\{f_i\}$ be a basis of H and $\{\psi_i\}$ be the dual basis of H^*, i.e., such that $\langle f_i, \psi_j\rangle = \delta_{ij}$ for all i, j. Then we have identities

$$\sum_i f_i\langle h, \psi_i\rangle = h, \ \sum_i \langle f_i, \varphi\rangle\psi_i = \varphi,$$

for all $h \in H$ and $\varphi \in H^*$, moreover the element of $\sum_i f_i \otimes \psi_i \in H \otimes H^*$ does not depend on the choice of $\{f_i\}$.

Let us define a linear map $\beta : \mathrm{End}(A \star H)_A \to (A \star H)\#H^*$ by

$$T \mapsto \sum_i [T(1_A \star f_{i2})(1_A \star S^{-1}(f_{i1}))]\#\psi_i.$$

Lemma 5.2. *The maps α and β are inverse of each other.*

Proof. We need to check that

$$\beta \circ \alpha = id_{(A\star H)\#H^*} \text{ and } \alpha \circ \beta = id_{\mathrm{End}(A\star H)_A}.$$

For all $x \in A, h \in H$ and $\varphi \in H^*$, we compute

$$\begin{aligned}
&\beta \circ \alpha((x \star h)\#\varphi)\\
&= \sum_i [\alpha((x \star h)\#\varphi)(1_A \star f_{i2})(1_A \star S^{-1}(f_{i1}))]\#\psi_i\\
&= \sum_i [(x \star h)(1_A \star \varphi \rightharpoonup f_{i2})(1_A \star S^{-1}(f_{i1}))]\#\psi_i\\
&= \sum_i (x(h_1 \rightharpoonup 1_A \leftharpoonup S(h_3)) \star h_2(\varphi \rightharpoonup f_{i2}))(1_A \star S^{-1}(f_{i1}))\#\psi_i\\
&= \sum_i (x(1_1 \rightharpoonup 1_A \leftharpoonup S(1'_2)) \star 1_2 1' h(\varphi \rightharpoonup f_{i2}))(1_A \star S^{-1}(f_{i1}))\#\psi_i\\
&= \sum_i ((1_1 \rightharpoonup x \leftharpoonup S(1_3)) \star 1_2 h(\varphi \rightharpoonup f_{i2}))(1_A \star S^{-1}(f_{i1}))\#\psi_i\\
&= \sum_i (x \star h(\varphi \rightharpoonup f_{i2}))(1_A \star S^{-1}(f_{i1}))\#\psi_i\\
&= \sum_i [x(h_1 f_{i2} \rightharpoonup 1_A \leftharpoonup S(h_3 f_{i4})) \star h_2 f_{i3} S^{-1}(f_{i1})]\#\psi_i\langle\varphi, f_{i5}\rangle\\
&= \sum_i (x \star h f_{i2} S^{-1}(f_{i1}))\#\psi_i\langle\varphi, f_{i3}\rangle = \sum_i (x \star h1_1)\#\psi_i\langle\varphi, 1_2 f_i\rangle\\
&= (x \star h1_1)\#\varphi_2\langle\varphi_1, 1_2\rangle = (x \star h(\varphi_1 \rightharpoonup 1))\#\varphi_2\\
&= (x \star S_{H^*}^{-1}(\varepsilon_t(\varphi_1)) \rightharpoonup h)\#\varphi_2 = S_{H^*}^{-1}(\varepsilon_t(\varphi_1)) \cdot (x \star h)\#\varphi_2\\
&= (x \star h) \cdot \varepsilon_t(\varphi_1)\#\varphi_2 = (x \star h)\#\varphi.
\end{aligned}$$

Also, for every $T \in \mathrm{End}(A \star H)_A, y \in A, g \in H$, we have

$$
\begin{aligned}
& \alpha \circ \beta(T)(y \star g) \\
&= \sum_i \alpha[T(1_A \star f_{i2})(1_A \star S^{-1}(f_{i1}))\#\psi_i](y \star g) \\
&= \sum_i T(1_A \star f_{i2})(1_A \star S^{-1}(f_{i1}))(y \star \langle \psi_i, g_2 \rangle g_1) \\
&= T(1_A \star g_3)(1_A \star S^{-1}(g_2))(y \star g_1) \\
&= T(1 \star g_5)(S^{-1}(g_4) \rightharpoonup y \leftharpoonup g_2 \star S^{-1}(g_3) g_1) \\
&= T(1 \star g_4)(S^{-1}(g_3) \rightharpoonup y \leftharpoonup g_2 \star S^{-1}(\varepsilon_s(g_1))) \\
&= T(1 \star g_3)(S^{-1}(g_2 1'_2) \rightharpoonup y \leftharpoonup g_1 1'_1 1_2 \star S^{-1}(1_1)) \\
&= T(1 \star g_3)(1_1 \rightharpoonup (S^{-1}(g_2) \rightharpoonup y \leftharpoonup g_1) \leftharpoonup S(1_3) \star 1_2) \\
&= T(1 \star g_3)(S^{-1}(g_2) \rightharpoonup y \leftharpoonup g_1 \star 1_H) \\
&= T((1 \star g_3)(S^{-1}(g_2) \rightharpoonup y \leftharpoonup g_1 \star 1_H)) \\
&= T((g_3 S^{-1}(g_2)) \rightharpoonup y \leftharpoonup (g_1 S(g_5)) \star g_4) \\
&= T(1_1 \rightharpoonup y \leftharpoonup g_1 S(g_3) \star 1_2 g_2) = T(1_1 \rightharpoonup y \leftharpoonup g_1 S(g_2) \star 1_2 g_3) \\
&= T(1_1 \rightharpoonup y \leftharpoonup S(1'_1) \star 1_2 1'_2 g) = T(1_1 \rightharpoonup y \leftharpoonup S(1'_2) \star 1_2 1'_1 g) \\
&= T(1_1 \rightharpoonup y \leftharpoonup S(1_3) \star 1_2 g) = T(y \star g).
\end{aligned}
$$

So we get α and β are inverse of each other. ∎

We now have the main result of this section as follows.

Theorem 5.3. *Let H be a finite dimensional weak Hopf algebra and $A \star H$ be a weak twisted smash product satisfying Eq. (5.1). Then there is a canonical isomorphism between the algebras $(A \star H)\#H^*$ and End$(A \star H)_A$.*

Remark 5.4. If A is a left weak H-module algebra and the right action is trivial. Then $A \star H$ is the weak smash product and from Theorem 5.3 we get the duality for weak smash product. There is a canonical isomorphism between the algebras $(A\#H)\#H^*$ and End$(A\#H)_A$. We can find the results in Nikshych [10, Throrem 3.3].

Acknowledgment

The author would like to thank the referee for his/her careful reading of this article and helpful comments. This work is supported by the Educational Ministry Science Technique Key Foundation of China (108154) and the NSF of CHINA (10871170).

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