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SEVERAL ANALYTIC INEQUALITIES IN SOME $Q-$SPACES

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Abstract. In this paper, we establish separate necessary and sufficient John-Nirenberg (JN) type inequalities for functions in $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ which imply Gagliardo-Nirenberg (GN) type inequalities in $Q_{\alpha}(\mathbb{R}^n)$. Consequently, we obtain Trudinger-Moser type inequalities and Brezis-Gallouet-Wainger type inequalities in $Q_{\alpha}(\mathbb{R}^n)$.

1. Introduction and Statement of Main Results

This paper studies several analytic inequalities in some Q spaces. We first establish John-Nirenberg type inequalities in $Q^{\beta}_{\alpha}(\mathbb{R}^n)(n \geq 2)$. Then we get Gagliardo-Nirenberg, Trudinger-Moser and Brezis-Gallouet-Wainger type inequalities in $Q_{\alpha}(\mathbb{R}^n)$. Here $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ is the set of all measurable complex-valued functions f on $\mathbb{R}^n$ satisfying

$$(1.1)\quad \|f\|_{Q^{\beta}_{\alpha}(\mathbb{R}^n)} = \sup_I \left((l(I))^{2(\alpha+\beta-1)-n} \int_I \int_I \frac{|f(x)-f(y)|^2}{|x-y|^{n+2(\alpha-\beta+1)}} dxdy \right)^{1/2} < \infty$$

for $\alpha \in (-\infty, \beta)$ and $\beta \in (1/2, 1]$, where the supremum is taken over all cubes I with edge length $l(I)$ and the edges parallel to the coordinate axes in $\mathbb{R}^n$. Obviously, $Q^1_{\alpha}(\mathbb{R}^n) = Q_{\alpha}(\mathbb{R}^n)$ which was introduced by Essen, Janson, Peng and Xiao in [9]. It has been found that $Q_{\alpha}(\mathbb{R}^n)$ is a useful and interesting concept, see, for example, Dafni and Xiao [6, 7], Xiao [19], Cui and Yang [5]. As a generalization of $Q_{\alpha}(\mathbb{R}^n)$, $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ is very useful in harmonic analysis and partial differential equations, see Yang and Yuan [20], Li and Zhai [14, 15] and Zhai [23] in which $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ was applied to study the well-posednes and regularity of mild solutions to fractional Navier-Stokes equations with fractional Laplacian $(-\triangle)^{\beta}$.

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JN type inequality is classical in modern analysis and widely applied in theory of partial differential equations. In [10], John and Nirenberg proved the JN inequality for $BMO(\mathbb{R}^n)$. In this paper, we establish JN type inequalities in $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ a special case of which implies Gagliardo-Nirenberg (GN) type inequalities meaning the continuous embeddings such as $L^r(\mathbb{R}^n)\cap Q_\alpha(\mathbb{R}^n)\subseteq L^p(\mathbb{R}^n)$ for $-\infty<\alpha<1$ and $1\le r\le p<\infty$. Moreover, from GN type inequalities in $Q_\alpha(\mathbb{R}^n)$, we get Trudinger-Moser and Brezis-Gallouet-Wainger type inequalities. See, for example, [1, 2, 8, 11, 12] for more information about Trudinger-Moser and Brezis-Gallouet-Wainger type inequalities. To achieve our main goals, we need the characterization of $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ in terms of the square mean oscillation over cubes.

We recall some facts about mean oscillation over cubes. For any cube I and an integrable function f on I, we define

$$f(I)=\frac{1}{|I|}\int_I f(x)dx \tag{1.2}$$

the mean of f on I, and for $1\le q<\infty$,

$$\Phi^q_f(I)=\frac{1}{|I|}\int_I |f(x)-f(I)|^q dx \tag{1.3}$$

the $q-$mean oscillation of f on I. Recall the well-known identities

$$\frac{1}{|I|}\int_I |f(x)-a|^2dx=\Phi^2_f(I)+|f(I)-a|^2 \tag{1.4}$$

for any complex number a, and

$$\frac{1}{|I|^2}\int_I\int_I |f(x)-f(y)|^2dxdy=2\Phi^2_f(I). \tag{1.5}$$

Moreover, if $I\subset J$, then we have

$$\Phi^2_f(I)\le\frac{|J|}{|I|}\Phi^2_f(J) \tag{1.6}$$

and

$$|f(I)-f(J)|^2\le\frac{|J|}{|I|}\Phi^2_f(J). \tag{1.7}$$

Let $\mathcal{D}_0=\mathcal{D}_0(\mathbb{R}^n)$ be the set of unit cubes whose vertices have integer coordinates, and let, for any integer $k\in\mathbb{Z}$, $\mathcal{D}_k=\mathcal{D}_k(\mathbb{R}^n)=\{2^{-k}I : I\in\mathcal{D}_0\}$, then the cubes in $\mathcal{D}=\cup_{-\infty}^{\infty}\mathcal{D}_k$ are called dyadic. Furthermore, if I is any cube, $\mathcal{D}_k(I)$, $k\ge 0$, denote the set of the 2^{kn} subcubes of edge length $2^{-k}l(I)$ obtained by k successive bipartitions of each edge of I. Moreover, put $\mathcal{D}(I)=\cup_0^{\infty}\mathcal{D}_k(I)$. For any cube I and a measurable function f on I, we define

$$\begin{aligned}\Psi_{f,\alpha,\beta}(I) &= (l(I))^{4\beta-4}\sum_{k=0}^{\infty}\sum_{J\in\mathcal{D}_k(I)} 2^{(2(\alpha-\beta+1)-n)k}\Phi_f^2(J) \\ &= (l(I))^{4\beta-4}\sum_{J\in\mathcal{D}(I)}\left(\frac{l(J)}{l(I)}\right)^{n-2(\alpha-\beta+1)}\Phi_f^2(J).\end{aligned} \tag{1.8}$$

We can prove the following proposition by a similar argument applied by Essen, Janson, Peng and Xiao for the case $\beta = 1$ in [9, Theorem 5.5]. The details are omitted here.

