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TRANSFERENCE OF BILINEAR OPERATORS BETWEEN JACOBI SERIES AND HANKEL TRANSFORMS

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Dedicated to Professor Yuichi Kanjin on his 60th birthday

Abstract. Fan-Sato[8] proved a tranceference theorem with respect to the multilinear operators on $\mathbf{R}^n$. Also Blasco-Villarroya[3] proved the similar result with repect to the biilinear operators on $\mathbf{Z}^2$. In this paper, we prove a tranceference theorem of the bilinear operators between Jacobi series and Hankel transforms.

1. INTRODUCTION

Let $0 < p, q, r < \infty$ with $1/p = 1/q + 1/r$, and $m(\xi, \eta)$ a bounded measurable function. The bilinear operator T from $L^q(\mathbf{R}) \times L^r(\mathbf{R})$ to $L^p(\mathbf{R})$ is defined by

$$T(f, g)(x) = \int_{\mathbf{R}^2} m(\xi, \eta)\hat{f}(\xi)\hat{g}(\eta)e^{2\pi ix(\xi+\eta)}d\xi d\eta,$$

where $\hat{f}(\xi) = \int_{\mathbf{R}} f(x)e^{-2\pi i\xi x}dx$. Recently, Lacey-Thiele ([11-13]) developed the study of the multilinear operators. They proved that the operator T is bounded if $1 < q, r < \infty,\ p > 2/3,\ m(\xi, \eta) = sgn(\xi + \alpha\eta),\ \alpha \in \mathbf{R}\backslash\{0, 1\}$, and solved the problem with respect to the cauchy integral. The study of those operators was started by Coifman-Meyer (cf. [4-6]). Also we would like to hope that the readers refer to [9].

Now Fan-Sato[8] proved the de Leeuw type Theorem with respect to the multilinear operator on $\mathbf{R}^n$. Also Blasco-Villarroya [3] proved the de Leeuw type Theorem with respect to the bilinear operators on $\mathbf{Z} \times \mathbf{Z}$.

In this paper, we treat the bilinear operators on Jacobi orthogonal systems and those on the modified Hankel transforms. Then we show a tranceference theorem

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among those orthogonal systems. The study of the transference thorem between Jacobi orthogonal system and the modified Hankel transform was begun by Igari [10]. After that, Connett-Schwartz [7] showed the weak type, and Betancor-Stempak [2], Stempak [16] developed the study. Also we refer to [14] and [15] in which we had the similar results.

Now we introduce the notations about Jacobi polynomials and the modified Hankel transforms. Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial of the degree n and the order $(\alpha,\beta),\ \alpha,\ \beta > -1$. It is defined by

$$(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\}.$$

Then the system $\{P_n^{(\alpha,\beta)}(\cos\theta)\}_{n=0}^{\infty}$ is an orthogonal system with respect to $L^2((0,\pi),\nu)$, where $d\nu(\theta) = (\sin\theta/2)^{2\alpha+1}(\cos\theta/2)^{2\beta+1}d\theta$. When we define $t_n^{(\alpha,\beta)} > 0$ by

$$(t_n^{(\alpha,\beta)})^{-2} = \int_0^{\pi}[P_n^{(\alpha,\beta)}(\cos\theta)]^2 d\nu(\theta),$$

$\{t_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta)\}_{n=0}^{\infty}$ is a complete orthonormal system of $L^2((0,\pi),\nu)$. Also let $\hat{f}(n)$ be defined by

$$\hat{f}(n) = \int_0^{\pi} f(\theta)t_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta)d\nu(\theta)$$

for $f \in L^1((0,\pi),\nu)$, and $||f||_{L^p(\nu)}$ the norm of f in $L^p((0,\pi),\nu)$ $(1 \le p < \infty)$, where $L^p((0,\pi),\nu)$ is the usual L^p-space with respect to the measure ν. For $\alpha > -1$, let $L^p((0,\infty),\mu)$ be the L^p-space on $(0,\infty)$ with respect to $d\mu(x) = x^{2\alpha+1}dx$, and $||f||_{L^p(\mu)}$ the norm of f in $L^p((0,\infty),\mu)$. Also for $f \in L^1((0,\infty),\mu)$

$$\mathcal{H}_{\alpha}f(x) = \int_0^{\infty} f(y)\mathcal{J}_{\alpha}(xy)d\mu(y),$$

where $\mathcal{J}_{\alpha}(x) = J_{\alpha}(x)/x^{\alpha}$ and J_{α} is the Bessel function of the first kind. Moreover, for any bounded continuous function $\phi(u,v)$ in $[0,\infty)\times[0,\infty)$, let $T = T_{\phi}$ be defined by

$$T(f,g)(x) = \int\int_{(0,\infty)\times(0,\infty)} \phi(u,v)\mathcal{H}_{\alpha}f(u)\mathcal{H}_{\alpha}g(v)\mathcal{J}_{\alpha}(xu)\mathcal{J}_{\alpha}(xv)d\mu(u)d\mu(v)$$

