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GLOBAL APPROXIMATION RESULTS FOR MODIFIED SZÁSZ-MIRAKJAN OPERATORS

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Abstract. This study is the continuation of our earlier works [3, 4]. Here, we mainly investigate the global approximation behavior of modified Szász-Mirakjan operators presented in the papers mentioned above.

1. Introduction

The classical Szász-Mirakjan operators are defined by

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!} \quad \text{for } x \geq 0,$$

and their modified versions has been constructed by the authors (see [3, 4]) as follows:

$$D_n(f;x) = e^{-nu_n(x)} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nu_n(x))^k}{k!} \quad \text{for } x \geq 0, \tag{1.1}$$

where $\{u_n(x)\}$ is a sequence of continuous non-negative real-valued functions on $[0,\infty)$. In [3], various uniform and pointwise approximation properties and a Voronovskaja-type theorem were obtained on the space

$$E := \left\{ f \in C[0,\infty) : \lim_{x\to+\infty} \frac{f(x)}{1+x^2} < \infty \right\}$$

endowed with the norm $\|f\|_* := \sup_{x\in[0,\infty)} \frac{f(x)}{1+x^2}$, and also on the space

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$$E^{*}:=\left\{f\in C[0,\infty):\lim_{x\to\infty}f(x)\text{ exists}\right\}$$

endowed with usual supremum norm on $[0,\infty)$ by taking the sequence $\{u_n^*(x)\}$ instead of $\{u_n(x)\}$, which is defined by

$$u_n^*(x)=\frac{-1+\sqrt{1+4n^2x^2}}{2n}\quad \text{for } x\geq 0 \text{ and } n\in\mathbb{N}:=\{1,2,...\}. \tag{1.2}$$

In this case, the corresponding modified Szász-Mirakjan operators were denoted by $D_n^*(f;x)$, i.e.,

$$D_n^*(f;x)=e^{-nu_n^*(x)}\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)\frac{(nu_n^*(x))^k}{k!}, \tag{1.3}$$

where the function sequence $\{u_n^*(x)\}$ is given by (1.2). Furthermore, in [4], some local approximation results are obtained for the operators D_n^* defined by (1.3). Recall that the operator D_n^* provides a better error estimation than the classical Szász-Mirakjan operators (see [3]).

In this paper, assuming

$$u_n(0)=0 \text{ and } 0<u_n(x)\leq x \text{ for } x>0 \text{ and } n\in\mathbb{N}, \tag{1.4}$$

we obtain global approximation results for the operators D_n given by (1.1) on an appropriate weighted space mentioned below. We should recall that such global approximations were established for the Bernstein polynomials by Lorentz [5] and for the Szász-Mirakjan and the Baskakov operators by Becker [1].

Let $p\in\mathbb{N}_0:=\{0,1,...\}$ and define the weight function μ_p as follows:

$$\mu_0(x):=1 \text{ and } \mu_p(x):=\frac{1}{1+x^p}\quad \text{for } x\geq 0 \text{ and } p\in\mathbb{N}_0. \tag{1.5}$$

Then, we consider the following (weighted) subspace $C_p[0,\infty)$ of $C[0,\infty)$ generated by μ_p :

$$C_p[0,\infty):=\{f\in C[0,\infty):\mu_p f \text{ is uniformly continuous and bounded on } [0,\infty)\}$$

endowed with the norm

$$\|f\|_p:=\sup_{x\in[0,\infty)}\mu_p(x)\,|f(x)|\quad \text{for}\quad f\in C_p[0,\infty).$$

In this case, we will need the following Lipschitz classes:

$$\begin{aligned}
\Delta_h^2 f(x) &:= f(x+2h)-2f(x+h)+f(x),\\
\omega_p^2(f,\delta) &:= \sup_{h\in(0,\delta]}\left\|\Delta_h^2 f\right\|_p,\\
\omega_p^1(f,\delta) &:= \sup\{\mu_p(x)\,|f(t)-f(x)|:|t-x|\leq\delta \text{ and } t,x\geq 0\}\\
Lip_p^2\alpha &:= \left\{f\in C_p[0,\infty):\omega_p^2(f;\delta)=O(\delta^{\alpha}) \text{ as } \delta\to 0^+\right\},
\end{aligned}$$

where $h > 0$ and $0 < \alpha \leq 2$.

