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HYBRID VISCOSITY-LIKE APPROXIMATION METHODS FOR GENERAL MONOTONE VARIATIONAL INEQUALITIES

Lu-Chuan Ceng, Q. H. Ansari and Juei-Ling Ho*

Abstract. In this paper, we introduce two implicit and explicit hybrid viscosity-like approximation methods for solving a general monotone variational inequality, which covers their monotone variational inequality with $C = H$ as a special case. We use the contractions to regularize the general monotone variational inequality, where the monotone operators are the generalized complements of nonexpansive mappings and the solutions are sought in the set of fixed points of another nonexpansive mapping. Such general monotone variational inequality includes some monotone inclusions and some convex optimization problems to be solved over the fixed point sets of nonexpansive mappings. Both implicit and explicit hybrid viscosity-like approximation methods are shown to be strongly convergent. In the meantime, these results are applied to deriving the strong convergence theorems for a general monotone variational inequality with minimization constraint. An application in hierarchical minimization is also included.

1. INTRODUCTION

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H and $\Gamma : C \to H$ be a nonlinear mapping. A variational inequality problem, denoted by $VI(\Gamma, C)$, is to find a point $x^* \in C$ such that

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*Corresponding author.

$$\langle \varGamma x^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

We say that the $VI(\varGamma, C)$ is monotone if the mapping $\varGamma$ is a monotone operator. Variational inequalities were initially studied by Stampacchia (See [24]) and ever since have been widely studied, since they cover as diverse disciplines as partial differential equations, optimal control, optimization, mathematical programming, mechanics, and finance. The reader is referred to [24-28, 35-40] and the references therein.

Very recently, Lu, Xu and Yin [23] were concerned with a special class of variational inequalities in which the mapping $\varGamma$ is the complement of a nonexpansive mapping and the constraint set is the set of fixed points of another nonexpansive mapping. That is, they considered the following type of monotone variational inequality problem of finding $x^* \in \mathrm{Fix}(T)$ such that

$$\langle (I - V)x^*, x - x^* \rangle \leq 0, \quad \forall x \in \mathrm{Fix}(T), \tag{1.2}$$

where $T, V : C \to C$ are nonexpansive mappings such that $\mathrm{Fix}(T) := \{x \in C : Tx = x\}$, is nonempty.

It is well-known that the $VI(\varGamma, C)$ is equivalent to the following fixed point equation

$$x^* = P_C(I - \lambda \varGamma)x^*, \tag{1.3}$$

where $\lambda > 0$ is an arbitrary fixed constant and P_C is the metric projection of H onto C.

It is also well-known that if $\varGamma$ is Lipschitzian and strongly monotone, then for small enough $\lambda > 0$, the mapping $P_C(I - \lambda \varGamma)$ is a contraction on C and so the sequence $\{x_n\}$ of Picard iterates, given by $x_{n+1} = P_C(I - \lambda \varGamma)x_n$ $(n \geq 0)$, converges strongly to the unique solution of the $VI(\varGamma, C)$.

In 2001, Yamada [1] introduced a hybrid steepest-descent method for solving the $VI(F, C)$ where $F : H \to H$ is Lipschitzian and strongly monotone and C is the fixed point set of a nonexpansive mapping $T : H \to H$, i.e., $C = \mathrm{Fix}(T)$. However, his method can not be applied to the variational inequality (1.2) since the mapping $I - V$ fails, in general, to be strongly monotone, though it is Lipschitzian. Therefore, other hybrid methods have to be sought.

In 2007, Mainge and Moudafi [2] introduced a hybrid viscosity approximation method for solving the variational inequality (1.2), which generates a sequence $\{x_n\}$ as follows:

$$x_{n+1} = \lambda_n f(x_n) + (1 - \lambda_n)[\alpha_n V x_n + (1 - \lambda_n) T x_n], \tag{1.4}$$

where the initial guess $x_0 \in C$, $f : C \to C$ is a contraction, and $\{\lambda_n\}$ and $\{\alpha_n\}$ are the sequences in $[0, 1]$ satisfying certain appropriate conditions.

Motivated by Mainge and Moudafi [2], Lu, Xu and Yin [23] investigated other hybrid viscosity approximation methods for solving the variational inequality (1.2). More precisely, assuming (1.2) is consistent and noticing the fact that if, for each $t \in (0,1),\ x_t \in C$ is a fixed point of the nonexpansive mapping $tV + (1-t)T$ then every weak accumulation point of $\{x_t\}$ as $t \to 0$ is a solution of the VI (1.2), they, upon the idea of regularization, introduced a new hybrid viscosity approximation method as follows:

$$z_{n+1} = \lambda_n[\alpha_n f(z_n) + (1-\alpha_n)Vz_n] + (1-\lambda_n)Tz_n, \tag{1.5}$$

where $\{\lambda_n\}$ and $\{\alpha_n\}$ are sequences in $(0,1)$, and $f : C \to C$ is a contraction. Their idea is to regularize the nonexpansive mapping V, instead of the nonexpansive mapping $W_t := tV + (1-t)T$ as done by Moudafi and Mainge [3]. Since Moudafi and Mainge's regularization depends upon t whereas theirs not, they derived their convergence result for the regularization under dramatically less restrictive conditions; as a matter of fact, the conditions (A1) and (A3) of Moudafi and Mainge [3] are completely removed. Moreover, they also applied both of their implicit and explicit schemes to solving a hierarchical minimization problem in a Hilbert space.

Inspired by the above research work going on in this field, we introduce two implicit and explicit hybrid viscosity-like approximation methods for solving a general monotone variational inequality, which covers the above monotone variational inequality (1.2) with $C = H$ as a special case. More precisely, let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa > 0,\ \eta > 0$, and $T : H \to H$ be nonexpansive with $\mathrm{Fix}(T) \neq \emptyset$. Given a nonexpansive mapping $V : H \to H$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma \leq \tau$, where $\tau = 1 - \sqrt{1-\mu(2\eta-\mu\kappa^2)}$. Consider the following general monotone variational inequality problem of finding $z^* \in \mathrm{Fix}(T)$ such that

$$\langle(\mu F - \gamma V)z^*, z - z^*\rangle \geq 0, \quad \forall z \in \mathrm{Fix}(T). \tag{1.6}$$

In particular, whenever $\mu = 2,\ F = \frac{1}{2}I,\ \gamma = \tau = 1$, the VI (1.6) reduces to the VI (1.2) with $C = H$.

We use the contractions to regularize the VI (1.6), where the monotone operator $\mu F - \gamma V$ are the generalized complement of nonexpansive mapping V and the solutions are sought in the fixed point set $\mathrm{Fix}(T)$ of another nonexpansive mapping T. Both implicit and explicit hybrid viscosity-like approximation methods are shown to be strongly convergent. In the meantime, we also apply both of our implicit and explicit hybrid viscosity-like approximation methods to solving a general monotone variational inequality with minimization constraint in a Hilbert space. An application in a hierarchical minimization problem is also included. All in all, the results presented in this paper extend Lu, Xu and Yin results to the case of the general monotone variational inequality when $C = H$.

2. PRELIMINARIES

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall the following concepts of mappings.

(i) A mapping $f : C \to C$ is a ρ-contraction if there is a constant $\rho \in [0, 1)$ such that

$$\|f(x) - f(y)\| \leq \rho\|x - y\|, \quad \forall x, y \in C.$$

(ii) A mapping $T : C \to C$ is nonexpansive provided

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

(iii) A mapping $\Gamma : C \to H$ is

(a) monotone if

$$\langle \Gamma x - \Gamma y, x - y\rangle \geq 0, \quad \forall x, y \in C;$$

(b) strictly monotone if

$$\langle \Gamma x - \Gamma y, x - y\rangle > 0, \quad \forall x, y \in C,\ x \neq y;$$

(c) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle \Gamma x - \Gamma y, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C.$$

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C).$$

Throughout this paper, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. The following lemmas are useful for our paper.

Lemma 2.1. *Given $x \in H$ and $z \in C$, there are the following statements:*

(i) *$z = P_C x$ if and only if there holds the relation:*

$$\langle x - z, y - z\rangle \leq 0, \quad \forall y \in C;$$

(ii) *$z = P_C x$ if and only if there holds the relation:*

$$\|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2, \quad \forall y \in C;$$

(iii) *there holds the relation*

$$\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H.$$

Consequently, P_C is monotone and nonexpansive.

The following lemma is not hard to prove.

Lemma 2.2 (cf. [4]). *Let $f : C \to C$ be a ρ-contraction with $\rho \in [0, 1)$ and $T : C \to C$ be a nonexpansive mapping. Then*

(i) *$I - f$ is $(1-\rho)$-strongly monotone:*

$$\langle (I-f)x - (I-f)y, x-y\rangle \geq (1-\rho)\|x-y\|^2, \quad \forall x, y \in C;$$

(ii) *$I - T$ is monotone:*

$$\langle (I-T)x - (I-T)y, x-y\rangle \geq 0, \quad \forall x, y \in C.$$

Lemma 2.3 (Demiclosedness Principle (cf. [5])). *Let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x = y$.*

The following fact is straightforward (but useful).

Lemma 2.4. *There holds the following inequality in an inner product space X:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in X.$$

The following lemma plays a key role in proving strong convergence of our algorithms.