$(f, g \in C_c^{\infty}(0,\infty))$, and $\varepsilon > 0$

$$\widetilde{T}_{\varepsilon}(F,G)(\theta) = \Sigma_{n,m=0}^{\infty}\phi(\varepsilon n,\varepsilon m)\hat{F}(n)\hat{G}(m)t_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta)t_m^{(\alpha,\beta)}P_m^{(\alpha,\beta)}(\cos\theta)$$

$(F,\ G \in C_c^{\infty}(0,\pi))$.

Let $\alpha = \beta = -1/2$. Then we have that $d\nu(\theta) = d\theta$ is the Lebesgue measure on $[0,\pi)$, $d\mu = dx$ the Lebesgue measure on the real line, $t_n^{(-1/2,-1/2)}P_n^{(-1/2,-1/2)}$

$(\cos\theta) = \sqrt{\frac{\pi}{2}}\cos n\theta,\ \hat{f}(0) = \sqrt{\frac{1}{\pi}}\int_0^\pi f(\theta)\,d\theta,\ \hat{f}(n) = \sqrt{\frac{2}{\pi}}\int_0^\pi f(\theta)\cos n\theta d\theta\ (n = 1, 2, \cdots)$, and $\mathcal{H}_{-1/2}f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(y)\cos xydy$. In this case, the transference theorem about the bilinear operators which we state later is showed by Fan-Sato[8]. In this article, we generalize the transfernce theorem about the bilinear operators in the cases of $\alpha, \beta \geq -\frac{1}{2}$. Here, we state our result in precise:

Theorem. *Let* $1 < p, q, r < \infty$ *with* $1/p = 1/q+1/r,\ \alpha, \beta \geq -1/2$, *and* $\phi(u, v)$ *a bounded continuous function on* $[0, \infty) \times [0, \infty)$. *If there exists a constant* $C > 0$ *such that for any* $\varepsilon > 0$

$$||\widetilde{T}_\varepsilon(F, G)||_{L^p(\nu)} \leq C||F||_{L^q(\nu)}||G||_{L^r(\nu)}\ (F, G \in C_c^\infty(0, \pi)),$$

then there exists a constant $C > 0$ *such that*

$$||T(f, g)||_{L^p(\mu)} \leq C||f||_{L^q(\mu)}||g||_{L^r(\mu)}\ (f, g \in C_c^\infty(0, \infty)).$$

In §3, we will show that T is a bounded bilinear operator from $L^q((0, \infty), \mu) \times L^r((0, \infty), \mu)$ to $L^p((0, \infty), \mu)$, when $\{\widetilde{T}_\varepsilon\}_{\varepsilon>0}$ are uniformly bounded from $L^q((0, \pi), \nu) \times L^r((0, \pi), \nu)$ to $L^p((0, \pi), \nu)$.

Throughout this paper, we may use varying a constant $C = C_{a,b,c,\cdots}$ which depends only on $a, b, c, \cdots$. Also we use the notation $O_{a,b,c,\cdots}(x)$ which means $|\frac{O_{a,b,c,\cdots}(x)}{x}| \leq C_{a,b,c,\cdots}$.

2. Some Lemmas

In this section, we prove some Lemmas for our Theorem, whose essential idea depends on Igari[10]. After that, we will give the proof of Theorem by showing some steps in §3. First for $f, g \in C_c^\infty(0, M)$, let M and ε be positive numbers such that $f, g \in C_c^\infty(0, \infty)$ and $\pi/\varepsilon > M$, and we define $f_\varepsilon(\theta) = f(\theta/\varepsilon)$ and $g_\varepsilon(\theta) = g(\theta/\varepsilon)$. Also for $\phi(u, v)$, let

$$\begin{aligned} &G(\theta/\varepsilon, 1/\varepsilon) \\ &= \sum_{n,m=0}^{\infty} \phi(\varepsilon n, \varepsilon m)\hat{f}_\varepsilon(n)\hat{g}_\varepsilon(m)t_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta)t_m^{(\alpha,\beta)}P_m^{(\alpha,\beta)}(\cos\theta). \end{aligned} \tag{1}$$

Then, by the assumption of Theorem, there exists a constant $C > 0$ such that

$$||G(\theta/\varepsilon, 1/\varepsilon)||_{L^p(\nu)} \leq C||f_\varepsilon||_{L^q(\nu)}||g_\varepsilon||_{L^r(\nu)}. \tag{2}$$