Proposition 1.1. *Let $-\infty < \alpha < \beta$ and $\beta \in (1/2, 1]$. Then $Q_\alpha^\beta(\mathbb{R}^n)$ equals the space of all measurable functions f on $\mathbb{R}^n$ such that $\sup_I \Psi_{f,\alpha,\beta}(I)$ is finite, where I ranges over all cubes in $\mathbb{R}^n$. Moreover, the square root of this supremum is a norm on $Q_\alpha^\beta(\mathbb{R}^n)$, equivalent to $\|f\|_{Q_\alpha^\beta(\mathbb{R}^n)}$ as defined above.*

Using this equivalent characterization of $Q_\alpha^\beta(\mathbb{R}^n)$, we can establish the following JN type inequalities.

Theorem 1.2. *Let $-\infty < \alpha < \beta$, $\beta \in (1/2, 1]$ and $0 \le p < 2$. If there exist positive constants B, C and c, such that, for all cubes $I \subset \mathbb{R}^n$, and any $t > 0$,*

$$(l(I))^{4\beta-4}\sum_{k=0}^{\infty} 2^{(2(\alpha-\beta+1)-n)k}\sum_{J\in\mathcal{D}_k(I)}\frac{m_J(t)}{|J|} \le B\max\left\{1, \left(\frac{C}{t}\right)^p\right\}\exp(-ct), \tag{1.9}$$

then f is a function in $Q_\alpha^\beta(\mathbb{R}^n)$. Here $m_I(t)$ is the distribution function of $f - f(I)$ on the cube I :

$$m_I(t) = |\{x \in I : |f(x) - f(I)| > t\}|. \tag{1.10}$$

Theorem 1.3. *Let $-\infty < \alpha < \beta$, $\beta \in (1/2, 1]$ and $f \in Q_\alpha^\beta(\mathbb{R}^n)$. Then there exist positive constants B and b, such that*

$$\begin{aligned}&(l(I))^{4\beta-4}\sum_{k=0}^{\infty} 2^{(2(\alpha-\beta+1)-n)k}\sum_{J\in\mathcal{D}_k(I)}\frac{m_J(t)}{|J|} \\ &\le B\max\left\{1, \left(\frac{\|f\|_{Q_\alpha^\beta}}{t}\right)^2\right\}\exp\left(\frac{-bt}{\|f\|_{Q_\alpha^\beta}}\right)\end{aligned} \tag{1.11}$$

holds for $t \le \|f\|_{Q_\alpha^\beta(\mathbb{R}^n)}$ and any cubes $I \subset \mathbb{R}^n$, or for $t > \|f\|_{Q_\alpha^\beta(\mathbb{R}^n)}$ and cubes $I \subset \mathbb{R}^n$ with $(l(I))^{2\beta-2} \ge 1$. Moreover, there holds

$$(l(I))^{4\beta-4}\sum_{k=0}^{\infty} 2^{(2(\alpha-\beta+1)-n)k}\sum_{J\in\mathcal{D}_k(I)}\frac{m_J(t)}{|J|} \le B \tag{1.12}$$

for $t > \|f\|_{Q^{\beta}_{\alpha}(\mathbb{R}^n)}$ *and cubes* $I \subset \mathbb{R}^n$ *with* $(l(I))^{2\beta-2} < 1$.

For $\beta = 1$, the JN inequality in $Q_\alpha(\mathbb{R}^n)$ was conjectured by Essen-Janson-Peng-Xiao in [9] and finally a modified version as in Theorems 1.2-1.3 was established by Yue-Dafni [21].

According to Essen, Janson, Peng and Xiao [9, Theorem 2.3] and Li and Zhai [14, Theorem 3.2], we know that if $-\infty < \alpha$ and $\max\{\alpha, 1/2\} < \beta \leq 1$, $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ is decreasing in α for a fixed β. Moreover, if $\alpha \in (-\infty, \beta-1)$, then all $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ equal to $Q^{\beta}_{-\frac{n}{2}+\beta-1}(\mathbb{R}^n) := BMO^{\beta}(\mathbb{R}^n)$. Thus, when $k = 0$ and $\alpha = -\frac{n}{2} + \beta - 1$, (1.11) implies a special JN type inequality, that is, for $f \in L^2(\mathbb{R}^n) \cap BMO^{\beta}(\mathbb{R}^n)$ and $t \leq \|f\|_{BMO^{\beta}(\mathbb{R}^n)}$,

$$|\{x \in \mathbb{R}^n : |f| > t\}| \leq \frac{B\|f\|^2_{L^2(\mathbb{R}^n)}}{t^2} \exp\left(\frac{-bt}{\|f\|_{BMO^{\beta}(\mathbb{R}^n)}}\right). \tag{1.13}$$

When $t > \|f\|_{BMO^{\beta}(\mathbb{R}^n)}$, we get a weaker form of (1.13).

Proposition 1.4. *Let* $\beta \in (1/2, 1]$. *If* $f \in BMO^{\beta}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, *then*

(i) (1.13) *holds for all* $t \leq \|f\|_{BMO^{\beta}(\mathbb{R}^n)}$;

(ii)

$$|\{x \in \mathbb{R}^n : f(x) > t\}| \leq \frac{B\|f\|^2_{L^2(\mathbb{R}^2)}}{\|f\|^2_{BMO^{\beta}(\mathbb{R}^n)}} \tag{1.14}$$

holds for all $t > \|f\|_{BMO^{\beta}(\mathbb{R}^n)}$.