Lemma 2.5 ([6, Lemma 2.1]). *Let $\{s_n\}$ be a sequence of nonnegative numbers satisfying the condition*

$$s_{n+1} \leq (1-\gamma_n)s_n + \gamma_n\delta_n, \quad \forall n \geq 0,$$

where $\{\gamma_n\}, \{\delta_n\}$ are sequences of real numbers such that

(i) *$\{\gamma_n\} \subset [0, 1]$ and $\sum_{n=0}^{\infty} \gamma_n = \infty$, or equivalently,*

$$\prod_{n=0}^{\infty}(1-\gamma_n) := \lim_{n\to\infty}\prod_{k=0}^{n}(1-\gamma_k) = 0;$$

(ii) *$\limsup_{n\to\infty} \delta_n \leq 0$, or*

(ii') *$\sum_{n=0}^{\infty} \gamma_n\delta_n$ is convergent.*
Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.6 ([6, Lemma 3.1]). *Let λ be a number in $(0, 1]$ and let $\mu > 0$. Let $F : H \to H$ be an operator on a Hilbert space H such that, for some* *constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone. Associating with a nonexpansive mapping $T : H \to H$, define the mapping $T^{\lambda} : H \to H$ by*

$$T^{\lambda}x := Tx - \lambda\mu F(Tx), \quad \forall x \in H.$$

Then T^{λ} is a contraction provided $\mu < 2\eta/\kappa^2$, that is,

$$\|T^{\lambda}x - T^{\lambda}y\| \leq (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in H,$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$.

Remark 2.1. Put $F = \frac{1}{2}I$, where I is the identity operator of H. Then we have $\mu < 2\eta/\kappa^2 = 4$. Also, put $\mu = 2$. Then it is easy to see that $\kappa = \eta = \frac{1}{2}$ and

$$\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} = 1 - \sqrt{1 - 2(2 \cdot \frac{1}{2} - 2(\frac{1}{2})^2)} = 1.$$

In particular, whenever $\lambda > 0$, we have $T^{\lambda}x := Tx - \lambda\mu F(Tx) = (1 - \lambda)Tx$.

3. Implicit Hybrid Viscosity-like Approximation Method

Suppose $F : H \to H$ is a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa > 0,\ \eta > 0$. Suppose $T : H \to H$ is nonexpansive with $\mathrm{Fix}(T) \neq \emptyset$. Consider the variational inequality problem of finding $x^* \in \mathrm{Fix}(T)$ such that

$$\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T). \tag{3.1}$$

Yamada [1] introduced the following hybrid steepest-descent method for solving the variational inequality (3.1), which generates a sequence $\{x_n\}$ via the following iterative algorithm:

$$x_{n+1} = Tx_n - \lambda_{n+1}\mu F(Tx_n), \quad \forall n \geq 0, \tag{3.2}$$

where $0 < \mu < 2\eta/\kappa^2$, the initial guess $x_0 \in H$ is arbitrary and the sequence $\{\lambda_n\}$ in $(0, 1)$ satisfies the conditions:

$$\lambda_n \to 0, \quad \sum_{n=0}^{\infty} \lambda_n = \infty \quad \text{and} \quad \sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$$

A key fact in Yamada's argument is that, for small enough $\lambda > 0$, the mapping

$$T^{\lambda}x := Tx - \lambda\mu F(Tx), \quad \forall x \in H$$

is a contraction, due to the κ-Lipschitz continuity and η-strong monotonicity of F.

Now given a nonexpansive mapping $V : H \to H$ and $t \in (0,1)$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma \le \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Then we consider a mapping W_t on H defined by

$$W_t x = t\gamma V x + (I - t\mu F)Tx, \quad \forall x \in H.$$

It is easy to see that W_t is a nonexpansive mapping. Indeed, we have

$$\begin{aligned}\|W_t x - W_t y\| &\le t\gamma\|Vx - Vy\| + \|(I - \mu t F)Tx - (I - \mu t F)Ty\| \\ &\le t\gamma\|x - y\| + (1 - t\tau)\|x - y\| \\ &= (1 - t(\tau - \gamma))\|x - y\|.\end{aligned}$$

Since $0 < \gamma \le \tau$, it is known that W_t is nonexpansive on H.

In our case, we consider the variational inequality problem of finding $z^* \in \mathrm{Fix}(T)$ such that

$$\langle (\mu F - \gamma V)z^*, z - z^*\rangle \ge 0, \quad \forall z \in \mathrm{Fix}(T). \tag{3.3}$$

where the mappings V, T, F and the parameters μ, γ are the same as above. In particular, whenever $\mu = 2,\ F = \frac{1}{2}I,\ \gamma = \tau = 1$, the VI (3.3) reduces to the following variational inequality problem of finding $z^* \in \mathrm{Fix}(T)$ such that

$$\langle (I - V)z^*, z - z^*\rangle \ge 0, \quad \forall z \in \mathrm{Fix}(T), \tag{3.3$'$}$$

which was considered and studied in Mainge and Moudafi [2] and Lu, Xu and Yin [23].

We remark that if we take $\varGamma = \mu F - \gamma V$ then the VI (3.3) is equivalent to the following variational inequality problem of finding $z^* \in \mathrm{Fix}(T)$ such that

$$\langle \varGamma z^*, z - z^*\rangle \ge 0, \quad \forall z \in \mathrm{Fix}(T).$$

Because it is possible that $\varGamma = \mu F - \gamma V$ is not strongly monotone, Yamada's argument fails to work. As a matter of fact, the variational inequality (3.3) is, in general, ill-posed and thus regularization is need. Inspired by Lu, Xu and Yin [23], our idea remains to regularize the nonexpansive mapping V by contractions, and is based on the following proposition.

Proposition 3.1. *Let S denote the solution set of the VI (3.3). Let $t \in (0,1)$ and let z_t be a fixed point of the mapping $W_t = t\gamma V + (I - t\mu F)T$; namely, $z_t = t\gamma V z_t + (I - t\mu F)Tz_t$. Assume $\{z_t\}$ remains bounded as $t \to 0$.*

- (i) *The solution set S of the variational inequality (3.3) is nonempty and each weak limit point (as $t \to 0$) of $\{z_t\}$ solves the VI (3.3).*
- (ii) *If $\mu F - \gamma V$ is strictly monotone, then the net $\{z_t\}$ converges weakly to the (unique) solution of the VI (3.3).*

(iii) *If $\mu F - \gamma V$ is strongly monotone (e.g., $\mu\eta > \gamma$), then the net $\{z_t\}$ converges strongly to the solution of the VI (3.3).*

To prove part (i) of Proposition 3.1, we need the following useful lemma.

Lemma 3.2. ([23]). *Assume that $\Gamma : C \to H$ is monotone and weakly continuous along segments (i.e., $\Gamma(x + ty) \rightharpoonup \Gamma x$ as $t \to 0$). Then the VI (1.1) is equivalent to the dual variational inequality problem of finding a point $x^* \in C$ such that*

$$\langle \Gamma x, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{3.4}$$

Proof of Proposition 3.1. Let W be the set of all weak accumulation points of $\{z_t\}$ as $t \to 0$; that is,

$$W = \{z : z_{t_n} \rightharpoonup z \text{ for some sequence } \{t_n\} \text{ in } (0,1) \text{ such that } t_n \to 0\}.$$

Then $W \neq \emptyset$ since $\{z_t\}$ is bounded.

To prove (i), we notice that the boundedness of $\{z_t\}$ implies that $W \neq \emptyset$ and

$$\|z_t - Tz_t\| = t\|\gamma V z_t - \mu F(Tz_t)\| \to 0 \quad \text{as } t \to 0.$$

It thus follows from Lemma 2.3 that $W \subset \mathrm{Fix}(T)$.

Observe that

$$\begin{aligned}
\langle F(z_t) - F(\hat{x}), z_t - \hat{x} \rangle &= \langle F(Tz_t) - F(\hat{x}), z_t - \hat{x} \rangle + \langle F(z_t) - F(Tz_t), z_t - \hat{x} \rangle \\
&= \langle F(Tz_t) - F(\hat{x}), Tz_t - \hat{x} \rangle + \langle F(Tz_t) - F(\hat{x}), z_t - Tz_t \rangle \\
&\quad + \langle F(z_t) - F(Tz_t), z_t - \hat{x} \rangle \\
&\leq \langle F(Tz_t) - F(\hat{x}), Tz_t - \hat{x} \rangle + 2\kappa \|z_t - \hat{x}\| \|z_t - Tz_t\|.
\end{aligned}$$

Hence, utilizing Lemmas 2.4 and 2.6, we deduce that, for any $\hat{x} \in \mathrm{Fix}(T)$,

$$\begin{aligned}
\|z_t - \hat{x}\|^2 &= \|(I - t\mu F)Tz_t - (I - t\mu F)\hat{x} + t(\gamma V z_t - \mu F(\hat{x}))\|^2 \\
&\leq \|(I - t\mu F)Tz_t - (I - t\mu F)\hat{x}\|^2 + 2t\langle \gamma V z_t - \mu F(\hat{x}), z_t - \hat{x} \rangle \\
&= \|Tz_t - \hat{x}\|^2 - 2t\mu \langle F(Tz_t) - F(\hat{x}), Tz_t - \hat{x} \rangle + t^2\mu^2 \|F(Tz_t) - F(\hat{x})\|^2 \\
&\quad + 2t(\langle \gamma V z_t - \mu F(z_t), z_t - \hat{x} \rangle + \mu \langle F(z_t) - F(\hat{x}), z_t - \hat{x} \rangle) \\
&\leq \|z_t - \hat{x}\|^2 - 2t\mu \langle F(Tz_t) - F(\hat{x}), Tz_t - \hat{x} \rangle + t^2\mu^2\kappa^2 \|z_t - \hat{x}\|^2 \\
&\quad + 2t(\langle \gamma V z_t - \mu F(z_t), z_t - \hat{x} \rangle + \mu \langle F(Tz_t) - F(\hat{x}), Tz_t - \hat{x} \rangle \\
&\quad + 2\mu\kappa \|z_t - \hat{x}\| \|z_t - Tz_t\|) \\
&= \|z_t - \hat{x}\|^2 + t^2\mu^2\kappa^2 \|z_t - \hat{x}\|^2 + 2t\langle \gamma V z_t - \mu F(z_t), z_t - \hat{x} \rangle \\
&\quad + 4t\mu\kappa \|z_t - \hat{x}\| \|z_t - Tz_t\| \\
&= (1 + t^2\mu^2\kappa^2)\|z_t - \hat{x}\|^2 + 2t\langle \gamma V z_t - \mu F(z_t), z_t - \hat{x} \rangle \\
&\quad + 4t\mu\kappa \|z_t - \hat{x}\| \|z_t - Tz_t\|.
\end{aligned}$$