Moreover, by Fatou's lemma and the change of variable $\theta = \varepsilon\tau$,

$$\begin{aligned} &\Big(\int_0^\infty \liminf_{\varepsilon\to 0}|\sum_{n,m=0}^{\infty}\phi(\varepsilon n, \varepsilon m)\hat{f}_\varepsilon(n)\hat{g}_\varepsilon(m)t_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\varepsilon\tau) \\ &\times t_m^{(\alpha,\beta)}P_m^{(\alpha,\beta)}(\cos\varepsilon\tau)|\tau^{2\alpha+1}d\tau\Big)^{1/p} \leq C||f||_{L^q(\mu)}||g||_{L^r(\mu)}. \end{aligned}$$

Also for a fixed number $K > 0$, there exists $\varepsilon_0 > 0$ such that

$$||\chi_{(0,K)}(\tau)G(\tau, 1/\varepsilon)||_{L^p(\mu)} \le C||f||_{L^q(\mu)}||g||_{L^r(\mu)} \tag{3}$$

$(0 < \varepsilon < \varepsilon_0)$, by $\theta = \varepsilon\tau$ the change of variable and simple calculation in (2). Here, we prepare the following which is proved by Stempak[16] and the estimates of Jacobi polynomial[17]. we omit the proof.

Lemma 2.1. (cf. [16; p. 486]). *Let n be a fixed natural number, and $\varepsilon > 0$. Then we have*

(i)

$$\begin{aligned} & t_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta)(\sin\theta/2)^{\alpha+1/2}(\cos\theta/2)^{\beta+1/2} \\ &= (n\theta)^{1/2} J_\alpha(n\theta) + \begin{cases} O(\theta) & \textit{if } (Cn^{-1} \le \theta \le \pi - \varepsilon') \\ O(\theta^{\alpha+1/2} n^{\alpha-1/2}) & \textit{if } (0 < \theta < Cn^{-1}) \end{cases} \end{aligned}$$

(ii)

$$\begin{aligned} & t_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta)(\sin\theta/2)^{2\alpha+1}(\cos\theta/2)^{2\beta+1} \\ &= \sqrt{2n} J_\alpha(n\theta)(\theta/2)^{\alpha+1} + \begin{cases} O(\theta^{\alpha+3/2}) & \textit{if } (Cn^{-1} \le \theta \le \pi - \varepsilon') \\ O(\theta^{2\alpha+1} n^{\alpha-1/2}) & \textit{if } (0 < \theta < Cn^{-1}) \end{cases} \end{aligned}$$

(iii)

$$\begin{aligned} & \varepsilon^\alpha t_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta) \\ &= (2\varepsilon/\theta)^\alpha \sqrt{2n} J_\alpha(n\theta) + \begin{cases} O(\theta^{1/2-\alpha}\varepsilon^\alpha) & \textit{if } (Cn^{-1} \le \theta \le \pi - \varepsilon') \\ O(n^{\alpha-1/2}\varepsilon^\alpha) & \textit{if } (0 < \theta < Cn^{-1}), \end{cases} \end{aligned}$$

where ε' is a fixed number with $0 < \varepsilon' < \pi$ and C is a constant.

Next we show the following:

Lemma 2.2. (cf. [10]). *Let $\varepsilon, \eta,\ K$ be positive numbers with $K\varepsilon < \pi$, and N a fixed natural number. Then there exists $0 < \delta < 1$ such that for any $\eta < \tau < K$,*

$$\begin{aligned} & \sum_{n=0}^{N[1/\varepsilon]} \hat{f}_\varepsilon(n) t_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\varepsilon\tau) \\ = & \sum_{n=0}^{N[1/\varepsilon]} \mathcal{H}_\alpha f(n\varepsilon)\mathcal{J}_\alpha(n\varepsilon\tau)(n\varepsilon)^{2\alpha+1}\varepsilon + O_{f,\eta,K}(N\varepsilon^\delta)\ (0 < \eta < \tau < K), \end{aligned}$$

where $O_{f,\eta,K}$ depends only on f, η, K.

Proof. Let $supp\ f$ be in $[\eta', M'] \subset (0, M)$. then by Lemma2.1 and the change of variable $\theta = \varepsilon\tau$ with the the behavior of Hankel transform and Bessel function at origin and infinity, we have

$$\begin{aligned}&(\varepsilon^{-\alpha}\hat{f}_\varepsilon(n))(\varepsilon^{\alpha}t_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\varepsilon\tau))\\ =\ &(\mathcal{H}_\alpha f(n\varepsilon)\mathcal{J}_\alpha(n\varepsilon\tau)(n\varepsilon)^{2\alpha+1}\varepsilon + O_f(\varepsilon^{5/2}))\\ &\times((2/\tau)^{\alpha}\sqrt{2n}J_\alpha(n\varepsilon\tau) + \begin{cases} O((\varepsilon\tau)^{1/2-\alpha}\varepsilon^{\alpha}) & \text{if } (Cn^{-1} \le \varepsilon\tau \le \pi - \varepsilon') \\ O(n^{\alpha-1/2}\varepsilon^{\alpha}) & \text{if } (0 < \varepsilon\tau < Cn^{-1}), \end{cases}).\end{aligned}$$

Hence, we estimate each part, and obtain the desired result. We omit the details. ■

It is easy to see next result by Lemma2.2.