When $\beta = 1$ and $t > \|f\|_{BMO(\mathbb{R}^n)}$, (1.13) also holds and implies the following GN type inequalities in $Q_\alpha(\mathbb{R}^n)$ which can also be deduced from [4, Theorem 2] and [9, Theorem 2.3]: for $-\infty < \alpha < 1$ and $1 \leq r \leq p < \infty$,

$$\|f\|_{L^p(\mathbb{R}^n)} \leq C_n p \|f\|^{r/p}_{L^r(\mathbb{R}^n)} \|f\|^{1-r/p}_{Q_\alpha(\mathbb{R}^n)}, \tag{1.15}$$

for $f \in L^r(\mathbb{R}^n) \cap Q_\alpha(\mathbb{R}^n)$. Here, $C_{*,\cdots,*}$ denotes a constant which depends only on the quantities appearing in the subscript indexes.

As an application of (1.15), we establish the Trudinger-Moser type inequality which implies a generalized JN type inequality.

Theorem 1.5.

(i) *There exists a positive constant* γ_n *such that for every* $0 < \zeta < \gamma_n$

$$\int_{\mathbb{R}^n} \Phi_p \left(\zeta \left(\frac{|f(x)|}{\|f\|_{Q_\alpha(\mathbb{R}^n)}}\right)\right) dx \leq C_{n,\zeta} \left(\frac{\|f\|_{L^p(\mathbb{R}^n)}}{\|f\|_{Q_\alpha(\mathbb{R}^n)}}\right)^p \tag{1.16}$$

holds for all

$$f \in L^p(\mathbb{R}^n) \cap Q_\alpha(\mathbb{R}^n) \quad \textit{with} \quad 1 < p < \infty \quad \textit{and} \quad -\infty < \alpha < 1.$$

Here Φ_p *is the function defined by*

$$\Phi_p(t) = e^t - \sum_{j<p, j\in\mathbb{N}\cup\{0\}} \frac{t^j}{j!}, t \in \mathbb{R}.$$

(ii) *There exists a positive constant* γ_n *such that*

$$|\{x \in \mathbb{R}^n : |f| > t\}| \leq C_n \frac{\|f\|^2_{L^2(\mathbb{R}^n)}}{\|f\|^2_{Q_\alpha(\mathbb{R}^n)}} \frac{1}{\left(\exp\left(\frac{t\gamma_n}{\|f\|_{Q_\alpha(\mathbb{R}^n)}}\right) - 1 - \frac{t\gamma_n}{\|f\|_{Q_\alpha(\mathbb{R}^n)}}\right)} \tag{1.17}$$

holds for all $t > 0$ *and*

$$f \in L^2(\mathbb{R}^n) \cap Q_\alpha(\mathbb{R}^n) \quad \textit{with} \quad -\infty < \alpha < 1.$$

In particular, we have

$$|\{x \in \mathbb{R}^n : |f| > t\}| \leq C_n \frac{\|f\|^2_{L^2(\mathbb{R}^n)}}{\|f\|^2_{Q_\alpha(\mathbb{R}^n)}} \exp\left(-\frac{t\gamma_n}{\|f\|_{Q_\alpha(\mathbb{R}^n)}}\right) \tag{1.18}$$

holds for all $t > \|f\|_{Q_\alpha(\mathbb{R}^n)}$ *and*

$$f \in L^2(\mathbb{R}^n) \cap Q_\alpha(\mathbb{R}^n) \quad \textit{with} \quad -\infty < \alpha < 1.$$

We can also get the following Brezis-Gallouet-Wainger type inequalities.

Proposition 1.6. *For every* $1 < q < \infty$ *and* $n/q < s < \infty$, *we have*

$$\begin{aligned} &\|f\|_{L^\infty(\mathbb{R}^n)} \\ &\leq C_{n,p,q,s} \left(1 + (\|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{Q_\alpha(\mathbb{R}^n)}) \log(e + \|(-\triangle)^{s/2} f\|_{L^q(\mathbb{R}^n)})\right) \end{aligned} \tag{1.19}$$

holds for all $(-\triangle)^{s/2} f \in L^q(\mathbb{R}^n)$ *satisfying*

$$f \in L^p(\mathbb{R}^n) \cap Q_\alpha(\mathbb{R}^n) \quad \textit{when} \quad 1 \leq p < \infty \quad \textit{and} \quad -\infty < \alpha < 1.$$

In the next section, we prove our main results. We verify Theorem 1.2-1.3 for $\beta \in (1/2, 1]$ by applying similar arguments in the proof of Yue and Dafni [21, Theorems 1-2] for $\beta = 1$. We deduce Proposition 1.4 from a special case of Theorem 1.3. Finally, we demonstrate Theorem 1.5 and Proposition 1.6 by applying (1.15) and the $L^p - L^q$ estimates for $e^{-t(-\triangle)^{s/2}}$.

2. Proofs of Main Results

2.1. Proof of Theorem 1.2

According to Proposition 1.1, it suffices to prove that $\Psi_{f,\alpha,\beta}(I)$ is bounded independent of I. More specially, we will prove for any $p < q$, we have

$$\Psi^q_{f,\alpha,\beta}(I) := (l(I))^{4\beta-4}\sum_{k=0}^{\infty} 2^{(2(\alpha-\beta+1)-n)k} \sum_{J\in\mathcal{D}_k(I)} \Phi^q_f(J) \le BK_{C,c,q,p}, \tag{2.1}$$

where B, C, c are the constants appearing in (1.9), and $K_{C,c,q,p}$ is a constant depending only on C, c, p, and q. When $q = 2$, $\Psi^q_{f,\alpha,\beta}(I) = \Psi_{f,\alpha,\beta}(I)$, so this implies the theorem.