It follows that

$$(3.5) \quad \langle \mu F(z_t) - \gamma V z_t, z_t - \hat{x} \rangle \leq \frac{t\mu^2\kappa^2}{2} \|z_t - \hat{x}\|^2 + 2\mu\kappa \|z_t - \hat{x}\| \|z_t - Tz_t\|.$$

Note that $0 < \gamma \leq \tau$ and

$$\begin{aligned} \mu\eta \geq \tau \ &\Leftrightarrow \ \mu\eta \geq 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \\ &\Leftrightarrow \ \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \geq 1 - \mu\eta \\ &\Leftrightarrow \ 1 - 2\mu\eta + \mu^2\kappa^2 \geq 1 - 2\mu\eta + \mu^2\eta^2 \\ &\Leftrightarrow \ \kappa^2 \geq \eta^2 \\ &\Leftrightarrow \ \kappa \geq \eta. \end{aligned}$$

It is clear that

$$\langle (\mu F - \gamma V)x - (\mu F - \gamma V)y, x - y \rangle \geq (\mu\eta - \gamma)\|x - y\|^2, \quad \forall x, y \in H.$$

Hence it follows from $0 < \gamma \leq \tau \leq \mu\eta$ that $\mu F - \gamma V$ is monotone. Thus we have

$$\langle (\mu F - \gamma V)z_t, z_t - \hat{x} \rangle \geq \langle (\mu F - \gamma V)\hat{x}, z_t - \hat{x} \rangle.$$

This together with (3.5) implies that

$$(3.6) \quad \langle (\mu F - \gamma V)\hat{x}, z_t - \hat{x} \rangle \leq \frac{t\mu^2\kappa^2}{2} \|z_t - \hat{x}\|^2 + 2\mu\kappa \|z_t - \hat{x}\| \|z_t - Tz_t\|.$$

Now if $\tilde{x} \in W \subset \mathrm{Fix}(T)$ and if $t_n \to 0$ is such that $x_{t_n} \rightharpoonup \tilde{x}$, then we conclude from (3.6) and $\|z_{t_n} - Tz_{t_n}\| \to 0$ that

$$(3.7) \quad \langle (\mu F - \gamma V)\hat{x}, \tilde{x} - \hat{x} \rangle \leq 0, \quad \forall \hat{x} \in \mathrm{Fix}(T).$$

By Lemma 3.2, we get $\tilde{x} \in S$.

To see (ii), we assume that $\{t'_n\}$ is another null sequence in $(0, 1)$ such that $x_{t'_n} \rightharpoonup \hat{x}$. Then $\hat{x} \in S$ and interchange $\tilde{x}$ and $\hat{x}$ in (3.7) to get

$$(3.8) \quad \langle (\mu F - \gamma V)\tilde{x}, \hat{x} - \hat{x} \rangle \leq 0.$$

Adding up (3.7) and (3.8) yields

$$\langle (\mu F - \gamma V)\tilde{x} - (\mu F - \gamma V)\hat{x}, \tilde{x} - \hat{x} \rangle \leq 0.$$

So the strict monotonicity of $\mu F - \gamma V$ implies that $\tilde{x} = \hat{x}$ and $\{z_t\}$ converges weakly.

Finally to prove (iii), we observe that the strong monotonicity of $\mu F - \gamma V$ and (3.5) imply that

$$\begin{aligned} &\alpha\|z_t - \hat{x}\|^2 + \langle(\mu F - \gamma V)\hat{x}, z_t - \hat{x}\rangle \\ &\le \frac{t\mu^2\kappa^2}{2}\|z_t - \hat{x}\|^2 + 2\mu\kappa\|z_t - \hat{x}\|\|z_t - Tz_t\|, \quad \hat{x} \in \mathrm{Fix}(T) \end{aligned} \tag{3.9}$$

where $\alpha > 0$ is the strong monotonicity coefficient of $\mu F - \gamma V$; that is,

$$\langle(\mu F - \gamma V)x - (\mu F - \gamma V)y, x - y\rangle \ge \alpha\|x - y\|^2, \quad \forall x, y \in H.$$

By part (ii), we have $z_t \rightharpoonup \tilde{z} \in S$, where $\tilde{z}$ is the unique solution of the VI (3.3). Replacing the $\hat{x}$ in (3.9) with $\tilde{z}$ and then letting $t \to 0$, we obtain that $z_t \to \tilde{z}$. ■

Proposition 3.1 says that every weak cluster point of $\{z_t\}$ solves the VI (3.3). This motivates us to regularize V in the following manner.

Let

$$V_t = tf + (1 - t)V,$$

where $f : H \to H$ is a contraction with coefficient $\rho \in [0, 1)$ and $t \in (0, 1)$. Then we consider the contraction $x \mapsto s\gamma V_t x + (I - s\mu F)Tx$, where $s \in (0, 1)$. Denote by $z_{s,t}$ the (unique) fixed point of this contraction. That is, $z_{s,t} \in H$ is the only solution of the fixed point equation

$$\begin{aligned} z_{s,t} &= s\gamma V_t z_{s,t} + (I - s\mu F)Tz_{s,t} \\ &= s\gamma[tf(z_{s,t}) + (1 - t)Vz_{s,t}] + (I - s\mu F)Tz_{s,t}. \end{aligned} \tag{3.10}$$

Indeed, in terms of Lemma 2.6 we obtain that for each $x, y \in H$

$$\begin{aligned} &\|[s\gamma V_t + (I - s\mu F)T]x - [s\gamma V_t + (I - s\mu F)T]y\| \\ &\le s\gamma\|V_t x - V_t y\| + \|(I - s\mu F)Tx - (I - s\mu F)Ty\| \\ &= s\gamma\|tf(x) + (1 - t)Vx - tf(y) - (1 - t)Vy\| + (1 - s\tau)\|x - y\| \\ &\le s\gamma[t\|f(x) - f(y)\| + (1 - t)\|Vx - Vy\|] + (1 - s\tau)\|x - y\| \\ &\le s\gamma[t\rho\|x - y\| + (1 - t)\|x - y\|] + (1 - s\tau)\|x - y\| \\ &= s\gamma(1 - t(1 - \rho))\|x - y\| + (1 - s\tau)\|x - y\| \\ &= \{1 - s[\tau - \gamma(1 - t(1 - \rho))]\}\|x - y\|. \end{aligned}$$

Since $0 < \gamma \le \tau,\ 0 \le \rho < 1$ and $0 < s, t < 1$, we have

$$\gamma(1 - t(1 - \rho)) < \gamma \le \tau,$$

and hence

$$0 < 1 - s[\tau - \gamma(1 - t(1 - \rho))] < 1.$$

This implies that $s\gamma V_t + (I - s\mu F)T$ is a contraction on H. Thus Banach's contraction principle guarantee's that there exists a unique $z_{s,t} \in H$ such that (3.10) holds.

Theorem 3.3. *Suppose $S \neq \emptyset$. Then the iterated $\lim_{t\to 0}\lim_{s\to 0} z_{s,t}$ exists in the norm topology and is the (unique) solution z^* of the VIP of finding $z^* \in S$ such that*

$$\langle(\mu F - \gamma f)z^*, z - z^*\rangle \geq 0, \quad \forall z \in S. \tag{3.11}$$

Equivalently, z^* is the unique fixed point of the contraction $P_S(I - \mu F + \gamma f)$; that is, $z^* = P_S(I - \mu F + \gamma f)z^*$.

Proof. First, let us show that the mapping $\mu F - \gamma V_t$ is strongly monotone. Indeed, observe that for each $x, y \in H$

$$\begin{aligned}
&\langle(\mu F - \gamma V_t)x - (\mu F - \gamma V_t)y, x - y\rangle \\
&= \mu\langle F(x) - F(y), x - y\rangle - \gamma\langle V_t x - V_t y, x - y\rangle \\
&\geq \mu\eta\|x - y\|^2 - \gamma(1 - t(1 - \rho))\|x - y\|^2 \\
&= [\mu\eta - \gamma(1 - t(1 - \rho))]\|x - y\|^2.
\end{aligned}$$

Since $0 < \gamma \leq \tau \leq \mu\eta,\ 0 \leq \rho < 1$ and $0 < t < 1$, we have

$$\gamma(1 - t(1 - \rho)) < \gamma \leq \tau \leq \mu\eta$$

and hence

$$\mu\eta - \gamma(1 - t(1 - \rho)) > 0.$$

This shows that $\mu F - \gamma V_t$ is strongly monotone.