Lemma 2.3. *Let $\varepsilon, \eta,\ K$ be positive numbers with $K\varepsilon < \pi$, and N a fixed natural number. Then we have the following:*

(i) *for any $0 < \delta < 1$, we have*

$$\begin{aligned}&\sum_{n=1}^{N[1/\varepsilon]} |\hat{f}_\varepsilon(n)t_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\varepsilon\tau)|\\ =&\sum_{n=1}^{N[1/\varepsilon]} |\mathcal{H}_\alpha f(n\varepsilon)\mathcal{J}_\alpha(n\varepsilon\tau)|(n\varepsilon)^{2\alpha+1}\varepsilon + O_{f,\eta,K}(N\varepsilon^{\delta})\ (0<\eta<\tau<K),\end{aligned}$$

where $O_{f,\eta,K}$ depends only on f, η, and K.

(ii) *for any natural number n, and $\tau\ (0 < \eta < \tau < K)$, we have*

$$\begin{aligned}&\hat{f}_\varepsilon(n)t_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\varepsilon\tau)\\ &= \mathcal{H}_\alpha f(n\varepsilon)\mathcal{J}_\alpha(n\varepsilon\tau)(n\varepsilon)^{2\alpha+1}\varepsilon + C_{f,\eta,K}(n,\varepsilon,\tau),\end{aligned}$$

where

$$C_{f,\eta,K}(n,\varepsilon,\tau) = \begin{cases} C_{f,\eta,K}\varepsilon^2 & \textit{if } (Cn^{-1} \le \varepsilon\tau \le \pi-\varepsilon') \\ C_{f,\eta,K}n^{2\alpha}\varepsilon^{2\alpha+2} & \textit{if } (0<\varepsilon\tau<Cn^{-1}), \end{cases},$$

and C, ε' are fixed numbers with $0 < \varepsilon' < \pi$.

3. The Proof of Theorem

We have seven steps for the proof of Theorem. For a natural number N, we define

$$G^N(\theta/\varepsilon, 1/\varepsilon) = \sum_{n,m=0}^{N[1/\varepsilon]} \phi(\varepsilon n, \varepsilon m)\hat{f}_\varepsilon(n)\hat{g}_\varepsilon(m)t_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta)t_m^{(\alpha,\beta)}P_m^{(\alpha,\beta)}(\cos\theta),$$

and

$$H^N(\theta/\varepsilon, 1/\varepsilon) = G(\theta/\varepsilon, 1/\varepsilon) - G^N(\theta/\varepsilon, 1/\varepsilon).$$

Step 1. For any $1 < s < \infty$, there exists a constant $C_{f,\eta,K,s} > 0$ such that

$$(4) \qquad || \sum_{N[1/\varepsilon]}^{\infty} |\hat{f}_\varepsilon(n) t_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\varepsilon\tau)| ||_{L^s([\eta,K],\mu)} \leq C_{f,\eta,K,s} N^{-1/2}.$$

In fact, we estimate

$$I = \int_\eta^K (\sum_{N[1/\varepsilon]+1}^{\infty} |\hat{f}_\varepsilon(n) t_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\varepsilon\tau)|)^s \tau^{2\alpha+1} d\tau.$$

By the change of variable $\tau = \theta/\varepsilon$ and the estimate

$$|t_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta)(\sin\theta/2)^{\alpha+1/2}(\cos\theta/2)^{\beta+1/2}| \leq C$$

(cf.[1]), we have

$$\begin{aligned} I &\leq C\varepsilon^{-(2\alpha+2)} \int_{\varepsilon\eta}^{\varepsilon K} \theta^{(2\alpha+1)(1-s/2)} d\theta \times (\sum_{N[1/\varepsilon]+1}^{\infty} |\hat{f}_\varepsilon(n)|)^s \\ &\leq C_{f,\eta,K,s} \varepsilon^{-(\alpha+1/2)s} (\sum_{N[1/\varepsilon]+1}^{\infty} |\hat{f}_\varepsilon(n)|)^s. \end{aligned}$$