For a fixed cube I, and any $J \in \mathcal{D}_k(I)$, let $\int_J |f(x)-f(J)|^q dx = q\int_0^\infty t^{q-1}m_J(t)dt$. Using the Monotone Convergence Theorem and the inequality (1.9), we have

$$\begin{aligned}
\Psi^q_{f,\alpha,\beta}(I) &= (l(I))^{4\beta-4}\sum_{k=0}^{\infty} 2^{(2(\alpha-\beta+1)-n)k} \sum_{J\in\mathcal{D}_k(t)} \frac{q}{|J|}\int_0^\infty t^{q-1}m_J(t)dt\\
&= q\int_0^\infty t^{q-1}\left((l(I))^{4\beta-4}\sum_{k=0}^{\infty} 2^{(2(\alpha-\beta+1)-n)k} \sum_{J\in\mathcal{D}_k(I)} \frac{m_J(t)}{|J|}\right)dt\\
&\le q\int_0^\infty t^{q-1}B(1+\left(\frac{C}{t}\right)^p)e^{-ct}dt\\
&= qB\left(c^{-q}\int_0^\infty u^{q-1}e^{-u}du + C^p c^{-(q-p)}\int_0^\infty u^{q-p-1}e^{-u}du\right)\\
&= qB(c^{-q}\Gamma(q) + C^p c^{-(q-p)}\Gamma(q-p))
\end{aligned}$$

where $\Gamma(y) = \int_0^\infty u^{y-1}e^{-u}du$. Since $0 \le p < q$, $\Gamma(q)$ and $\Gamma(q-p)$ are finite. Thus, we can get the desired inequality by taking $K_{C,c,p,q} = q(c^{-q}\Gamma(q) + C^p c^{-(q-p)}\Gamma(q-p))$.

2.2. Proof of Theorem 1.3

Assume that f is a nontrivial element of $Q^\beta_\alpha(\mathbb{R}^n)$. Then $\gamma = \sup_I(\Psi_{f,\alpha,\beta}(I))^{1/2} < \infty$. For all cubes I we have

$$\begin{aligned}
&(l(I))^{2\beta-2}\frac{1}{|I|}\int_I |f(x)-f(I)|dx\\
&\le ((l(I))^{4\beta-4}\Phi^2_f(I))^{1/2} \le (\Psi_{f,\alpha,\beta}(I))^{1/2} \le \gamma.
\end{aligned} \tag{2.2}$$

For a cube I and each $J \in \mathcal{D}_k(I)$, we have by the Chebyshev inequality, for $t > 0$,

$$m_J(t) \le t^{-2}\int_J |f(x)-f(J)|^2dx.$$

Thus we get

$$(l(I))^{4\beta-4}\sum_{k=0}^{\infty}2^{(2(\alpha-\beta+1)-n)k}\sum_{J\in\mathcal{D}_k(I)}\frac{m_J(t)}{|J|}\le t^{-2}\Psi_{f,\alpha,\beta}(I)\le t^{-2}\gamma^2. \tag{2.3}$$

Thus, if $t\le\gamma$, then (1.11) holds with $B=e$ and $b=1$.

To consider the case of $t>\gamma$, we need the Calderón-Zygmund decomposition, see Calderón and Zygmund [3], and Neri [17].

Lemma 2.1. *Assume that f is a nonnegative function in $L^1(\mathbb{R}^n)$ and ξ is a positive constant. There is a decomposition $\mathbb{R}^n=P\cup\Omega$, $P\cap\Omega=\emptyset$, such that*

(a) $\Omega=\cup_{k=1}^{\infty}I_k$, where I_k is a collection of cubes whose interiors are disjoint;

(b) $f(x)\le\xi$ for a.e. $x\in P$;

(c) $\xi<\frac{1}{|I|}\int_I f(x)dx\le 2^n\xi$, for all I in the collection $\{I_k\}$.

(d) $\xi|\triangle|\le\int_\triangle f(x)dx\le 2^n\xi|\triangle|$, if $\triangle$ is any union of cubes I from $\{I_k\}$.

In the following we fix a cube I. For $\xi=t(l(I))^{2-2\beta}$ with any $t>0$, we apply the Calderón-Zygmund decomposition to $|f(x)-f(J)|$ on a subcube $J\in\mathcal{D}_k(I)$. Set $\Omega=\Omega_J(t)$, $P=J\backslash\Omega_J(t)$.

From Cauchy-Schwarz inequality and (d) of Lemma 2.1, we get

$$(t(l(I))^{2-2\beta})^2|\triangle|\le\int_\triangle|f(x)-f(J)|^2dx \tag{2.4}$$

for any union $\triangle$ of the cubes K in the decomposition of $\Omega_J(t)$. Inequality (2.4) with $\triangle=\Omega_J(t)$ gives us a variant of inequality (2.3):

$$\begin{aligned}&(l(I))^{4\beta-4}\sum_{k=0}^{\infty}2^{(2(\alpha-\beta+1)-n)k}\sum_{J\in\mathcal{D}_k(I)}\frac{|\Omega_J(t)|}{|J|}\\&\le\frac{\Psi_{f,\alpha,\beta}(I)}{(t(l(I))^{2-2\beta})^2}\le\left(\frac{\gamma}{(t(l(I))^{2-2\beta})}\right)^2\end{aligned} \tag{2.5}$$

for all $t>0$.

When $t\ge\gamma$, we can strengthen the estimate (c) in Lemma 2.1 as follows:

$$t(l(I))^{2-2\beta}<\frac{1}{|K|}\int_K|f(x)-f(J)|dx\le(2^n\gamma+t)(l(I))^{2-2\beta} \tag{2.6}$$

for all cubes K in the decomposition of $\Omega_J(t)$. In fact, note that K is such a cube, then $K\ne J$. Otherwise, (2.2) implies

$$\frac{1}{|J|}\int_J|f(x)-f(J)|dx\le\gamma(l(I))^{2-2\beta}\le t(l(I))^{2-2\beta}.$$

This contradicts (c). It follows from the proof of the Calderón-Zygmund decomposition (see, Stein [18]) that K must have a "parent" cube $K^* \subset J$ satisfying $K \in \mathcal{D}_1(K^*)$, $l(K^*) = 2l(K)$ and

$$|f(K^*) - f(J)| \leq |K^*|^{-1} \int_{K^*} |f(x) - f(J)|dx \leq t(l(I))^{2-2\beta}.$$