According to Proposition 3.1 (iii), we have that, for each fixed $t \in (0, 1)$,

$$\|\cdot\| - \lim_{s\to 0} z_{s,t} =: z_t$$

exists and solves the variational inequality problem of finding $z_t \in \mathrm{Fix}(T)$ such that

$$\langle(\mu F - \gamma V_t)z_t, y - z_t\rangle \geq 0, \quad \forall y \in \mathrm{Fix}(T). \tag{3.12}$$

Equivalently, $z_t \in \mathrm{Fix}(T)$ satisfies

$$t\langle(\mu F - \gamma f)z_t, y - z_t\rangle + (1 - t)\langle(\mu F - \gamma V)z_t, y - z_t\rangle \geq 0, \quad \forall y \in \mathrm{Fix}(T). \tag{3.13}$$

In particular, for any $y^* \in S$,

$$t\langle(\mu F - \gamma f)z_t, y^* - z_t\rangle + (1 - t)\langle(\mu F - \gamma V)z_t, y^* - z_t\rangle \geq 0. \tag{3.14}$$

However, since $y^* \in S$, we have

$$\langle (\mu F - \gamma V)z_t, y^* - z_t\rangle \le \langle (\mu F - \gamma V)y^*, y^* - z_t\rangle \le 0. \tag{3.15}$$

It follows from (3.14) that

$$\langle (\mu F - \gamma f)z_t, y^* - z_t\rangle \ge 0, \quad \forall y^* \in S. \tag{3.16}$$

It turns out that

$$\begin{aligned}
\mu\eta\|z_t - y^*\|^2 &\le \mu\langle F(z_t) - F(y^*), z_t - y^*\rangle \\
&\le \langle \mu F(y^*) - \gamma f(z_t), y^* - z_t\rangle \\
&= \langle (\mu F - \gamma f)y^*, y^* - z_t\rangle + \gamma\langle f(y^*) - f(z_t), y^* - z_t\rangle \\
&\le \langle (\mu F - \gamma f)y^*, y^* - z_t\rangle + \gamma\rho\|y^* - z_t\|^2.
\end{aligned}$$

Therefore,

$$\|z_t - y^*\|^2 \le \frac{1}{\mu\eta - \gamma\rho}\langle (\mu F - \gamma f)y^*, y^* - z_t\rangle. \tag{3.17}$$

In particular,

$$\|z_t - y^*\| \le \frac{1}{\mu\eta - \gamma\rho}\|\mu F - \gamma f)y^*\|$$

and $\{z_t\}$ is thus bounded.

Next let us show that $\omega_w(\{z_t\}) \subset S$; that is, if $\{t_j\}$ is a null sequence in $(0, 1)$ such that $z_{t_j} \rightharpoonup \tilde{z}$ as $j \to \infty$, then $\tilde{z} \in S$. To see this, we combine (3.14) and (3.15) to obtain, for all $y^* \in S$,

$$\langle (\mu F - \gamma V)y^*, y^* - z_t\rangle \ge \frac{t}{1-t}\langle (\mu F - \gamma f)z_t, z_t - y^*\rangle.$$

Since $\{z_t\}$ is bounded, we may let $t = t_j \to 0$ (as $j \to \infty$) in the last inequality to get

$$\langle (\mu F - \gamma V)y^*, y^* - \tilde{z}\rangle \ge 0, \quad \forall y^* \in S.$$

This implies that $\tilde{z} \in S$.

Now from (3.17) we have

$$\|z_{t_j} - \tilde{z}\|^2 \le \frac{1}{\mu\eta - \gamma\rho}\langle (\mu F - \gamma f)\tilde{z}, \tilde{z} - z_{t_j}\rangle.$$

Taking the limit as $j \to \infty$, we see that $z_{t_j} \to \tilde{z}$. Moreover, letting $t = t_j \to 0$ in (3.16), we get

$$\langle (\mu F - \gamma f)\tilde{z}, y^* - \tilde{z}\rangle \ge 0, \quad \forall y^* \in S.$$

This shows that $\tilde{z} \in S$ solves the VI (3.11). By uniqueness, we have $\tilde{z} = z^*$. Therefore, $z_t \to z^*$ as $t \to 0$. ■

Remark 3.4. Although, using a diagonal argument, for any given null sequence $\{s_n\}$ in $(0,1)$, we can find another null sequence $\{t_n\}$ in $(0,1)$ such that $z_{s_n,t_n} \to z^*$, we wonder whether or not the limit of $\{z_{s,t}\}$ exists in norm as $(s,t) \to (0,0)$ jointly.

Remark 3.5. Moudafi and Mainge [3] studied the VI (3.3)′ by regularizing the mapping $W_t := tV + (1-t)T$ and defined $x_{s,t}$ as the unique solution of the fixed point equation:

$$x_{s,t} = sf(x_{s,t}) + (1-s)[tVx_{s,t} + (1-t)Tx_{s,t}]. \tag{3.18}$$

Since Moudafi and Mainge's regularization depends on t, the convergence of the scheme (3.18) is very complicated. But, Lu, Xu and Yin [23] studied the VI (3.3)′ by regularizing the mapping V and defined $z_{s,t}$ as the unique solution of the fixed point equation:

$$z_{s,t} = s[tf(z_{s,t}) + (1-t)Vz_{s,t}] + (1-s)Tz_{s,t}. \tag{3.18$'$}$$

Thus, the convergence of the scheme (3.18)′ is very simple. In the meantime, we investigate the VI (3.3) by regularizing the mapping V and define $x_{s,t}$ as the unique solution of the fixed point equation:

$$z_{s,t} = s\gamma[tf(z_{s,t}) + (1-t)Vz_{s,t}] + (I - s\mu F)Tz_{s,t}.$$

Whenever $\gamma = 1,\ \mu = 2$ and $F = \frac{1}{2}I$, our scheme reduces to the scheme (3.18)′. In addition, the convergence of our scheme is very simple as well. Indeed, Moudafi and Mainge [3] (see also [8] for improvements) proved the strong convergence of the iterated $\lim_{s\to 0}\lim_{t\to 0} x_{s,t}$ under much more restrictive assumptions, two of which are ((A1) and (A3) in [3])

(a) For each $t \in (0,1)$, the fixed point set of $W_t,\ \mathrm{Fix}(W_t)$, is nonempty and the set $\{\mathrm{Fix}(W_t) : 0 < t < 1\}$ is bounded;

(b) $\emptyset \neq S \subset \|\cdot\| - \liminf_{t\to 0} \mathrm{Fix}(W_t) := \{z : \exists z_t \in \mathrm{Fix}(W_t) \text{ such that } z_t \to z \text{ in norm as } t \to 0\}$.

In our regularization and Lu, Xu and Yin one, these conditions (1) and (2) have completely been removed.

4. Explicit Hybrid Viscosity-like Approximation Method

Our variational inequality (3.3) involves two nonexpansive mappings T and V. Our explicit hybrid viscosity-like approximation method is motivated by our implicit hybrid viscosity-like approximation method investigated in the last section and the recent investigation on iterative methods for nonexpansive mappings (see more details in [9-12,7,13-16,6,17-19,4,29-34]).

Our explicit iterative scheme generates a sequence $\{z_n\}$ from an arbitrary initial guess $z_0 \in C$ and via the recursive formula:

$$z_{n+1} = \lambda_n \gamma V_n z_n + (I - \lambda_n \mu F) T z_n, \tag{4.1}$$

where the mappings V, T, F and the parameters μ, γ are the same as in Section 3, $\{\lambda_n\}$ and $\{\alpha_n\}$ are sequences in $(0, 1)$, and $V_n = \alpha_n f + (1 - \alpha_n)V$ with f being a contraction of C with coefficient $\rho \in [0, 1)$. In particular, whenever $\mu = 2,\ F = \frac{1}{2}I,\ \gamma = 1$, the scheme (4.1) reduces to the following one

$$z_{n+1} = \lambda_n V_n z_n + (1 - \lambda_n) T z_n. \tag{4.1$'$}$$

Such a scheme was introduced and studied by Lu, Xu and Yin [23]. Similar iterative methods can be found in [2] (see also [20]). The convergence of the scheme (4.1) is not easy to discuss. However, we have the following result.

Theorem 4.1. *Suppose the solution set S of the VI (3.3) is nonempty. Suppose the following conditions hold:*

(*i*) $\lambda_n \to 0$ *and* $\alpha_n \to 0$*;*
(*ii*) $\sum_{n=1}^{\infty} \alpha_n \lambda_n = \infty$*;*
(*iii*) $|\alpha_n \lambda_n - \alpha_{n-1} \lambda_{n-1}| / \alpha_n \lambda_n^2 \to 0$*;*
(*iv*) $|\lambda_n - \lambda_{n-1}| / \alpha_n \lambda_n^2 \to 0$*;*
(*v*) *there are constants* $\theta > 0$ *and* $\bar{k} > 0$ *satisfying* $\|x - Tx\| \geq \bar{k}[d(x, \mathrm{Fix}(T))]^\theta$ *for* $x \in H$ *and for some* $\theta > 0$*;*
(*vi*) $\lambda_n^{1/\theta} / \alpha_n \to 0$.

Suppose also that the sequence $\{z_n\}$ defined by the algorithm (4.1) is bounded. Then $\{z_n\}$ converges in norm to the unique fixed point z^* of the contraction $P_S(I - \mu F + \gamma f)$, or the unique solution of the variational inequality (3.11).

Proof. Let z^* be the unique fixed point of the contraction $P_S(I - \mu F + \gamma f)$; that is, z^* is the unique solution of the variational inequality (3.11).

We divide our proof into the following steps

(1) $\|z_{n+1} - z_n\| \to 0$.
(2) $\|z_n - Tz_n\| \to 0$; hence $\omega_w(\{z_n\}) \subset \mathrm{Fix}(T)$.
(3) $\|z_{n+1} - z_n\| / \lambda_n \to 0$.
(4) $\limsup_{n\to\infty} \langle (\gamma f - \mu F) z^*, z_n - z^* \rangle \leq 0$.
(5) $\limsup_{n\to\infty} \alpha_n^{-1} \langle (\gamma V - \mu F) z^*, z_{n+1} - z^* \rangle \leq 0$.
(6) $z_n \to z^*$.