Here, we remark

$$\sum_{N[1/\varepsilon]+1}^{\infty} |\hat{f}_\varepsilon(n)| \leq C N^{-1/2} \varepsilon^{\alpha+1/2},$$

because we have

$$\sum_{N[1/\varepsilon]+1}^{\infty} |\hat{f}_\varepsilon(n)| \leq (\sum_{N[1/\varepsilon]+1}^{\infty} |n\hat{f}_\varepsilon(n)|^2)^{1/2} (\sum_{N[1/\varepsilon]+1}^{\infty} \frac{1}{n^2})^{1/2},$$

and $(\sum_{N[1/\varepsilon]+1}^{\infty} |n\hat{f}_\varepsilon(n)|^2)^{1/2} \leq C\varepsilon^\alpha$ by Igari's method[10;p.203]. Therefore, we have

$$I \leq C\varepsilon^{-(\alpha+1/2)s} \varepsilon^{(\alpha+1/2)s} N^{-s/2} = CN^{-s/2},$$

and get the desired result (4).

Step 2. Let $\varepsilon, \eta,\ K$ be positive numbers with $K\varepsilon < \pi$, and N a fixed natural number. Then there exists $C = C_{f,g,\eta,K} > 0$ and $0 < \delta < 1$ such that

$$(5) \qquad ||H^N(\tau, 1/\varepsilon)||_{L^p([\eta,K],\mu)} \leq C(N^{-1/2} + N^{1/2}\varepsilon^\delta),$$

where C is independent of ε, N.
In fact, we divide $H^N(\tau, 1/\varepsilon)$ into the three parts:

$$\begin{aligned}
&H^N(\tau, 1/\varepsilon)\\
&= \sum_{n=0}^{n[1/\varepsilon]} \sum_{m=N[1/\varepsilon]+1}^{\infty} \phi(\varepsilon n, \varepsilon m) \hat{f}_\varepsilon(n) \hat{g}_\varepsilon(m) t_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\varepsilon\tau) t_m^{(\alpha,\beta)} P_m^{(\alpha,\beta)}(\cos\varepsilon\tau)\\
&+ \sum_{m=0}^{N[1/\varepsilon]} \sum_{n=N[1/\varepsilon]+1}^{\infty} \phi(\varepsilon n, \varepsilon m) \hat{f}_\varepsilon(n) \hat{g}_\varepsilon(m) t_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\varepsilon\tau) t_m^{(\alpha,\beta)} P_m^{(\alpha,\beta)}(\cos\varepsilon\tau)\\
&+ \sum_{n,m=N[1/\varepsilon]+1}^{\infty} \phi(\varepsilon n, \varepsilon m) \hat{f}_\varepsilon(n) \hat{g}_\varepsilon(m) t_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\varepsilon\tau) t_m^{(\alpha,\beta)} P_m^{(\alpha,\beta)}(\cos\varepsilon\tau)\\
&= \textstyle\sum_1 + \sum_2 + \sum_3,
\end{aligned}$$

say.

First we estimate $|| \sum_1 ||^p_{L^p([\eta,K],\mu)}$. By Lemma 2.3(i),

$$\left\| \sum_{n=0}^{N[1/\varepsilon]} \hat{f}_\varepsilon(n) t_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\varepsilon\tau) \right\| \leq C_{f,\eta,K}(1 + N\varepsilon^\delta) \ \ (\eta < \tau < K), \tag{6}$$

and

$$\begin{aligned}
&\|\textstyle\sum_1\|^p_{L^p([\eta,K],\mu)}\\
&\leq C_{f,\eta,K}(1+N\varepsilon^\delta)^p \left\| \sum_{m=N[1/\varepsilon]}^{\infty} \right\| \hat{g}_\varepsilon(m) t_m^{(\alpha,\beta)} P_m^{(\alpha,\beta)}(\cos\varepsilon\tau) |||^p_{L^p([\eta,K],\mu)}\\
&\leq C_{f,g,\eta,K}(1+N\varepsilon^\delta)^p N^{-p/2}
\end{aligned}$$

by (6) and Step 1. Therefore, we obtain

$$\|\textstyle\sum_1\|^p_{L^p([\eta,K],\mu)} \leq C_{f,g,\eta,K}(N^{-1/2} + N^{1/2}\varepsilon^\delta)^p. \tag{7}$$

We similarly get

$$|| \textstyle\sum_2 ||^p_{L^p([\eta,K],\mu)} \leq C_{f,g,\eta,K}(N^{-1/2} + N^{1/2}\varepsilon^\delta)^p. \tag{8}$$