Then (2.2) implies

$$\begin{aligned} t(l(I))^{2-2\beta} &< \frac{1}{|K|} \int_K |f(x) - f(J)|dx \\ &\leq \frac{1}{|K|} \int_K |f(x) - f(K^*)|dx + |f(K^*) - f(J)| \\ &\leq \frac{2^n}{|K^*|} \int_{K^*} |f(x) - f(K^*)|dx + t(l(I))^{2-2\beta} \\ &\leq (2^n\gamma + t)(l(I))^{2-2\beta}. \end{aligned}$$

There holds $\Omega_J(t') \subset \Omega_J(t)$ for $0 < t < t'$. In fact, for any cube $K \in \Omega_J(t')\backslash\Omega_J(t)$, we get $K \subset J\backslash\Omega_J(t)$. So, property (b) tells us

$$t(l(I))^{2-2\beta} \geq \frac{1}{|K|} \int_K |f(x) - f(J)|dx > t'(l(I))^{2-2\beta}.$$

This is a contradiction.

Letting $t' = t + 2^{n+1}\gamma$ for $t \geq \gamma$, we claim that

$$|\Omega_J(t')| \leq 2^{-n}|\Omega_J(t)|. \tag{2.7}$$

To prove this, take a cube K in the decomposition for $\Omega_J(t)$. Then (2.6) implies that

$$\frac{1}{|K|} \int_K |f(x) - f(J)|dx \leq (2^n\gamma + t)(l(I))^{2-2\beta} < t'(l(I))^{2-2\beta}.$$

Thus, K is not a cube in the decomposition of $\Omega_J(t')$, and was further subdivided. Set $\triangle' = K \cap \Omega_J(t')$. If $\triangle' \neq \emptyset$, it must be a union of cubes from the decomposition of $\Omega_J(t')$. Thus, according to (d) of Lemma 2.1, (2.2) and (2.6),

$$\begin{aligned} t'(l(I))^{2-2\beta} &\leq |\triangle'|^{-1} \int_{\triangle'} |f(x) - f(J)|dx \\ &\leq |\triangle'|^{-1} \int_{\triangle'} |f(x) - f(K)|dx + |f(K) - f(J)| \\ &\leq |\triangle'|^{-1}|K| \frac{1}{|K|} \int_{\triangle'} |f(x) - f(K)|dx + \frac{1}{|K|} \int_K |f(x) - f(J)|dx \\ &\leq |\triangle'|^{-1}|K|\gamma(l(K))^{2-2\beta} + (2^n\gamma + t)(l(I))^{2-2\beta} \\ &\leq |\triangle'|^{-1}|K|\gamma(l(I))^{2-2\beta} + (2^n\gamma + t)(l(I))^{2-2\beta} \end{aligned}$$

since $2-2\beta>0$ and $K\subset I$. Replacing t' by $t+2^{n+1}\gamma$, dividing by $(l(I))^{2-2\beta}$, subtracting t and dividing by γ, we have

$$(2^{n+1}-2^n)\le|\triangle'|^{-1}|K|\quad\text{and}\quad|K\cap\Omega_J(t')|=|\triangle'|\le 2^{-n}|K|$$

for any cube K in the decomposition of $\Omega_J(t)$. Summing over all such K, and noting that $\Omega_J(t')=\Omega_J(t)\cap\Omega_J(t')$, we prove (2.7).

For each $J\in\mathcal{D}_k(I)$, property (b) of the decomposition for $|f-f(J)|$ implies that

$$m_J(t(l(I))^{2-2\beta})=|\{x\in J:|f(x)-f(J)|>t(l(I))^{2-2\beta}\}|\le|\Omega_J(t)|. \tag{2.8}$$

For $t>\gamma$, let j be the integer part of $\frac{t-\gamma}{2^{n+1}\gamma}$ and $s=(1+j2^{n+1})\gamma$. Then $\gamma\le s\le t$. Thus one obtains from (2.8) that

$$\begin{aligned}
&(l(I))^{4\beta-4}\sum_{k=0}^{\infty}2^{(2(\alpha-\beta+1)-n)k}\sum_{J\in\mathcal{D}_k(I)}\frac{m_J(t)}{|J|}\\
&=(l(I))^{4\beta-4}\sum_{k=0}^{\infty}2^{(2(\alpha-\beta+1)-n)k}\sum_{J\in\mathcal{D}_k(I)}\frac{m_J((l(I))^{2-2\beta}t(l(I))^{2\beta-2})}{|J|}\\
&\le(l(I))^{4\beta-4}\sum_{k=0}^{\infty}2^{(2(\alpha-\beta+1)-n)k}\sum_{J\in\mathcal{D}_k(I)}\frac{m_J((l(I))^{2-2\beta}s(l(I))^{2\beta-2})}{|J|}\\
&\le(l(I))^{4\beta-4}\sum_{k=0}^{\infty}2^{(2(\alpha-\beta+1)-n)k}\sum_{J\in\mathcal{D}_k(I)}\frac{|\Omega_J((1+j2^{n+1})\gamma(l(I))^{2\beta-2})|}{|J|}\\
&\le(l(I))^{4\beta-4}\sum_{k=0}^{\infty}2^{(2(\alpha-\beta+1)-n)k}\sum_{J\in\mathcal{D}_k(I)}\frac{|\Omega_J(\gamma(l(I))^{2\beta-2}+j2^{n+1}\gamma)|}{|J|}\\
&\le 2^{-n}(l(I))^{4\beta-4}\sum_{k=0}^{\infty}2^{(2(\alpha-\beta+1)-n)k}\sum_{J\in\mathcal{D}_k(I)}\frac{|\Omega_J(\gamma(l(I))^{2\beta-2}+(j-1)2^{n+1}\gamma)|}{|J|}
\end{aligned}$$

if $(l(I))^{2\beta-2}\ge1$, by using (2.7) for

$$t=((l(I))^{2\beta-2}+(j-1)2^{n+1})\gamma\quad\text{and}\quad t'=((l(I))^{2\beta-2}+j2^{n+1})\gamma.$$

