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MEAN LIPSCHITZ SPACES CHARACTERIZATION VIA MEAN OSCILLATION

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Abstract. In this paper we characterize mean Lipschitz spaces in terms of some L^p-mean oscillation on the unit disc.

1. Introduction

Let $\mathbb{D}$ be the unit disc and $\mathbb{T}$ the unit circle. For a measurable function f on $\mathbb{D}$ and $0 < r < 1$, we define the L^p-mean $M_p(r, f)$ of f by

$$M_p(r,f) = \left(\int_{-\pi}^{\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}.$$

The Hardy space H^p is the collection of holomorphic functions f such that

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(r,f) < \infty.$$

For a measurable function f on $\mathbb{T}$, we define the L^p-norm on $\mathbb{T}$ by

$$\|f\|_{L^p} = \left(\int_{-\pi}^{\pi} |f(e^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}.$$

For $1 \leq p$ and $f \in L^p(\mathbb{T})$, we denote by $P[f]$ the Poisson integral of f :

$$P[f](re^{i\theta}) = \int_{-\pi}^{\pi} P(r, \theta - t) f(e^{it}) \frac{dt}{2\pi},$$

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where $P(r,t) = \text{Re}\,\{(1+re^{it})/(1-re^{it})\} = (1-r^2)/(1-2r\cos t + r^2)$, and $re^{i\theta} \in \mathbb{D}$. It is well known that $u = P[f]$ is harmonic in $\mathbb{D}$, that the integral means $M_p(r,u)$ are bounded, and that the radial limit

$$u^*(e^{i\theta}) = \lim_{r\to 1^-} u(re^{i\theta})$$

exists and equals $f(e^{i\theta})$ for a.e. real θ. Conversely, if u is any harmonic function on $\mathbb{D}$ for which the integral means $M_p(r,u)$ are bounded, then $f = u^*$ exists a.e., belongs to $L^p(\mathbb{T})$, and $u = P[f]$. In short, if h^p denotes the class of harmonic functions u for which $M_p(r,u)$ is bounded, then the radial limit map $u \to u^*$ establishes an isometric isomorphism between h^p, taken in the natural norm imposed by its definition, and $L^p(\mathbb{T})$.

Let

$$\Delta_t f(e^{i\theta}) = f(e^{i(\theta+t)}) - f(e^{i\theta}),$$

$$\Delta_t^2 f(e^{i\theta}) = f(e^{i(\theta+t)}) - 2f(e^{i\theta}) + f(e^{i(\theta-t)}).$$

For $1 \le p,q < \infty$ and $0 < \alpha < 1$ we define the mean Lipschitz space $\Lambda_\alpha^{p,q}$ to be the collection of $f \in H^p$ such that

$$\Delta_\alpha^{p,q}(f) = \left(\int_{-\pi}^{\pi} \frac{\|\Delta_t f\|_{L^p}^q}{|t|^{1+\alpha q}} \frac{dt}{2\pi}\right)^{1/q} < \infty.$$

Now for $\alpha = 1$ we define the mean Lipschitz space $\Lambda_*^{p,q}$ to be the collection of $f \in H^p$ such that

$$\Delta_*^{p,q}(f) = \left(\int_{-\pi}^{\pi} \frac{\|\Delta_t^2 f\|_{L^p}^q}{|t|^{1+q}} \frac{dt}{2\pi}\right)^{1/q} < \infty.$$

Let $1 \le p,q < \infty$. For $\beta > 0$ and $g \in L^p(\mathbb{D})$ let

$$L_\beta^{p,q}(g) = \left(\int_0^1 M_p(r,g)^q (1-r)^{\beta q - 1} dr\right)^{1/q}.$$

The following theorem is a characterization of the mean Lipschitz space in terms of of the integral mean function.

Theorem 1.1. ([4], [1]). *Let* $1 \le p,q < \infty$. *Let* $f \in H^p$.

(i) *Let* $0 < \alpha < 1$. *Then*

$$\|f\|_{H^p} + \Delta_\alpha^{p,q}(f) \sim \|f\|_{H^p} + L_{1-\alpha}^{p,q}(f').$$