Proof of (1). We compute

$$\begin{aligned}
& z_{n+1} - z_n \\
&= \lambda_n \gamma V_n z_n + (I - \lambda_n \mu F) T z_n - [\lambda_{n-1} \gamma V_{n-1} z_{n-1} \\
&\quad + (I - \lambda_{n-1} \mu F) T z_{n-1}] \\
&= \alpha_n \lambda_n \gamma [f(z_n) - f(z_{n-1})] + \lambda_n \gamma (1 - \alpha_n)(V z_n - V z_{n-1}) \\
&\quad + [(I - \lambda_n \mu F) T z_n - (I - \lambda_n \mu F) T z_{n-1}] \\
&\quad + (\alpha_n \lambda_n - \alpha_{n-1} \lambda_{n-1}) \gamma [f(z_{n-1}) - V z_{n-1}] \\
&\quad + (\lambda_n - \lambda_{n-1}) [\gamma V z_{n-1} - \mu F(T z_{n-1})].
\end{aligned} \tag{4.2}$$

Since $\{z_n\}$ is bounded, we can find a constant $M > 0$ satisfying

$$M \geq \sup_{n \geq 0} \{\gamma \|f(z_n) - V z_n\|, \|\gamma V z_n - \mu F(T z_n)\|\}.$$

Now utilizing Lemma 2.6 we conclude from (4.2) and $0 < \gamma \leq \tau$ that

$$\begin{aligned}
& \|z_{n+1} - z_n\| \\
&\leq \alpha_n \lambda_n \gamma \|f(z_n) - f(z_{n-1})\| + \lambda_n \gamma (1 - \alpha_n) \|V z_n - V z_{n-1}\| \\
&\quad + \|(I - \lambda_n \mu F) T z_n - (I - \lambda_n \mu F) T z_{n-1}\| \\
&\quad + |\alpha_n \lambda_n - \alpha_{n-1} \lambda_{n-1}| \gamma \|f(z_{n-1}) - V z_{n-1}\| \\
&\quad + |\lambda_n - \lambda_{n-1}| \|\gamma V z_{n-1} - \mu F(T z_{n-1})\| \\
&\leq \alpha_n \lambda_n \gamma \rho \|z_n - z_{n-1}\| + \lambda_n \gamma (1 - \alpha_n) \|z_n - z_{n-1}\| \\
&\quad + (1 - \lambda_n \tau) \|z_n - z_{n-1}\| \\
&\quad + |\alpha_n \lambda_n - \alpha_{n-1} \lambda_{n-1}| M + |\lambda_n - \lambda_{n-1}| M \\
&= [1 - \lambda_n (\tau - \gamma + \alpha_n \gamma (1 - \rho))] \|z_n - z_{n-1}\| \\
&\quad + M(|\alpha_n \lambda_n - \alpha_{n-1} \lambda_{n-1}| + |\lambda_n - \lambda_{n-1}|) \\
&\leq (1 - (1 - \rho) \alpha_n \lambda_n \gamma) \|z_n - z_{n-1}\| \\
&\quad + M(|\alpha_n \lambda_n - \alpha_{n-1} \lambda_{n-1}| + |\lambda_n - \lambda_{n-1}|) \\
&= (1 - \alpha_n \lambda_n \gamma (1 - \rho)) \|z_n - z_{n-1}\| \\
&\quad + \alpha_n \lambda_n \gamma (1 - \rho) \cdot M(|\alpha_n \lambda_n - \alpha_{n-1} \lambda_{n-1}| \\
&\quad + |\lambda_n - \lambda_{n-1}|) / \alpha_n \lambda_n \gamma (1 - \rho).
\end{aligned} \tag{4.3}$$

Conditions (iii) and (iv) imply that

$$\lim_{n \to \infty} M(|\alpha_n \lambda_n - \alpha_{n-1} \lambda_{n-1}| + |\lambda_n - \lambda_{n-1}|) / \alpha_n \lambda_n \gamma (1 - \rho) = 0.$$

Hence, applying Lemma 2.5 to (4.3) yields $\|z_{n+1} - z_n\| \to 0$.

Proof of (2). By the definition of algorithm (4.1), we get immediately

$$\|z_{n+1} - Tz_n\| = \lambda_n \|\gamma V_n z_n - \mu F(Tz_n)\| \to 0 \quad \text{as } \{z_n\} \text{ is bounded}.$$

Hence, $\|z_n - Tz_n\| \le \|z_n - z_{n+1}\| + \|z_{n+1} - Tz_n\| \to 0$. From the demiclosedness of $I - T$ (Lemma 2.3) it follows that $\omega_w(\{z_n\}) \subset \mathrm{Fix}(T)$.

Proof of (3). Utilizing (iv) we know that there exists an integer $n_0 \ge 1$ such that

$$\frac{|\lambda_n - \lambda_{n-1}|}{\alpha_n \lambda_n^2 \gamma(1-\rho)} < \frac{1}{2}, \quad \forall n \ge n_0.$$

Hence from (4.3) it follows that for all $n \ge n_0$

$$\begin{aligned}
&\frac{\|z_{n+1} - z_n\|}{\lambda_n} \\
&= (1 - \alpha_n\lambda_n\gamma(1-\rho))\frac{\|z_n - z_{n-1}\|}{\lambda_{n-1}} \\
&\quad + (1 - \alpha_n\lambda_n\gamma(1-\rho))\|z_n - z_{n-1}\|(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n-1}}) \\
&\quad + M\frac{|\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}| + |\lambda_n - \lambda_{n-1}|}{\lambda_n} \\
&\le (1 - \alpha_n\lambda_n\gamma(1-\rho))\frac{\|z_n - z_{n-1}\|}{\lambda_{n-1}} + \|z_n - z_{n-1}\|\left|\frac{1}{\lambda_n} - \frac{1}{\lambda_{n-1}}\right| \\
&\quad + M\frac{|\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}| + |\lambda_n - \lambda_{n-1}|}{\lambda_n} \\
&= (1 - \alpha_n\lambda_n\gamma(1-\rho))\frac{\|z_n - z_{n-1}\|}{\lambda_{n-1}} \\
&\quad + \alpha_n\lambda_n\gamma(1-\rho) \cdot \frac{\|z_n - z_{n-1}\|}{\lambda_{n-1}} \frac{|\lambda_n - \lambda_{n-1}|}{\alpha_n\lambda_n^2\gamma(1-\rho)} \\
&\quad + M\frac{|\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}| + |\lambda_n - \lambda_{n-1}|}{\lambda_n} \\
&\le (1 - \alpha_n\lambda_n\gamma(1-\rho)/2)\frac{\|z_n - z_{n-1}\|}{\lambda_{n-1}} \\
&\quad + M\frac{|\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}| + |\lambda_n - \lambda_{n-1}|}{\lambda_n} \\
&= (1 - \alpha_n\lambda_n\gamma(1-\rho)/2)\frac{\|z_n - z_{n-1}\|}{\lambda_{n-1}} \\
&\quad + \frac{\alpha_n\lambda_n\gamma(1-\rho)}{2} \cdot M\frac{|\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}| + |\lambda_n - \lambda_{n-1}|}{\alpha_n\lambda_n^2\gamma(1-\rho)/2}.
\end{aligned} \tag{4.4}$$

By virtue of conditions (ii), (iii) and (iv), we can apply Lemma 2.5 to (4.4) to conclude $\|z_{n+1} - z_n\|/\lambda_n \to 0$.

Proof of (4). We verify that $\omega_w(\{z_n\}) \subset S$; that is, every weak limit point of $\{z_n\}$ solves the variational inequality (3.3). To see this, we rewrite z_{n+1} as

$$z_{n+1} = \alpha_n\lambda_n\gamma f(z_n) + \lambda_n\gamma(1-\alpha_n)Vz_n + (I-\lambda_n\mu F)Tz_n$$

so that

$$\begin{aligned} &z_n - z_{n+1} \\ &= \alpha_n\lambda_n(I-\gamma f)z_n \\ &\quad +\lambda_n(1-\alpha_n)(I-\gamma V)z_n + (1-\lambda_n)(I-T)z_n - \lambda_n(I-\mu F)Tz_n \\ &= \alpha_n\lambda_n(\mu F-\gamma f)z_n+\lambda_n(1-\alpha_n)(\mu F-\gamma V)z_n+(1-\lambda_n)(I-T)z_n \\ &\quad +\lambda_n[(I-\mu F)z_n - (I-\mu F)Tz_n]. \end{aligned} \tag{4.5}$$

Set $y_n = \frac{z_n - z_{n+1}}{\lambda_n(1-\alpha_n)}$. It is then easily seen from (4.5) that

$$\begin{aligned} y_n &= (\mu F-\gamma V)z_n + \frac{\alpha_n}{1-\alpha_n}(\mu F-\gamma f)z_n + \frac{1-\lambda_n}{\lambda_n(1-\alpha_n)}(I-T)z_n \\ &\quad +\frac{1}{1-\alpha_n}[(I-\mu F)z_n - (I-\mu F)Tz_n]. \end{aligned}$$

Utilizing the monotonicity of I–T and μF–γV, we deduce that for each $x' \in \mathrm{Fix}(T)$,

$$\begin{aligned} &\langle y_n, z_n - x'\rangle \\ &= \langle(\mu F-\gamma V)z_n, z_n - x'\rangle + \frac{\alpha_n}{1-\alpha_n}\langle(\mu F-\gamma f)z_n, z_n-x'\rangle \\ &\quad +\frac{1-\lambda_n}{\lambda_n(1-\alpha_n)}\langle(I-T)z_n - (I-T)x', z_n - x'\rangle \\ &\quad +\frac{1}{1-\alpha_n}\langle(I-\mu F)z_n - (I-\mu F)Tz_n, z_n - x'\rangle \\ &\geq \langle(\mu F-\gamma V)x', z_n - x'\rangle + \frac{\alpha_n}{1-\alpha_n}\langle(\mu F-\gamma f)z_n, z_n-x'\rangle \\ &\quad +\frac{1}{1-\alpha_n}\langle(I-\mu F)z_n - (I-\mu F)Tz_n, z_n - x'\rangle. \end{aligned} \tag{4.6}$$

Note that $\|z_n - Tz_n\| \to 0$ (Step (2)) implies $\|(I-\mu F)z_n - (I-\mu F)Tz_n\| \to 0$. Also, since $y_n \to 0$ (Step (3)), $\alpha_n \to 0$, and $\{z_n\}$ is bounded, we obtain from (4.6) that

$$\limsup_{n\to\infty}\langle(\mu F-\gamma V)x', z_n - x'\rangle \leq 0, \quad \forall x' \in \mathrm{Fix}(T). \tag{4.7}$$

This suffices to guarantee that $\omega_w(\{z_n\}) \subset S$. Indeed, if $\tilde{z} \in \omega_w(\{z_n\})$ and if $z_{m_j} \rightharpoonup \tilde{z}$ for some subsequence $\{z_{m_j}\}$ of $\{z_n\}$, then we conclude from (4.7) that

$$\langle(\mu F-\gamma V)x', \tilde{z} - x'\rangle \leq \limsup_{n\to\infty}\langle(\mu F-\gamma V)x', z_n - x'\rangle \leq 0, \quad \forall x' \in \mathrm{Fix}(T).$$

Therefore, by the dual version of (3.3) (see Lemma 3.2), $\tilde{z} \in S$.