Next we estimate $|| \sum_3 ||_{L^p([\eta,K],\mu)}$. By the Schwarz inequality and Step1, we have that

$$
\begin{aligned}
& || \textstyle\sum_3 ||_{L^p([\eta,K],\mu)} \\
& \le C \int_\eta^K \left(\sum_{N[1/\varepsilon]+1}^{\infty} |\hat{f}_\varepsilon(n) t_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos \varepsilon\tau)| \right)^p \\
& \quad \left(\sum_{N[1/\varepsilon]+1}^{\infty} |\hat{g}_\varepsilon(m) t_m^{(\alpha,\beta)} P_m^{(\alpha,\beta)}(\cos \varepsilon\tau)| \right)^p d\mu \\
& \le C \left\| \sum_{N[1/\varepsilon]+1}^{\infty} |\hat{f}_\varepsilon(n) t_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos \varepsilon\tau)| \right\|_{L^{2p}([\eta,K],\mu)}^p \\
& \times || \sum_{N[1/\varepsilon]+1}^{\infty} |\hat{g}_\varepsilon(n) t_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos \varepsilon\tau)| ||_{L^{2p}([\eta,K],\mu)}^p \\
& = C_{f,g,\eta,K,p} N^{-p}.
\end{aligned}
$$

Then, we obtain

$$
|| \textstyle\sum_3 ||_{L^p([\eta,K],\mu)}^p \le C_{f,g,\eta,K,p} N^{-p}, \tag{9}
$$

and by (7), (8), and (9)

$$
\begin{aligned}
& ||H^N(\tau, 1/\varepsilon)||_{L^p([\eta,K],\mu)} \\
& \le \|\textstyle\sum_1\|_{L^p([\eta,K],\mu)} + || \textstyle\sum_2 ||_{L^p([\eta,K],\mu)} + || \textstyle\sum_3 ||_{L^p([\eta,K],\mu)} \\
& \le C_{f,g,\eta,K}(N^{-1/2} + N^{1/2}\varepsilon^\delta + N^{-1}).
\end{aligned}
$$

Step 3. There exist $\{\varepsilon_j\}$ with $\varepsilon_j \downarrow 0 (j \to \infty)$ and $G(\tau)$ a function such that

$$
G(\tau, 1/\varepsilon_j) \to G(\tau)
$$

in the weak* topology in $L^p((0, K), \mu)$ for all $K > 0$.

In fact, we get the above result by (3) and the diagonal argument.

Step 4. There exists a subsequence $\{\varepsilon_{j_n}\}$ of $\{\varepsilon_j\}$ and $G^N(\tau)$ a function such that

$$
G^N(\tau, 1/\varepsilon_{j_n}) \to G^N(\tau)
$$

in the weak* topology in $L^p([\eta, K], \mu)$ for any N and $0 < \eta < K$.
Because by Step2, there exist $\{\varepsilon_j'\}$ a subsequence of $\{\varepsilon_j\}$ and $H^N(\tau)$ a function such that

$$
H^N(\tau, 1/\varepsilon_{j'}) \to H^N(\tau)
$$

in the weak* topology in $L^p([\eta, K], \mu)$ for all N and $0 < \eta \leq \tau \leq K$, and

$$||H^N||_{L^p([\eta,K],\mu)} \leq C_{f,g,\eta,K} N^{-1/2}. \tag{10}$$

Also by (3) and Step2, we have

$$\begin{aligned} &||G^N(\tau, 1/\varepsilon)\chi_{(\eta,K)}(\tau)||_{L^p(\mu)} \\ \leq\ & ||G(\tau, 1/\varepsilon)\chi_{(\eta,K)}(\tau)||_{L^p(\mu)} + ||H^N(\tau, 1/\varepsilon)\chi_{(\eta,K)}(\tau)||_{L^p(\mu)} \\ \leq\ & C||f||_{L^q(\mu)}||g||_{L^r(\mu)} + C(N^{-1/2} + N^{1/2}\varepsilon^{\delta}), \end{aligned}$$

since $G(\tau, 1/\varepsilon) = G^N(\tau, 1/\varepsilon) + H^N(\tau, 1/\varepsilon)$. So we obtain

$$||G^N(\tau, 1/\varepsilon)\chi_{(\eta,K)}||_{L^p(\mu)} \leq C_{N,f,g,\eta,K}.$$

Then there exist a subsequence $\{\varepsilon_{j_n}\}$ of $\{\varepsilon_j\}$ and $G^N(\tau)$ a function such that

$$G^N(\tau, 1/\varepsilon_{j_n}) \to G^N(\tau)$$

in the weak* topology in $L^p([\eta, K], \mu)$ for all $0 < \eta < K$ and $G = G^N + H^N$.