(ii) *For* $\alpha = 1$, *we have*

$$\|f\|_{H^p} + \Delta_*^{p,q}(f) \sim \|f\|_{H^p} + L_1^{p,q}(f'').$$

Suppose $f \in L^1(\mathbb{T})$. If I is a subinterval of $[-\pi, \pi]$, let $|I|$ denote its length, and write

$$f_I = \frac{1}{|I|} \int_I f(e^{i\theta}) \frac{d\theta}{2\pi}.$$

The space BMO of functions of bounded mean oscillation is the collection of $f \in L^1(\mathbb{T})$ for which $\|f\|_{BMO} < \infty$ (see [7]) where

$$\|f\|_{BMO} = \sup \left\{ \frac{1}{|I|} \int_I |f(e^{i\theta}) - f_I| \frac{d\theta}{2\pi} : I \text{ a subinterval of } [-\pi, \pi] \right\}.$$

In this paper we characterize Lipschitz type spaces in terms of certain mean oscillation as in the BMO space. Let $I_s = [-s, s]$ with $0 < s < \pi$. We define the L^p-moduli of continuity in terms of certain mean oscillation as follows,

$$\omega_p(f, s) = \left(\int_{-\pi}^{\pi} \left(\frac{1}{|I_s|} \int_{I_s} |\Delta_t f(e^{i\theta})| \frac{dt}{2\pi} \right)^p \frac{d\theta}{2\pi} \right)^{1/p};$$

$$\omega_p^*(f, s) = \left(\int_{-\pi}^{\pi} \left(\frac{1}{|I_s|} \int_{I_s} |\Delta_t^2 f(e^{i\theta})| \frac{dt}{2\pi} \right)^p \frac{d\theta}{2\pi} \right)^{1/p}.$$

The following is our result characterizating the Besov spaces in terms of these mean oscillations.

Theorem 1.2. *Let* $1 \le p, q < \infty$ *and* $f \in H^p$.

(i) *Let* $0 < \alpha < 1$. *Then*

$$\|f\|_{H^p} + \Delta_\alpha^{p,q}(f) \sim \|f\|_{H^p} + \left(\int_0^\pi \frac{1}{s^{1+\alpha q}} \left(\omega_p(f, s)\right)^q ds \right)^{1/q}.$$

(ii) *For* $\alpha = 1$, *we have*

$$\|f\|_{H^p} + \Delta_*^{p,q}(f) \sim \|f\|_{H^p} + \left(\int_0^\pi \frac{1}{s^{1+q}} \left(\omega_p^*(f, s)\right)^q ds \right)^{1/q}.$$

Remark 1.3. In [2], Dyakonov characterized the mean Lipschitz spaces in terms of the Garsian type norm (see [3]) that is another kind of the mean oscillation. However, he considered only the case $0 < \alpha < 1$.

2. Proof of Theorem

We need a couple of lemmas.

Lemma 2.1. (Hardy's inequality). *Let* $p \ge 1, r > 0$ *and* h *be a non-negative function. Then we have*

$$\left[\int_0^1 \left(\int_0^x h(y) dy \right)^p x^{-r-1} dx \right]^{1/p} \le \frac{p}{r} \left(\int_0^1 (y h(y))^p y^{-r-1} dy \right)^{1/p}.$$

See [6] for a proof of Hardy's inequality.

Lemma 2.2.

$$M_p(r, f_\theta) \le M_p(r, f_{\theta\theta}).$$

Proof. Note that

$$\begin{aligned} f_\theta(z) &= izf'(z) \\ f_{\theta\theta}(z) &= -z^2 f''(z) - zf'(z). \end{aligned} \tag{2.1}$$

Thus we have

$$\begin{aligned} f'_\theta(z) &= izf''(z) + if'(z) \\ &= \frac{1}{iz} f_{\theta\theta}(z). \end{aligned} \tag{2.2}$$

Since

$$|f_\theta(re^{i\theta})| \le \int_0^r |f'_\theta(\rho e^{i\theta})| d\rho,$$

by Minkowski's inequality, it follows that

$$M_p(r, f_\theta) \le \int_0^r M_p(\rho, f'_\theta) d\rho \le r M_p(r, f'_\theta).$$

By (2.2), we have

$$M_p(r, f'_\theta) = \frac{1}{r} M_p(r, f_{\theta\theta}).$$

Therefore

$$M_p(r, f_\theta) \le M_p(r, f_{\theta\theta}). \qquad \blacksquare$$

Proof of Theorem 1.2(i): If $f \in \Lambda_\alpha^{p,q}$, then, by Minkowski's inequality,

$$\left(\int_{-\pi}^{\pi} \left(\frac{1}{|I_s|} \int_{I_s} |\Delta_t f(e^{i\theta})| \frac{dt}{2\pi} \right)^p \frac{d\theta}{2\pi} \right)^{1/p} \lesssim \frac{1}{|I_s|} \int_{I_s} \|\Delta_t f\|_{L^p} \frac{dt}{2\pi}.$$