Iterating the previous estimate j times and using (2.5) with $t=\gamma(l(I))^{2\beta-2}$, one has

$$\begin{aligned}
&(l(I))^{4\beta-4}\sum_{k=0}^{\infty}2^{(2(\alpha-\beta+1)-n)k}\sum_{J\in\mathcal{D}_k(I)}\frac{m_J(t)}{|J|}\\
&\le 2^{-nj}(l(I))^{4\beta-4}\sum_{k=0}^{\infty}2^{(2(\alpha-\beta+1)-n)k}\sum_{J\in\mathcal{D}_k(I)}\frac{|\Omega_J(\gamma(l(I))^{2\beta-2})|}{|J|}
\end{aligned}$$

$$
\begin{aligned}
&\leq 2^{-nj}\gamma^2\gamma^{-2} \\
&\leq \ 2^{-n\left(\frac{t-\gamma}{2^{n+1}\gamma}-1\right)} \\
&= 2^{-\frac{n}{2^{n+1}}(t/\gamma)} 2^{\frac{n}{2^{n+1}}+n}.
\end{aligned}
$$

Taking $B = 2^{n/2^{n+1}+n}$ and $b = \frac{n}{2^{n+1}}\ln 2$, we get (1.11) when $(l(I))^{2\beta-2} \geq 1$.

If $(l(I))^{2\beta-2} < 1$, using (2.8) and (2.4), one has

$$
\begin{aligned}
&(l(I))^{4\beta-4}\sum_{k=0}^{\infty} 2^{(2(\alpha-\beta+1)-n)k} \sum_{J\in\mathcal{D}_k(I)} \frac{m_J(t)}{|J|} \\
\leq\ & (l(I))^{4\beta-4}\sum_{k=0}^{\infty} 2^{(2(\alpha-\beta+1)-n)k} \sum_{J\in\mathcal{D}_k(I)} \frac{|\Omega_J(t(l(I))^{2\beta-2})|}{|J|} \\
\leq\ & \gamma^2 t^{-2} \leq 1
\end{aligned}
$$

which yields (1.12).

2.3. Proof of Proposition 1.4

Taking $k = 0$ and $\alpha = -\frac{n}{2} + \beta - 1$ in (1.11), we get that

$$
(l(I))^{4\beta-4}\frac{m_I(t)}{|I|} \leq B\frac{\|f\|^2_{BMO^\beta(\mathbb{R}^n)}}{t^2}\exp\left(\frac{-bt}{\|f\|_{BMO^\beta(\mathbb{R}^n)}}\right)
$$

holds for $t \leq \|f\|_{BMO^\beta(\mathbb{R}^n)}$ and any cube I. Thus for $t \leq \|f\|_{BMO^\beta(\mathbb{R}^n)}$ and any cube I, we have

$$
\begin{aligned}
&(l(I))^{4\beta-4}\frac{m_I(t)}{|I|}\int_I |f(x)-f(I)|^2dx \\
\leq\ & B\frac{\|f\|^2_{BMO^\beta(\mathbb{R}^n)}}{t^2}\exp\left(\frac{-bt}{\|f\|_{BMO^\beta(\mathbb{R}^n)}}\right)\int_I |f(x)-f(I)|^2dx \\
\leq\ & B\frac{\|f\|^2_{BMO^\beta(\mathbb{R}^n)}}{t^2}\exp\left(\frac{-bt}{\|f\|_{BMO^\beta(\mathbb{R}^n)}}\right)\int_I |f(x)|^2dx \\
\leq\ & B\frac{\|f\|^2_{BMO^\beta(\mathbb{R}^n)}}{t^2}\exp\left(\frac{-bt}{\|f\|_{BMO^\beta(\mathbb{R}^n)}}\right)\int_{\mathbb{R}^n} |f(x)|^2dx.
\end{aligned}
$$

This tells us

$$
\begin{aligned}
(2.9)\qquad & m_I(t)\frac{(l(I))^{4\beta-4}}{|I|}\int_I |f(x)-f(I)|^2dx \\
& \leq B\frac{\|f\|^2_{BMO^\beta(\mathbb{R}^n)}}{t^2}\exp\left(\frac{-bt}{\|f\|_{BMO^\beta(\mathbb{R}^n)}}\right)\int_{\mathbb{R}^n}|f(x)|^2dx.
\end{aligned}
$$

According to the definition of $BMO^\beta(\mathbb{R}^n)$, see Li and Zhai [14], we have

$$
f\in BMO^\beta(\mathbb{R}^n)\Longleftrightarrow \|f\|^2_{BMO^\beta(\mathbb{R}^n)}=\sup_I\frac{(l(I))^{4\beta-4}}{|I|}\int_I|f(x)-f(I)|^2dx<\infty.
$$

Thus, we get

$$
\begin{aligned}
& m_I(t)\|f\|^2_{BMO^\beta(\mathbb{R}^n)} \\
\leq\quad & B\frac{\|f\|^2_{BMO^\beta(\mathbb{R}^n)}}{t^2}\exp\left(\frac{-bt}{\|f\|_{BMO^\beta(\mathbb{R}^n)}}\right)\int_{\mathbb{R}^n}|f(x)|^2dx,
\end{aligned}
$$

for $t\leq\|f\|_{BMO^\beta(\mathbb{R}^n)}$. Then, taking an increasing sequence of cubes covering $\mathbb{R}^n$, we obtain

$$
(2.10)\qquad |\{x\in\mathbb{R}^n:|f(x)|>t\}|\leq\frac{B}{t^2}\exp\left(\frac{-bt}{\|f\|_{BMO^\beta(\mathbb{R}^n)}}\right)\int_{\mathbb{R}^n}|f(x)|^2dx
$$

for $t\leq\|f\|_{BMO^\beta(\mathbb{R}^n)}$, since $f(I)\longrightarrow 0$ as $l(I)\longrightarrow\infty$. Finally, we get (1.13). Similarly, we can prove (1.14) since $\exp\left(\frac{-bt}{\|f\|_{BMO^\beta(\mathbb{R}^n)}}\right)\leq 1$ for $t>\|f\|_{BMO^\beta(\mathbb{R}^n)}$.