Step 5. For a fixed $\tau > 0$, we have

$$\begin{aligned} &\lim_{\varepsilon\to 0} G^N(\tau, 1/\varepsilon) \\ =\ & \int_0^N \int_0^N \phi(u,v)\mathcal{H}_\alpha f(u)\mathcal{J}_\alpha(\tau u)\mathcal{H}_\alpha g(v)\mathcal{J}_\alpha(\tau v)d\mu(u)d\mu(v) (= G^N(\tau)). \end{aligned}$$

In fact, since we have

$$G^N(\tau, 1/\varepsilon) = \sum_{n,m=0}^{\infty} \phi(\varepsilon n, \varepsilon m)\hat{f}_\varepsilon(n)\hat{g}_\varepsilon(m)t_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\varepsilon\tau)t_m^{(\alpha,\beta)}P_m^{(\alpha,\beta)}(\cos\varepsilon\tau)$$

and Lemma2.3(ii), we obtain

$$\begin{aligned} & G^N(\tau, 1/\varepsilon) \\ =\ & \sum_{n,m=0}^{N[1/\varepsilon]} \phi(\varepsilon n, \varepsilon m)\mathcal{H}_\alpha f(n\varepsilon)\mathcal{J}_\alpha(n\varepsilon)(n\varepsilon)^{2\alpha+1}\varepsilon\mathcal{H}_\alpha g(n\varepsilon)\mathcal{J}_\alpha(n\varepsilon)(n\varepsilon)^{2\alpha+1}\varepsilon \\ +\ & \sum_{n,m=1}^{N[1/\varepsilon]} \phi(\varepsilon n, \varepsilon m)\mathcal{H}_\alpha f(n\varepsilon)\mathcal{J}_\alpha(n\varepsilon)(n\varepsilon)^{2\alpha+1}\varepsilon C_{g,\eta,K}(m,\varepsilon,\tau) \\ +\ & \sum_{n,m=1}^{N[1/\varepsilon]} \phi(\varepsilon n, \varepsilon m)\mathcal{H}_\alpha g(m\varepsilon)\mathcal{J}_\alpha(m\varepsilon)(m\varepsilon)^{2\alpha+1}\varepsilon C_{f,\eta,K}(n,\varepsilon,\tau) \\ +\ & \sum_{n,m=1}^{N[1/\varepsilon]} \phi(\varepsilon n, \varepsilon m)C_{f,\eta,K}(n,\varepsilon,\tau)C_{g,\eta,K}(m,\varepsilon,\tau) + O_{f,g,\eta,K}(N\varepsilon) \\ =\ & I_1 + I_2 + I_3 + I_4 + O_{f,g,\eta,K}(N\varepsilon), \text{ say.} \end{aligned}$$

Then, by the definition of the Riemann integral we obtain that

$$\lim_{\varepsilon\to 0} I_1 = \int_0^N \int_0^N \phi(u,v)\mathcal{H}_\alpha f(u)\mathcal{J}_\alpha(\tau u)\mathcal{H}_\alpha g(v)\mathcal{J}_\alpha(\tau v)d\mu(u)d\mu(v).$$

Also by Lemma 2.3(ii) we get that $I_2 = I_3 = O_{f,g,\eta,K}(N\varepsilon^\delta)$, and $I_4 = O_{f,g,\eta,K}((N\varepsilon^\delta)^2)$ $(0<\delta<1)$. Then, we have

$$\lim_{\varepsilon\to 0} G^N(\tau, 1/\varepsilon) = \int\int_{(0,N)\times(0,N)} \phi(u,v)\mathcal{H}_\alpha f(u)\mathcal{J}_\alpha(\tau u)\mathcal{H}_\alpha g(v)\mathcal{J}_\alpha(\tau v)d\mu(u)d\mu(v).$$

Step 6. We have

$$G(\tau) = \int_0^\infty \int_0^\infty \phi(u,v)\mathcal{H}_\alpha f(u)\mathcal{J}_\alpha(\tau u)\mathcal{H}_\alpha g(v)\mathcal{J}_\alpha(\tau v)d\mu(u)\mu(v)\ a.e.\ \tau.$$

In fact, let $0<\eta'<K'$ be fixed. Since $\{|G^N(\tau,1/\varepsilon)|\}_\varepsilon$ is uniformly bounded on $[\eta',K']$ by (6), we have

$$\lim_{k\to\infty}\int G^N(\tau,1/\varepsilon_{j_k})h(\tau)d\mu(\tau) = \int G^N(\tau)h(\tau)d\mu(\tau)$$

for any $h\in C_c^\infty(\eta',K')$. Then by the Lebesgue' s convergence theorem and Step5 we have

$$\begin{aligned}&\int G^N(\tau)h(\tau)d\mu(\tau)\\ =&\int_0^\infty\{\int_0^N\int_0^N \phi(u,v)\mathcal{H}_\alpha f(u)\mathcal{J}_\alpha(\tau u)\mathcal{H}_\alpha g(v)\mathcal{J}_\alpha(\tau v)d\mu(u)\mu(v)\}h(\tau)d\mu(\tau)\end{aligned}$$

for any $h\in C_c^\infty(0,\infty)$, and

$$G^N(\tau) = \int_0^N\int_0^N \phi(u,v)\mathcal{H}_\alpha f(u)\mathcal{J}_\alpha(\tau u)\mathcal{H}_\alpha g(v)\mathcal{J}_\alpha(\tau v)d\mu(u)\mu(v)\ a.e.. \tag{11}$$