Thus, by Hardy's inequality, we have

$$\begin{aligned} \int_0^\pi \frac{1}{s^{1+\alpha q}} \big(\omega_p(f, s)\big)^q ds &\lesssim \int_0^\pi \frac{1}{s^{1+\alpha q}} \left(\frac{1}{|I_s|} \int_{I_s} \|\Delta_t f\|_{L^p} \frac{dt}{2\pi} \right)^q ds \\ &\lesssim \int_0^\pi \frac{1}{s^{1+\alpha q+q}} \left(\int_0^s \|\Delta_t f\|_{L^p} \frac{dt}{2\pi} \right)^q ds \\ &\lesssim \int_0^\pi \frac{\|\Delta_t f\|_{L^p}^q}{t^{1+\alpha q}} \frac{dt}{2\pi} \\ &= \left(\Delta_\alpha^{p,q}(f) \right)^q. \end{aligned}$$

For the converse, by Cauchy formula,

$$f'(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{[f(e^{it}) - f(e^{i\theta})]e^{it}}{(e^{it} - re^{i\theta})^2} dt, \quad 0 < r < 1.$$

Thus

$$\begin{aligned} |f'(re^{i\theta})| &\lesssim \int_{-\pi}^{\pi} \frac{|\Delta_t f(e^{i\theta})|}{|e^{i(t+\theta)} - re^{i\theta}|^2} \frac{dt}{2\pi} \\ &= \int_{-\pi}^{\pi} \frac{|\Delta_t f(e^{i\theta})|}{|e^{it} - r|^2} \frac{dt}{2\pi}. \end{aligned}$$

For $0 \le r < 1$, let $N = N(r)$ be the positive integer such that

$$(1-r)2^N < \pi \le (1-r)2^{N+1}. \tag{2.3}$$

Then,

$$\int_{-\pi}^{\pi} \frac{|\Delta_t f(e^{i\theta})|}{|e^{it} - r|^2} \frac{dt}{2\pi} = \sum_{k=0}^{N+1} \int_{J_k} \frac{|\Delta_t f(e^{i\theta})|}{|e^{it} - r|^2} \frac{dt}{2\pi},$$

where

$$\begin{aligned} J_0 &= \{t : |t| < 1 - r\}, \\ J_k &= \{t : 2^{k-1}(1-r) \le |t| < 2^k(1-r)\} \quad (k = 1, \cdots, N) \\ J_{N+1} &= \{t : 2^N(1-r) \le |t| \le \pi\}. \end{aligned}$$

Note that $|e^{it} - r| \ge 1 - r$ and for $t \in J_k$

$$|e^{it} - r| \ge |e^{it} - 1| - (1-r) \sim 2^k(1-r) - (1-r) \ge 2^{k-1}(1-r).$$

Therefore, for $k = 0, 1, ..., N+1$ we have

$$\int_{J_k} \frac{|\Delta_t f(e^{i\theta})|}{|e^{it} - r|^2} \frac{dt}{2\pi} \lesssim \frac{1}{2^k(1-r)} \frac{1}{|J_k|} \int_{J_k} |\Delta_t f(e^{i\theta})| \frac{dt}{2\pi}$$

and

$$|f'(re^{i\theta})| \lesssim \sum_{k=0}^{N+1} \frac{1}{2^k(1-r)} F_k(\theta), \qquad F_k(\theta) \equiv \frac{1}{|J_k|} \int_{J_k} |\Delta_t f(e^{i\theta})| \frac{dt}{2\pi}. \tag{2.4}$$

Note that for $k = 0, ..., N(r)$

$$||F_k||_{L^p} = \left(\int_{-\pi}^{\pi} \left(\frac{1}{|J_k|} \int_{J_k} |\Delta_t f(e^{i\theta})| \frac{dt}{2\pi} \right)^p \frac{d\theta}{2\pi} \right)^{1/p} = \omega_p(f, 2^k(1-r)),$$