Now take a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ satisfying

$$\limsup_{n\to\infty}\langle(\gamma f-\mu F)z^*, z_n-z^*\rangle = \limsup_{i\to\infty}\langle(\gamma f-\mu F)z^*, z_{n_i}-z^*\rangle.$$

With no loss of generality, we may further assume that $z_{n_j} \rightharpoonup \tilde{z}$; then $\tilde{z} \in S$. Therefore, noticing that z^* is the solution of the VI (3.11), we get

$$\limsup_{n\to\infty}\langle(\gamma f-\mu F)z^*, z_n-z^*\rangle = \limsup_{i\to\infty}\langle(\gamma f-\mu F)z^*, \tilde{z}-z^*\rangle \le 0.$$

Proof of (5). We have

$$\begin{aligned} &\langle(\gamma V-\mu F)z^*, z_{n+1}-z^*\rangle \\ &= \langle(\gamma V-\mu F)z^*, z_{n+1}-P_{\mathrm{Fix}(T)}z_{n+1}\rangle+\langle(\gamma V-\mu F)z^*, P_{\mathrm{Fix}(T)}z_{n+1}-z^*\rangle \\ &\le \langle(\gamma V-\mu F)z^*, z_{n+1}-P_{\mathrm{Fix}(T)}z_{n+1}\rangle \\ &\le \|(\gamma V-\mu F)z^*\|d(z_{n+1},\mathrm{Fix}(T)) \\ &\le \|(\gamma V-\mu F)z^*\|(\frac{1}{k}\|z_{n+1}-Tz_{n+1}\|)^{1/\theta}. \end{aligned} \tag{4.8}$$

We also have

$$\begin{aligned} \|z_{n+1}-Tz_{n+1}\| &\le \|z_{n+1}-Tz_n\|+\|z_{n+1}-z_n\| \\ &= \lambda_n\|\gamma V_n z_n-\mu F(Tz_n)\|+\|z_{n+1}-z_n\| \\ &\le M\lambda_n+\|z_{n+1}-z_n\|. \end{aligned}$$

Hence for a big enough constant $k_1>0$, we have

$$\begin{aligned} \tfrac{1}{\alpha_n}\langle(\gamma V-\mu F)z^*, z_{n+1}-z^*\rangle &\le \frac{k_1}{\alpha_n}(\lambda_n+\|z_{n+1}-z_n\|)^{1/\theta} \\ &\le \frac{k_1\lambda_n^{1/\theta}}{\alpha_n}(1+\frac{\|z_{n+1}-z_n\|}{\lambda_n})^{1/\theta}. \end{aligned} \tag{4.9}$$

By Step (3) and condition (vi), we obtain $\limsup_{n\to\infty}\frac{1}{\alpha_n}\langle(\gamma V-\mu F)z^*, z_{n+1}-z^*\rangle \le 0$.

Proof of (6). Observe that

$$\begin{aligned} &z_{n+1}-z^* \\ &= [(I-\lambda_n\mu F)Tz_n-(I-\lambda_n\mu F)z^*]+\alpha_n\lambda_n\gamma(f(z_n)-f(z^*)) \\ &\quad +\lambda_n(1-\alpha_n)\gamma(Vz_n-Vz^*)+\alpha_n\lambda_n(\gamma f-\mu F)z^*+\lambda_n(1-\alpha_n)(\gamma V-\mu F)z^*. \end{aligned}$$

Noticing $0<\gamma\le\tau$ and utilizing Lemma 2.6, we give the following estimation

$$
\begin{aligned}
&\|z_{n+1}-z^*\|^2\\
&=\|[(I-\lambda_n\mu F)Tz_n-(I-\lambda_n\mu F)z^*]+\alpha_n\lambda_n\gamma(f(z_n)-f(z^*))\\
&\quad+\lambda_n(1-\alpha_n)\gamma(Vz_n-Vz^*)\\
&\quad+\alpha_n\lambda_n(\gamma f-\mu F)z^*+\lambda_n(1-\alpha_n)(\gamma V-\mu F)z^*\|^2\\
&\le\|[(I-\lambda_n\mu F)Tz_n-(I-\lambda_n\mu F)z^*]+\alpha_n\lambda_n\gamma(f(z_n)-f(z^*))\\
&\quad+\lambda_n(1-\alpha_n)\gamma(Vz_n-Vz^*)\|^2\\
&\quad+2\langle\alpha_n\lambda_n(\gamma f-\mu F)z^*+\lambda_n(1-\alpha_n)(\gamma V-\mu F)z^*,z_{n+1}-z^*\rangle\\
&\le[1-\lambda_n\tau+\alpha_n\lambda_n\gamma\rho+\lambda_n(1-\alpha_n)\gamma]^2\|z_n-z^*\|^2\\
&\quad+2\alpha_n\lambda_n\langle(\gamma f-\mu F)z^*,z_{n+1}-z^*\rangle\\
&\quad+2\lambda_n(1-\alpha_n)\langle(\gamma V-\mu F)z^*,z_{n+1}-z^*\rangle\\
&\le[1-\alpha_n\lambda_n\gamma(1-\rho)]\|z_n-z^*\|^2+2\alpha_n\lambda_n\langle(\gamma f-\mu F)z^*,z_{n+1}-z^*\rangle\\
&\quad+2\lambda_n(1-\alpha_n)\langle(\gamma V-\mu F)z^*,z_{n+1}-z^*\rangle\\
&=[1-\alpha_n\lambda_n\gamma(1-\rho)]\|z_n-z^*\|^2+\alpha_n\lambda_n\gamma(1-\rho)\\
&\quad\cdot\frac{2}{\gamma(1-\rho)}(\langle(\gamma f-\mu F)z^*,z_{n+1}-z^*\rangle\\
&\quad+\frac{1-\alpha_n}{\alpha_n}\langle(\gamma V-\mu F)z^*,z_{n+1}-z^*\rangle).
\end{aligned}\tag{4.10}
$$

Setting $\gamma_n=\alpha_n\lambda_n\gamma(1-\rho)$ and

$$
\delta_n=\frac{2}{\gamma(1-\rho)}(\langle(\gamma f-\mu F)z^*,z_{n+1}-z^*\rangle+\frac{1-\alpha_n}{\alpha_n}\langle(\gamma V-\mu F)z^*,z_{n+1}-z^*\rangle),
$$

we can rewrite (4.10) as

$$
\|z_{n+1}-z^*\|^2\le(1-\gamma_n)\|z_n-z^*\|^2+\gamma_n\delta_n. \tag{4.11}
$$

From Steps (4) and (5), we have $\limsup_{n\to\infty}\delta_n\le 0$. Since $\sum_n\gamma_n=\infty$, we can apply Lemma 2.5 to (4.11) to conclude that $\|z_n-z^*\|\to 0$ as $n\to\infty$. ■

Remark 4.2. As pointed out in Lu, Xu and Yin [23], whenever the sequences $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen as

$$
\alpha_n=\frac{1}{(n+1)^\alpha}\quad\text{and}\quad\lambda_n=\frac{1}{(n+1)^\lambda},
$$

conditions (i)-(iv) of Theorem 4.1 are satisfied provided $0<\alpha,\lambda<1$ and $\alpha+2\lambda\le 1$. Also conditions (vi) is satisfied provided $\lambda/\alpha>\theta$.

5. Applications

Let H he a Hilbert space and let $\varphi_0, \varphi_1 : H \to \mathrm{R} := (-\infty, \infty]$ be proper lower semicontinuous convex functions. Consider the following hierarchical minimization

$$\min_{x \in H} \varphi_0(x) \quad \text{and} \quad \min_{x \in S_0} \varphi_1(x), \tag{5.1}$$

where $S_0 := \mathrm{argmin}_{x \in H} \varphi_0(x)$. (Here we always assume that S_0 is nonempty.) Let $S = \mathrm{argmin}_{x \in S_0} \varphi_1(x)$ and assume $S \neq \emptyset$. Assume φ_0 and φ_1 are differentiable and their gradients are Lipschitz continuous:

$$\|\nabla\varphi_0(x) - \nabla\varphi_0(y)\| \leq L_0\|x - y\| \quad \text{and} \quad \|\nabla\varphi_1(x) - \nabla\varphi_1(y)\| \leq L_1\|x - y\|,$$

where $D(\nabla\varphi_0) = D(\nabla\varphi_1) = H$.

Let

$$T = I - \gamma_0 \nabla\varphi_0 \quad \text{and} \quad V = I - \gamma_1 \nabla\varphi_1, \tag{5.2}$$

where $\gamma_0 > 0$ and $\gamma_1 > 0$.