2.4. Proof of Theorem 1.5

(i) According to (1.15), we have

$$
\begin{aligned}
\int_{\mathbb{R}^n}\Phi_p\left(\zeta\frac{|f(x)|}{\|f\|_{Q_\alpha(\mathbb{R}^n)}}\right)dx &= \int_{\mathbb{R}^n}\sum_{j\geq p,j\in\mathbb{N}}\frac{\zeta^j}{j!}\left(\frac{|f(x)|}{\|f\|_{Q_\alpha(\mathbb{R}^n)}}\right)^j dx \\
&\leq \sum_{j\geq p,j\in\mathbb{N}}\frac{\zeta^j}{j!}\frac{\|f\|^j_{L^j(\mathbb{R}^n)}}{\|f\|^j_{Q_\alpha(\mathbb{R}^n)}} \\
&\leq \sum_{j\geq p,j\in\mathbb{N}}\frac{\zeta^j}{j!}\frac{\left(C_nj\|f\|^{p/j}_{L^p(\mathbb{R}^n)}\|f\|^{1-p/j}_{Q_\alpha(\mathbb{R}^n)}\right)^j}{\|f\|^j_{Q_\alpha(\mathbb{R}^n)}} \\
&\leq \sum_{j\geq p,j\in\mathbb{N}}a_j(\zeta C_n)^j\left(\frac{\|f\|_{L^p(\mathbb{R}^n)}}{\|f\|_{Q_\alpha(\mathbb{R}^n)}}\right)^p
\end{aligned}
$$

with $a_j = \frac{j^j}{j!}$. Since $\lim_{j\longrightarrow\infty} \frac{a_j}{a_{j+1}} = e^{-1}$, the power series of the above right hand side converges provided $\zeta C_n < e^{-1}$ i.e. $\zeta < \gamma_n := (C_n e)^{-1}$.
(ii) According to (i) with $p = 2$, we have

$$\int_{\mathbb{R}^n} \left(\exp\left(\gamma_n \frac{|f(x)|}{\|f\|_{Q_\alpha(\mathbb{R}^n)}} \right) - 1 - \gamma_n \frac{|f(x)|}{\|f\|_{Q_\alpha(\mathbb{R}^n)}} \right) dx \leq C_n \frac{\|f\|^2_{L^2(\mathbb{R}^n)}}{\|f\|^2_{Q_\alpha(\mathbb{R}^n)}}.$$

On the other hand, since the distribution function $m(t) = |\{x \in \mathbb{R}^n : |f(x)| > t\}|$ is non-increasing, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(\exp\left(\gamma_n \frac{|f(x)|}{\|f\|_{Q_\alpha(\mathbb{R}^n)}} \right) - 1 - \gamma_n \frac{|f(x)|}{\|f\|_{Q_\alpha(\mathbb{R}^n)}} \right) dx \\
= \ & \sum_{j=2}^{\infty} \frac{\gamma_n^j}{j!} \frac{\|f\|^j_{L^j(\mathbb{R}^n)}}{\|f\|^j_{Q_\alpha(\mathbb{R}^n)}} \\
= \ & \sum_{j=2}^{\infty} \frac{\gamma_n^j}{j!} \frac{j}{\|f\|^j_{Q_\alpha(\mathbb{R}^n)}} \int_0^\infty m(s) s^{j-1} ds \\
\geq \ & m(t) \sum_{j=2}^{\infty} \frac{\gamma_n^j}{j!} \frac{j}{\|f\|^j_{Q_\alpha(\mathbb{R}^n)}} \int_0^t s^{j-1} ds \\
= \ & m(t) \sum_{j=2}^{\infty} \frac{1}{j!} \left(\frac{\gamma_n t}{\|f\|_{Q_\alpha(\mathbb{R}^n)}} \right)^j \\
= \ & m(t) \left(\exp\left(\frac{\gamma_n t}{\|f\|^j_{Q_\alpha(\mathbb{R}^n)}} \right) - 1 - \frac{\gamma_n t}{\|f\|_{Q_\alpha(\mathbb{R}^n)}} \right)
\end{aligned}$$

for all $t > 0$. Thus, we have

$$m(t) \leq C_n \frac{\|f\|^2_{L^2(\mathbb{R}^n)}}{\|f\|^2_{Q_\alpha(\mathbb{R}^n)}} \frac{1}{\left(\exp\left(\frac{\gamma_n t}{\|f\|^j_{Q_\alpha(\mathbb{R}^n)}} \right) - 1 - \frac{\gamma_n t}{\|f\|_{Q_\alpha(\mathbb{R}^n)}} \right)}.$$

2.5. Proof of Proposition 1.6

We will use some facts about the factional heat equations

$$\partial_t v(t,x) + (-\triangle)^{s/2} v(t,x) = 0 \quad \text{for} \quad (t,x) \in (0,\infty) \times \mathbb{R}^n$$

with initial data $v(0,x) = g(x)$ for $x \in \mathbb{R}^n$. The fractional heat equations have been studied by Miao-Yuan-Zhang [16], Zhai [22, 24] and references therein. Here

$$\mathcal{F}((-\triangle)^{s/2} v(t,x))(\xi) = |\xi|^s \mathcal{F} v(t,\xi)$$

and $v_g(t,x) = e^{-t(\triangle)^{s/2}}g(x) = K_t^s(x) * g(x)$ with $K_t^s(\cdot) = \mathcal{F}^{-1}(e^{-t|\cdot|^s})$ where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transformation and its inverse. We need the $L^p \longrightarrow L^q$ estimates for the semigroup $\{e^{-t(-\triangle)^{s/2}}\}_{t\geq 0}$. For the proof, see, for example, Kozono-Wadade [13, Lemma 3.4] or Miao-Yuan-Zhang [16, Lemma 3.1].