and

$$||F_{N+1}||_{L^p} \lesssim \|f\|_{H^p}.$$

Therefore, by (2.4) and (2.3) we have

$$\begin{aligned}
L^{p,q}_{1-\alpha}(f') &= \left(\int_0^1 M_p(r,f')^q (1-r)^{(1-\alpha)q-1} dr\right)^{1/q} \\
&\le \left(\int_0^1 \left|\left|\sum_{k=0}^{N(r)+1} \frac{F_k}{2^k(1-r)}\right|\right|_{L^p}^q (1-r)^{(1-\alpha)q-1} dr\right)^{1/q} \\
&\le \left(\int_0^1 \left(\sum_{k=0}^{N(r)+1} \left|\left|\frac{F_k}{2^k(1-r)}\right|\right|_{L^p}\right)^q (1-r)^{(1-\alpha)q-1} dr\right)^{1/q} \\
&\lesssim \left(\int_0^1 \left(\|f\|_{H^p} + \sum_{k=0}^{N(r)} \left|\left|\frac{F_k}{2^k(1-r)}\right|\right|_{L^p}\right)^q (1-r)^{(1-\alpha)q-1} dr\right)^{1/q} \\
&\approx \|f\|_{H^p} + \left(\int_0^1 \left(\sum_{k=0}^{N(r)} \frac{1}{2^k}\omega_p(f, 2^k(1-r))\right)^q \frac{dr}{(1-r)^{1+\alpha q}}\right)^{1/q}.
\end{aligned}$$

For notational conveniences we let $\omega_p(f,t) = 0$ if $t > \pi$. With this notation, we have

$$\begin{aligned}
&\left(\int_0^1 \left(\sum_{k=0}^{N(r)} \frac{1}{2^k}\omega_p(f, 2^k(1-r))\right)^q \frac{dr}{(1-r)^{1+\alpha q}}\right)^{1/q} \\
&\le \left(\int_0^1 \left(\sum_{k=0}^{\infty} \frac{1}{2^k}\omega_p(f, 2^k(1-r))\right)^q \frac{dr}{(1-r)^{1+\alpha q}}\right)^{1/q} \\
&\le \sum_{k=0}^{\infty} \frac{1}{2^k}\left(\int_0^1 \left(\omega_p(f, 2^k(1-r))\right)^q \frac{dr}{(1-r)^{1+\alpha q}}\right)^{1/q} \\
&\le \sum_{k=0}^{\infty} \frac{1}{2^k}\left(\int_0^\pi (\omega_p(f,s))^q \frac{dr}{2^{-\alpha q}s^{1+\alpha q}}\right)^{1/q}
\end{aligned}$$

Since $0 < \alpha < 1$, we thus have

$$\begin{aligned}
L^{p,q}_{1-\alpha}(f') &\lesssim \|f\|_{H^p} + \left(\int_0^1 \left(\sum_{k=0}^{N(r)} \frac{1}{2^k}\omega_p(f, 2^k(1-r))\right)^q \frac{dr}{(1-r)^{1+\alpha q}}\right)^{1/q} \\
&\lesssim \|f\|_{H^p} + \left(\int_0^\pi (\omega_p(f,s))^q \frac{dr}{s^{1+\alpha q}}\right)^{1/q}.
\end{aligned}$$

Proof of Theorem 1.2(ii): If $f \in \Lambda_*^{p,q}$, then by Minkowski's inequality,

$$\left(\int_{-\pi}^{\pi}\left(\frac{1}{|I_s|}\int_{I_s}|\Delta_t^2 f(e^{i\theta})|\frac{dt}{2\pi}\right)^p\frac{d\theta}{2\pi}\right)^{1/p} \lesssim \frac{1}{|I_s|}\int_{I_s}\|\Delta_t^2 f\|_{L^p}\frac{dt}{2\pi}.$$

Thus, by Hardy's inequality, we have

$$\begin{aligned}\int_0^{\pi}\frac{1}{s^{1+q}}\big(\omega_p^*(f,s)\big)^q ds &\lesssim \int_0^{\pi}\frac{1}{s^{1+q}}\left(\frac{1}{|I_s|}\int_{I_s}\|\Delta_t^2 f\|_{L^p}\frac{dt}{2\pi}\right)^q ds\\ &\lesssim \int_0^{\pi}\frac{1}{s^{1+2q}}\left(\int_0^{s}\|\Delta_t^2 f\|_{L^p}\frac{dt}{2\pi}\right)^q ds\\ &\lesssim \int_0^{\pi}\frac{\|\Delta_t^2 f\|_{L^p}^q}{t^{1+q}}dt \lesssim \Delta_*^{p,q}(f).\end{aligned}$$

For the converse, since $f \in H^1$, it can be represented as a Poisson integral

$$f(re^{i\theta}) = \int_{-\pi}^{\pi} P(r,\theta-t)f(e^{it})\frac{dt}{2\pi}, \tag{2.5}$$

where $P(r,t) = \text{Re}\,\{(1+re^{it})/(1-re^{it})\} = (1-r^2)/(1-2r\cos t + r^2)$, and $re^{i\theta} \in \mathbb{D}$. Note that

$$1-2r\cos t + r^2 = (1-r)^2 + 2r(1-\cos t) \gtrsim ((1-r)+|t|)^2.$$

Therefore

$$|P_{\theta\theta}(r,t)| \lesssim \frac{1}{|t|(1-r)}P(r,t). \tag{2.6}$$

The estimate (2.6) appears in [5] (see p.157 of [5]).