It is readily seen that $S_0 = \mathrm{Fix}(T)$. It is also known that T and V are both nonexpansive if $0 < \gamma_0 < 2/L_0$ and $0 < \gamma_1 < 2/L_1$ (we always restrict γ_0 and γ_1 to such ranges). To see this, we need a result of [21] which says that the Lipschitz continuity of $\nabla\varphi_0$ implies that it is inverse strongly monotone; that is, the following inequality holds:

$$\langle x - y, \nabla\varphi_0(x) - \nabla\varphi_0(y)\rangle \geq \frac{1}{L_0}\|\nabla\varphi_0(x) - \nabla\varphi_0(y)\|^2$$

for all $x, y \in D(\nabla\varphi_0) = H$. Now it follows that

$$\begin{aligned}
&\|Tx - Ty\|^2 \\
&= \|(x - y) - \gamma_0(\nabla\varphi_0(x) - \nabla\varphi_0(y))\|^2 \\
&= \|x - y\|^2 - 2\gamma_0\langle x - y, \nabla\varphi_0(x) - \nabla\varphi_0(y)\rangle + \gamma_0^2\|\nabla\varphi_0(x) - \nabla\varphi_0(y)\|^2 \\
&\leq \|x - y\|^2 - \gamma_0(\frac{2}{L_0} - \gamma_0)\|\nabla\varphi_0(x) - \nabla\varphi_0(y)\|^2 \\
&\leq \|x - y\|^2.
\end{aligned}$$

Hence, T is nonexpansive. Similarly, V is nonexpansive.

The optimality condition for $x^* \in S_0$ to be a solution of the hierarchical minimization (5.1) is to find $x^* \in S_0$ such that

$$\langle \nabla\varphi_1(x^*), x - x^*\rangle \geq 0, \quad \forall x \in S_0,$$

or, equivalently, the variational inequality problem of finding $x^* \in \mathrm{Fix}(T)$ such that

$$(5.3) \qquad \langle (I - V)x^*, x - x^* \rangle \leq 0, \quad \forall x \in \mathrm{Fix}(T).$$

On the other hand, assume $F : H \to H$ is a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa > 0,\ \eta > 0$. Let $T, V : H \to H$ be the same as in (5.2). Let $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma \leq \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Then we consider the general variational inequality problem of finding $x^* \in \mathrm{Fix}(T)$ such that

$$(5.4) \qquad \langle (\mu F - \gamma V)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T).$$

In particular, whenever $\mu = 2,\ F = \frac{1}{2}I,\ \gamma = \tau = 1$, the VI (5.4) reduces to the VI (5.3).

In the remainder of this paper, we always assume $F : H \to H$ is a strongly positive bounded linear operator. That is, there is a constant $\eta > 0$ with the property

$$\langle Fx, x \rangle \geq \eta \|x\|^2, \quad \forall x \in H.$$

In this case, F is a κ-Lipschitzian and η-strongly monotone operator with $\kappa = \|F\|$. Therefore, Theorems 3.3 and 4.1 can be applied to the VI (5.4). In particular, taking $f = 0$, we have the following result.

Theorem 5.1. *Let S denote the solution set of the VI (5.4).*

(I) *Given $s, t \in (0,1)$. Define $z_{s,t} \in H$ by the fixed point equation*

$$(5.5) \qquad \begin{aligned} z_{s,t} &= -(s\mu F - \gamma s(1-t)I)^{-1}[\gamma\gamma_1 s(1-t)\nabla\varphi_1(z_{s,t}) \\ &\quad + \gamma_0(I - s\mu F)\nabla\varphi_0(z_{s,t})]. \end{aligned}$$

Then the iterated $\lim_{t\to 0}\lim_{s\to 0} z_{s,t}$ exists in the norm topology and is the unique solution $\tilde{z}$ to the VIP of finding $\tilde{z} \in S$ such that

$$\langle F\tilde{z}, z - \tilde{z} \rangle \geq 0, \quad \forall z \in S.$$

Equivalently, $\tilde{z}$ is the unique fixed point of the contraction $P_S(I - \mu F)$; that is, $\tilde{z} = P_S(I - \mu F)\tilde{z}$.

(II) *Define a sequence $\{z_n\}$ by the recursive algorithm:*

$$(5.6) \qquad \begin{aligned} z_{n+1} &= [(1 + \gamma\lambda_n(1 - \alpha_n))I - \lambda_n\mu F]z_n \\ &\quad - \gamma_0(I - \lambda_n\mu F)\nabla\varphi_0(z_n) - \gamma\gamma_1\lambda_n(1 - \alpha_n)\nabla\varphi_1(z_n), \end{aligned}$$

where we assume that $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfy the conditions (i)-(iv). Assume also that there are constants $\bar{k} > 0, \theta > 0$ satisfying

$$(5.7) \qquad \|\nabla\varphi_0(x)\| \geq \bar{k}[d(x, S_0)]^\theta, \quad \forall x \in D(\nabla\varphi_0) = H.$$

Moreover, assume $\lambda_n^{1/\theta}/\alpha_n \to 0$. *Then, if* $\{z_n\}$ *is bounded,* $\{z_n\}$ *converges in norm to the unique solution* $\tilde{z} \in S$ *to the VIP given in the case (I).*

Proof. (I) For this particular case, the implicit scheme (3.10) is reduced to

$$\begin{aligned} z_{s,t} &= s\gamma(1-t)Vz_{s,t} + (I - s\mu F)Tz_{s,t} \\ &= s\gamma(1-t)(I-\gamma_1\nabla\varphi_1)z_{s,t} + (I - s\mu F)(I-\gamma_0\nabla\varphi_0)z_{s,t} \\ &= s\gamma(1-t)z_{s,t} - s\gamma(1-t)\gamma_1\nabla\varphi_1(z_{s,t}) + z_{s,t} - \gamma_0\nabla\varphi_0(z_{s,t}) \\ &\quad -s\mu F z_{s,t} + \gamma_0 s\mu F\nabla\varphi_0(z_{s,t}) \\ &= z_{s,t} - (s\mu F - s\gamma(1-t)I)z_{s,t} - \gamma\gamma_1 s(1-t)\nabla\varphi_1(z_{s,t}) \\ &\quad -\gamma_0(I - s\mu F)\nabla\varphi_0(z_{s,t}), \end{aligned}$$

which is equivalent to

$$(s\mu F - s\gamma(1-t)I)z_{s,t} = -\gamma\gamma_1 s(1-t)\nabla\varphi_1(z_{s,t}) - \gamma_0(I - s\mu F)\nabla\varphi_0(z_{s,t}). \tag{5.8}$$

Since F is a strongly positive bounded linear operator with constant $\eta > 0$, F is a κ-Lipschitzian and η-strongly monotone operator with $\kappa = \|F\|$. Also, since $0 < \gamma \le \tau \le \mu\eta$, we have

$$0 < s\gamma(1-t) < s\gamma \le s\tau \le s\mu\eta.$$

This implies that the mapping $s\mu F - s\gamma(1-t)I$ is $s(\mu\eta - \gamma(1-t))$-strongly monotone. In the meantime, it is clear that the mapping $s\mu F - s\gamma(1-t)I$ is a linear bounded operator. Thus the mapping $s\mu F - s\gamma(1-t)I$ is a topological isomorphism from H onto itself, and so is the mapping $(s\mu F - s\gamma(1-t)I)^{-1}$. This implies that $z_{s,t}$ satisfies Eq. (5.5). By Theorem 3.3, we conclude that the iterated $\lim_{t\to 0}\lim_{s\to 0} z_{s,t} =: \tilde{z}$ exists and is the unique solution to the VIP of finding $\tilde{z} \in S$ such that

$$\langle F\tilde{z}, z - \tilde{z}\rangle \ge 0, \quad \forall z \in S.$$

Equivalently, $\tilde{z}$ is the unique fixed point of the contraction $P_S(I - \mu F)$; that is, $\tilde{z} = P_S(I - \mu F)\tilde{z}$.

(II) For this particular case, the explicit scheme (4.1) is reduced to

$$\begin{aligned} z_{n+1} &= \lambda_n\gamma V_n z_n + (I - \lambda_n\mu F)Tz_n \\ &= \lambda_n\gamma(1-\alpha_n)Vz_n + (I - \lambda_n\mu F)Tz_n \\ &= \lambda_n\gamma(1-\alpha_n)(I-\gamma_1\nabla\varphi_1)z_n + (I - \lambda_n\mu F)(I-\gamma_0\nabla\varphi_0)z_n \\ &= \gamma\lambda_n(1-\alpha_n)z_n - \gamma\gamma_1\lambda_n(1-\alpha_n)\nabla\varphi_1(z_n) + (I - \lambda_n\mu F)z_n \\ &\quad\ -\gamma_0(I - \lambda_n\mu F)\nabla\varphi_0(z_n) \\ &= [(1+\gamma\lambda_n(1-\alpha_n))I - \lambda_n\mu F]z_n - \gamma_0(I - \lambda_n\mu F)\nabla\varphi_0(z_n) \\ &\quad\ -\gamma\gamma_1\lambda_n(1-\alpha_n)\nabla\varphi_1(z_n). \end{aligned}$$

That is, $\{z_n\}$ is defined by the algorithm (5.6).

Since $I - T = \gamma_0 \nabla\varphi_0$ and $\mathrm{Fix}(T) = S_0$, we find that condition (5.7) implies condition (v) of Theorem 4.1.

Hence, all conditions (i)-(vi) of Theorem 4.1 are satisfied. Therefore, $\{z_n\}$ converges in norm to a point $\tilde{z} \in S$ which is indeed the unique solution $\tilde{z}$ to the VIP in the solution set S as argued in the case (I). ■

Corollary 5.2 (See [23, Theorem 5.1]). *Let S denote the solution set of the VI (5.3) (i.e., the solution set of the hierarchical minimization (5.1)).*

(I) *Given $s, t \in (0, 1)$. Define $z_{s,t} \in H$ by the fixed point equation*

$$z_{s,t} = -\frac{1}{st}[\gamma_0(1-s)\nabla\varphi_0(z_{s,t}) + \gamma_1 s(1-t)\nabla\varphi_1(z_{s,t})].$$

Then the iterated $\lim_{t\to 0}\lim_{s\to 0} z_{s,t}$ exists in the norm topology and is the minimum-norm solution of the hierarchical minimization (5.1).