Lemma 2.2. *For every $0 < s < \infty$, there exists a constant $C_{n,s}$ depending only on n and s such that*

$$\|e^{-t(-\triangle)^{s/2}}g\|_{L^q(\mathbb{R}^n)} \leq C_{n,s}t^{-\frac{n}{s}\left(\frac{1}{p}-\frac{1}{q}\right)}\|g\|_{L^p(\mathbb{R}^n)}.$$

holds for all $g \in L^p(\mathbb{R}^n)$, $t > 0$ and $1 \leq p \leq q \leq \infty$.

For any $g(x)$ in the Schwartz class of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^n)$, define $v_g(t,x) = e^{-(\triangle)^{s/2}}g(x)$ as the solution of fractional heat equation

$$\partial_t v(t,x) + (-\triangle)^{s/2}v(t,x) = 0$$

with initial data g. Fix $f \in L^2(\mathbb{R}^n) \cap Q_\alpha^\beta(\mathbb{R}^n)$ with $(-\triangle)^{s/2}f \in L^q$. Then

$$\begin{aligned}\int_0^t \langle -(-\triangle)^{s/2}f(x), v(s,x)\rangle ds &= \int_0^t \langle f(x), -(-\triangle)^{s/2}v(s,x)\rangle ds\\ &= \int_0^t \langle f(x), \partial_s v(s,x)\rangle dt\\ &= \langle f(x), v(t,x)\rangle - \langle f(x), g(x)\rangle.\end{aligned}$$

Thus

$$|\langle f,g\rangle| \leq |\langle f(x), v(t,x)\rangle| + \int_0^t |\langle(-\triangle)^{s/2}f(x), v(s,x)\rangle|ds = I_1 + I_2$$

for all $t > 0$. Here $\langle\cdot,\cdot\rangle$ denote the inner-product in L^2. Thus Hölder inequality, Lemma 2.2 and (1.15) imply that

$$\begin{aligned}I_1 &\leq \|f\|_{L^{q_1}(\mathbb{R}^n)}\|v(t,\cdot)\|_{L^{q_1'}(\mathbb{R}^n)} = \|f\|_{L^{q_1}(\mathbb{R}^n)}\|e^{-t(-\triangle)^{s/2}}g\|_{L^{q_1'}(\mathbb{R}^n)}\\ &\leq C_{n,s}q_1 t^{-\frac{n}{sq_1}}(\|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{Q_\alpha^\beta(\mathbb{R}^n)})\|g\|_{L^1(\mathbb{R}^n)}\end{aligned}$$

for all $t > 0$ and $p \leq q_1 < \infty$. Similarly, we have

$$\begin{aligned}I_2 &\leq \int_0^t \|(-\triangle)^{s/2}f\|_{L^q(\mathbb{R}^n)}\|v(s,\cdot)\|_{L^{q'}(\mathbb{R}^n)}ds\\ &= \|(-\triangle)^{s/2}f\|_{L^q(\mathbb{R}^n)}\int_0^t \|e^{-t(-\triangle)^{s/2}}g\|_{L^{q'}(\mathbb{R}^n)}ds\\ &\leq C_{n,s,q}\|(-\triangle)^{s/2}f\|_{L^q(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)}\int_0^t s^{-\frac{n}{sq}}ds\\ &\leq C_{n,s,q}t^{1-\frac{n}{sq}}\|(-\triangle)^{s/2}f\|_{L^q(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)}\end{aligned}$$

for all $t > 0$. Combing the duality argument and these two estimates, we have

$$\begin{aligned}\|f\|_{L^\infty(\mathbb{R}^n)} &= \sup_{\|g\|_{L^1(\mathbb{R}^n)}\le 1, g\in\mathcal{S}} |\langle f,g\rangle| \\ &\le C_{n,s,q}\left(q_1 t^{-\frac{n}{sq_1}}\left(\|f\|_{L^p(\mathbb{R}^n)}+\|f\|_{Q_\alpha(\mathbb{R}^n)}\right)+t^{1-\frac{n}{sq}}\|(-\triangle)^{s/2}f\|_{L^q(\mathbb{R}^n)}\right)\end{aligned}$$

for all $t > 0$ and $p \le q_1 < \infty$. Take

$$q_1 = \log(1/t), \quad t = \left(e^p + \|(-\triangle)^{s/2}f\|_{L^q(\mathbb{R}^n)}^{\left(1-\frac{n}{sq}\right)^{-1}}\right)^{-1}.$$

Then $t^{-n/(sq_1)} = (t^{1/\log t})^{n/s} = e^{n/s}$ and

$$\begin{aligned}& t^{1-\frac{n}{sq}}\|(-\triangle)^{s/2}f\|_{L^q(\mathbb{R}^n)} \\ &= \left(e^p + \|(-\triangle)^{s/2}f\|_{L^q(\mathbb{R}^n)}^{\left(1-\frac{n}{sq}\right)^{-1}}\right)^{-\left(1-\frac{n}{sq}\right)} \|(-\triangle)^{s/2}f\|_{L^q(\mathbb{R}^n)} \le 1.\end{aligned}$$

Since we can find constant $C_{n,s,p,q}$ such that $q_1 \le C_{n,s,p,q}\log\left(e+\|(-\triangle)^{s/2}f\|_{L^q(\mathbb{R}^n)}\right)$, (1.19) holds.

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