Note that $\int_{-\pi}^{\pi} P_{\theta\theta}(r,t)dt = 0$ since the derivative of a constant function is 0, and straightforward calculations show $P_{\theta\theta}(r,t)$ is an even function in t variable. We thus have

$$f_{\theta\theta}(re^{i\theta}) = \int_0^{\pi} P_{\theta\theta}(r,t)[f(e^{i(\theta+t)}) - 2f(e^{i\theta}) + f(e^{i(\theta-t)})]\frac{dt}{2\pi}.$$

Therefore

$$\begin{aligned}|f_{\theta\theta}(re^{i\theta})| &\lesssim \int_0^{\pi}|P_{\theta\theta}(r,t)||\Delta_t^2 f(e^{i\theta})|\frac{dt}{2\pi}\\ &\lesssim \sum_{k=0}^{N+1}\int_{J_k}|P_{\theta\theta}(r,t)||\Delta_t^2 f(e^{i\theta})|\frac{dt}{2\pi}\\ &\lesssim \sum_{k=0}^{N+1}\int_{J_k}\frac{1}{|t|(1-r)}P(r,t)|\Delta_t^2 f(e^{i\theta})|\frac{dt}{2\pi}.\end{aligned}$$

Since $P(r,t) \le \frac{(1-r)}{(2^k(1-r))^2}$ on J_k, we have

$$|f_{\theta\theta}(re^{i\theta})| \lesssim \sum_{k=0}^{N+1} \frac{1}{[2^k(1-r)]^2} G_k(\theta), \qquad G_k(\theta) \equiv \frac{1}{|J_k|}\int_{J_k} |\Delta_t^2 f(e^{i\theta})| \frac{dt}{2\pi}.$$

Therefore, following the argument as the proof of the case (i)with G_k instead of F_k, we have

$$\begin{aligned} L_1^{p,q}(f_{\theta\theta}) &= \left(\int_0^1 M_p(r, f_{\theta\theta})^q (1-r)^{q-1} dr\right)^{1/q} \\ &\lesssim \left(\int_0^1 \left(\left\|\sum_{k=0}^{N+1} \frac{G_k}{[2^k(1-r)]^2}\right\|_{L^p}\right)^q (1-r)^{q-1} dr\right)^{1/q} \\ &\lesssim \left(\int_0^1 \left(\left\|\sum_{k=0}^{N+1} \frac{G_k}{[2^k(1-r)]^2}\right\|_{L^p}\right)^q (1-r)^{q-1} dr\right)^{1/q} \\ &\lesssim \|f\|_{H^p} + \left(\int_0^1 \left(\sum_{k=0}^{N(r)} \frac{1}{4^k}\omega_p^*(f, 2^k(1-r))\right)^q \frac{dr}{(1-r)^{1+\alpha q}}\right)^{1/q}. \end{aligned}$$

Also, following the argument as the proof of the case (i) with ω instead of ω^*, we have

$$\left(\int_0^1 \left(\sum_{k=0}^{N(r)} \frac{1}{4^k}\omega_p^*(f, 2^k(1-r))\right)^q \frac{dr}{(1-r)^{1+\alpha q}}\right)^{1/q} \lesssim \left(\int_0^\pi \omega_p^*(f,s)^q \frac{ds}{s^{1+q}}\right)^{1/q}.$$

Therefore, we have

$$L_1^{p,q}(f_{\theta\theta}) \lesssim \|f\|_{H^p} + \left(\int_0^1 \frac{1}{s^{1+q}}\omega_p^*(f,s)^q \, ds\right)^{1/q}. \tag{2.7}$$

By Cauchy estimates we have $\sup_{|z|\le 1/2} |f''(z)| \lesssim \|f\|_{H^p}$, and by (2.1)

$$f''(re^{i\theta}) = \frac{1}{r^2 e^{2i\theta}}[if_\theta(re^{i\theta}) - f_{\theta\theta}(re^{i\theta})].$$

Therefore, by Lemma 2.2 and (2.7) we have

$$L_1^{p,q}(f'') \lesssim \|f\|_{H^p} + L_1^{p,q}(f_\theta) + L_1^{p,q}(f_{\theta\theta}) \lesssim \|f\|_{H^p} + L_1^{p,q}(f_{\theta\theta}).$$

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