Here, there exists $\{N_j\}_j$ such that $H^{N_j}(\tau)\to 0$ $a.e.$ τ by (10). Therefore, we remark

$$G^{N_j}(\tau)\to G(\tau)\ a.e.\ \tau, \tag{12}$$

since we have $G = G^N + H^N$.
Now for any natural number N, let

$$F_N(u,v) = \phi(u,v)\mathcal{H}_\alpha f(u)\mathcal{J}_\alpha g(v)\mathcal{J}_\alpha(\tau v)\chi_{(0,N)\times(0,N)}(u,v).$$

It is known that

$$\begin{aligned}\mathcal{H}_\alpha f(x) &= O(x^{-\ell})\ (\ell = 1, 2, \cdots)(x \to \infty),\\ \mathcal{H}_\alpha f(x) &= O(1)\ (x \to 0),\\ J_\alpha f(x) &= O(x^{\alpha})\ (x \to 0),\ \text{and}\\ J_\alpha f(x) &= O(x^{-1/2})\ (x \to \infty).\end{aligned}$$

By those properties, there exists $F \in L^1((0,\infty)\times(0,\infty), \mu\times\mu)$ such that $|F_N| \le F\ (0 < \tau < K)$.Then, by Lebesgue's convergence theorem and (11), we have

$$\begin{aligned}&\lim_{j\to\infty} G^{N_j}(\tau)\\ &= \int\int_{(0,\infty)\times(0,\infty)} \phi(u,v)\mathcal{H}_\alpha f(u)\mathcal{J}_\alpha(\tau u)\mathcal{H}_\alpha g(v)\mathcal{J}_\alpha(\tau v)d\mu(u)\mu(v).\end{aligned} \tag{13}$$

After all, we have

$$G(\tau) = \int\int_{(0,\infty)\times(0,\infty)} \phi(u,v)\mathcal{H}_\alpha f(u)\mathcal{J}_\alpha(\tau u)\mathcal{H}_\alpha g(v)\mathcal{J}_\alpha(\tau v)d\mu(u)\mu(v)\ a.e.\ \tau$$

by (11), (12), and (13).

Step 7. This step completes the proof of Theorem. For the sake of this, we use all notations in the former steps. Let $h \in C_c^\infty(0,\infty)$ with $supp\ h \subset [\eta, K]\ (0 < \eta < K)$. By Step2 and Step4, we have

$$\begin{aligned}&\left\|\int_\eta^K G^N(\tau, 1/\varepsilon)h(\tau)d\mu(\tau)\right\|\\ &\le C||f||_{L^q(\mu)}||g||_{L^r(\mu)}||h||_{L^{p'}(\mu)} + ||H^N(\tau,1/\varepsilon)||_{L^p((0,K),d\mu)}||h||_{L^{p'}((0,K),d\mu)}\\ &\le C||f||_{L^q(\mu)}||g||_{L^r(\mu)}||h||_{L^{p'}(\mu)} + C_{\eta,K,f,g}(N^{-1/2} + N^{1/2}\varepsilon^{\delta})||h||_{L^{p'}((0,K),d\mu)}.\end{aligned}$$

Then, by $\varepsilon \to 0$ and Steps 5 and 6, we have

$$\left|\int_\eta^K G^N(\tau)h(\tau)d\mu(\tau)\right| \le C||f||_{L^q(\mu)}||g||_{L^r(\mu)}||h||_{L^{p'}(\mu)} + O_{f,g,\eta,K}(N^{-1/2}),$$

and

$$\left|\int_\eta^K G(\tau)h(\tau)d\mu(\tau)\right| \le C||f||_{L^q(\mu)}||g||_{L^r(\mu)}||h||_{L^{p'}(\mu)}$$

by $N = N_j \to \infty (j \to \infty)$. Therefore, we obtain

$$||G||_{L^p(\mu)} \le C||f||_{L^q(\mu)}||g||_{L^r(\mu)},$$

and

$$||T(f,g)||_{L^p(\mu)} \leq C||f||_{L^q(\mu)}||g||_{L^r(\mu)}.$$

We finish our proof. ■

Remark. The reverse transference Theorem of Igari's Theorem[10] is an unsolved problem, since Igari[10] proved the transference theorem. The converse transference Theorem of our Theorem is not known, too.

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