(II) *Define a sequence $\{z_n\}$ by the recursive algorithm:*

$$z_{n+1} = (1-\alpha_n\lambda_n)z_n - \gamma_0(1-\lambda_n)\nabla\varphi_0(z_n) - \gamma_1\lambda_n(1-\alpha_n)\nabla\varphi_1(z_n)$$

where we assume that $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfy the conditions (i)-(iv). Assume also that there are constants $\bar{k} > 0, \theta > 0$ satisfying

$$\|\nabla\varphi_0(x)\| \geq \bar{k}[d(x, S_0)]^\theta, \quad \forall x \in D(\nabla\varphi_0) = H.$$

Moreover, assume $\lambda_n^{1/\theta}/\alpha_n \to 0$. Then, if $\{z_n\}$ is bounded, $\{z_n\}$ converges in norm to the minimum-norm solution of the hierarchical minimization (5.1).

Proof. In Theorem 5.1, put $\mu = 2,\ F = \frac{1}{2}I$ and $\gamma = \tau = 1$. Then the VI (5.4) reduces to the VI (5.3).

(I) For any given $s, t \in (0, 1)$, the fixed point equation (5.5) reduces to the following

$$z_{s,t} = -\frac{1}{st}[\gamma_0(1-s)\nabla\varphi_0(z_{s,t}) + \gamma_1 s(1-t)\nabla\varphi_1(z_{s,t})].$$

In terms of Theorem 5.1 (I), we deduce that the iterated $\lim_{t\to 0}\lim_{s\to 0} z_{s,t}$ exists in the norm topology and is the unique solution $\tilde{z}$ to the VIP of finding $\tilde{z} \in S$ such that

$$\langle \tilde{z}, z - \tilde{z}\rangle \geq 0, \quad \forall z \in S. \tag{5.9}$$

This is equivalent to the fact that $\tilde{z} = P_S(0)$; namely, $\tilde{z} \in S$ fulfills the property: $\|\tilde{z}\| = \min\{\|z\| : z \in S\}$.

(II) In this case, the recursive algorithm (5.6) reduces to the following

$$z_{n+1} = (1-\alpha_n\lambda_n)z_n - \gamma_0(1-\lambda_n)\nabla\varphi_0(z_n) - \gamma_1\lambda_n(1-\alpha_n)\nabla\varphi_1(z_n).$$

In terms of Theorem 5.1 (II), we know that $\{z_n\}$ converges in norm to the unique solution $\tilde{z} \in S$ to the VI (5.9), namely, the minimum-norm solution of the hierarchical minimization (5.1).

Remark 5.3. As reminded in [23], see [22] for the nonsmooth case in a finite-dimensional Hilbert space.

REFERENCES

1. I. Yamada, The hybrid steepest descent for the variational inequality problems over the intersection of fixed point sets of nonexpansive mappings, in: *Inherently Parallel Algorithms in Feasibility and Optimization and Their applications*, Elservier, New York, 2001, pp. 473-504.
2. P. E. Mainge and A. Moudafi, Strong convergence of an iterative method for hierarchical fixed-points problems, *Pacific J. Optim.*, **3** (2007), 529-538.
3. A. Moudafi and P. E. Mainge, *Towards viscosity approximations of hierarchical fixed-points problems*, Fixed Point Theory Appl., Vol. 2006, Article ID 95453, 1-10.
4. H. K. Xu, Viscosity approximation methods for nonexpansive mappings, *J. Math. Anal. Appl.*, **298** (2004), 279-291.
5. K. Geobel and W. A. Kirk, *Topics on Metric Fixed-Point Theory*, Cambridge University Press, Cambridge, England, 1990.
6. H. K. Xu and T. H. Kim, Convergence of hybrid steepest-descent methods for variational inequalities, *J. Optim. Theory Appl.*, **119(1)** (2003), 185-201.
7. A. Moudafi, Viscosity approximation methods for fixed-points problems, *J. Math. Anal. Appl.*, **241** (2000), 46-55.
8. H. K. Xu, Viscosity methods for hierarchical fixed point approach to variational inequalities, *Taiwanese J. Math.*, **14(2)** (2010), 463-478.
9. F. E. Browder, Convergence of approximation to fixed points of nonexpansive nonlinear mappings in Hilbert spaces, *Arch. Ration. Mech. Anal.*, **24** (1967), 82-90.
10. B. Halpern, Fixed points of nonexpansive maps, *Bull. Amer. Math. Math.*, **73** (1967), 957-961.
11. P. L. Lions, Approximation de points fixes de contractions, *C. R. Acad. Sci. Ser. A-B Paris*, **284** (1977), 1357-1359.
12. G. Marino and H. K. Xu, A general iterative method for nonexpansive mappings in Hilbert spaces, *J. Math. Anal. Appl.*, **318** (2006), 43-52.
13. S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, *J. Math. Anal. Appl.*, **75** (1980), 287-292.
14. S. Reich, Approximating fixed points of nonexpansive mappings, *Panamer. Math. J.*, **4(2)** (1994), 23-28.
15. N. Shioji and W. Takahashi, Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces, *Proc. Amer. Math. Soc.*, **125** (1997), 3641-3645.

16. R. Wittmann, Approximation of fixed points of nonexpansive mappings, *Arch. Math.*, **58** (1992), 486-491.

17. H. K. Xu, Another control condition in an iterative method for nonexpansive mappings, *Bull. Austral. Math. Soc.*, **65** (2002), 109-113.

18. H. K. Xu, Remarks on an iterative method for nonexpansive mappings, *Comm. Appl. Nonlinear Anal.*, **10(1)** (2003), 67-75.

19. H. K. Xu, An iterative approach to quadratic optimization, *J. Optim. Theory Appl.*, **116** (2003), 659-678.

20. G. Marino and H. K. Xu, *Explicit hierarchical fixed point approach to variational inequalities*, Preprint.

21. J. B. Baillon and G. Haddad, Quelques proprietes des operateurs angle-bornes et n-cycliquement monotones, *Isr. J. Math.*, **26** (1977), 137-150.

22. A. Cabot, Proximal point algorithm controlled by a slowly vanishing term: Applications to hierarchical minimization, *SIAM J. Optim.*, **15** (2005), 555-572.

23. X. W. Lu, H. K. Xu and X. M. Yin, Hybrid methods for a class of monotone variational inequalities, *Nonlinear Anal.*, **71** (2009), 1032-1041.

24. D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, SIAM, Philadelphia, 2000.

25. R. Glowinski, J. L. Lions and R. Tremoliers, *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, Holland, 1981.

26. P. Jaillet, D. Lamberton and B. Lapeyre, Variational inequalities and the pricing of American options, *Acta Appl. Math.*, **21** (1990), 263-289.

27. J. T. Oden, *Qualitative Methods on Nonlinear Mechanics*, Prentice-Hall, Englewood Cliffs, New Jersey, 1986.

28. E. Zeidler, *Nonlinear Functional Analysis ant Its Applications*, III: Variational Methods and Applications, Springer, New York, 1985.

29. L. C. Zeng and J. C. Yao, Implicit iteration scheme with perturbed mapping for common fixed points of a finite family of nonexpansive mappings, *Nonlinear Anal.*, **64** (2006), 2507-2515.

30. L. C. Ceng, P. Cubiotti and J. C. Yao, Strong convergence theorems for finitely many nonexpansive mappings and applications, *Nonlinear Anal.*, **67** (2007), 1464-1473.

31. L. C. Zeng, N. C. Wong and J. C. Yao, Convergence analysis of modified hybrid steepest-descent methods with variable parameters for variational inequalities, *J. Optim. Theory Appl.*, **132** (2007), 51-69.

32. L. C. Ceng and J. C. Yao, Relaxed viscosity approximation methods for fixed point problems and variational inequality problems, *Nonlinear Anal.*, **69** (2008), 3299-3309.

33. L. C. Ceng, H. K. Xu and J. C. Yao, The viscosity approximation method for asymptotically nonexpansive mappings in Banach spaces, *Nonlinear Anal.*, **69** (2008), 1402-1412.

34. L. C. Ceng, Q. H. Ansari and J. C. Yao, Mann-type steepest-descent and modified hybrid steepest-descent methods for variational inequalities in Banach spaces, *Numer. Funct. Anal. Optim.*, **29(9-10)** (2008), 987-1033.

35. L. C. Ceng, M. Teboulle and J. C. Yao, On Weak Convergence of an Iterative Method for Pseudomonotone Variational Inequalities and Fixed Point Problems, *Journal of Optimization Theory and Application*, **146** (2010), 19-31.

36. L. C. Ceng, B. S. Mordukhovich and J. C. Yao, Hybrid approximate proximal method with auxiliary variational inequality for vector optimization, *Journal of Optimization Theory and Application*, **146** (2010), 267-303.

37. S. L. Li, C. Li, Y. C. Liou and J. C. Yao, Existence of solutions for variational inequalities on Riemannian manifolds, *Nonlinear Analysis Series A: Theory, Methods & Applications*, **71** (2009), 5695-5706.

38. L. C. Ceng, Q. H. Ansari and J. C. Yao, On relaxed viscosity iterative methods for variational inequalities in Banach spaces, *Journal of Computational and Applied Mathematics*, **230** (2009), 813-822.

39. L. C. Ceng, S. Schaible and J. C. Yao, Strong Convergence of Iterative Algorithms for Variational Inequalities in Banach Spaces, *Journal of Optimization Theory and Applications*, **141** (2009), 265-283.

40. N. C. Wong, D. R. Sahu and J. C. Yao, Solving variational inequalities involving nonexpansive type mappings, *Nonlinear Analysis Series A: Theory, Methods & Applications*, **69** (2008), 4732-4753.

Lu-Chuan Ceng
Department of Mathematics
Shanghai Normal University
Shanghai 200234
and
Scientific Computing Key Laboratory of Shanghai Universities
P. R. China
E-mail: zenglc@hotmail.com

Q. H. Ansari
Department of Mathematics
Aligarh Muslim University
Aligarh 202 002
India
E-mail: qhansari@gmail.com

Juei-Ling Ho
Department of Finance
Tainan University of Technology
Tainan, Taiwan
E-mail: t20054@mail.tut.edu.tw