With this terminology, we obtain the following main result, which gives the global approximation behavior of the operators D_n.

Theorem 1.1. *Let D_n be given by (1.1) and (1.4). Assume that $u_n'(x)$ exists and $u_n' \neq 0$ on $[0,\infty)$. Then, for every $p \in \mathbb{N}_0$, $n \in \mathbb{N}$, $f \in C_p[0,\infty)$ and $x \in [0,\infty)$, there exists an absolute constant $M_p > 0$ such that*

$$\mu_p(x)\left|D_n\left(f;x\right)-f(x)\right| \leq M_p\omega_p^2\left(f,\sqrt{(u_n(x)-x)^2+\frac{u_n(x)}{n}}\right) \\ +\omega_p^1(f;x-u_n(x)),$$

where μ_p is the same as in (1.5). Particularly, if $f \in Lip_p^2\alpha$ for some $\alpha \in (0,2]$, then

$$\mu_p(x)\left|D_n\left(f(t);x\right)-f(x)\right| \leq M_p\left((u_n(x)-x)^2+\frac{u_n(x)}{n}\right)^{\frac{\alpha}{2}} \\ +\omega_p^1(f;x-u_n(x))$$

holds.

Remark. If the sequence $\{u_n(x)\}$ in (1.4) also satisfies

(1.6) $$\lim_{n\to\infty} u_n(x) = x \quad \text{for every } x \in [0,\infty),$$

then it follows from Theorem 1.1 that

$$\lim_{n\to\infty} \mu_p(x)\left|D_n\left(f;x\right)-f(x)\right| = 0 \quad \text{for every } x \in [0,\infty)$$

holds true provided that $f \in C_p[0,\infty)$ or $f \in Lip_p^2\alpha$ for some $\alpha \in (0,2]$. Furthermore, we will see that our operators D_n map $C_p[0,\infty)$ into itself (see Lemma 2.5 in the second section). Hence, if the convergence in (1.6) is uniform on $[0,\infty)$, then we have

$$\lim_{n\to\infty} \|D_nf-f\|_p = 0.$$

2. Auxiliary Results

In this section, we will get some lemmas which are quite effective in proving our Theorem 1.1.

Lemma 2.1. *Let $\{u_n(x)\}$ be a sequence of continuous positive valued functions on $[0,\infty)$. If $u_n'(x)$ exists and $u_n' \neq 0$ on $[0,\infty)$, then we get, for every $x \in [0,\infty)$, $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$, that*

$$\left(\frac{k}{n} - x\right) p_{n,k}(x) = \frac{u_n(x)}{nu_n'(x)} p_{n,k}'(x) + (u_n(x) - x)\, p_{n,k}(x), \tag{2.1}$$

where

$$p_{n,k}(x) := e^{-nu_n(x)} \frac{(nu_n(x))^k}{k!}.$$

Proof. It is easy to see that

$$p_{n,k}'(x) = p_{n,k}(x) \left(\frac{ku_n'(x)}{u_n(x)} - nu_n'(x)\right),$$

or

$$\frac{u_n(x)}{nu_n'(x)} p_{n,k}'(x) = p_{n,k}(x) \left(\frac{k}{n} - u_n(x)\right),$$

whence the result. ∎

Lemma 2.2. *Let $\{D_n\}$ be given by (1.1) and (1.4), and let $\varphi(y) := y - x$ for each $x \in [0,\infty)$. Assume that $u_n'(x)$ exists and $u_n' \neq 0$ on $[0,\infty)$. Then, we have, for each $x \in [0,\infty)$ and $n \in \mathbb{N}$, that*

(*i*) $D_n(\varphi^0; x) = 1,$

(*ii*) $D_n(\varphi^1; x) = u_n(x) - x,$

(*iii*) $D_n(\varphi^2; x) = (u_n(x) - x)^2 + \dfrac{u_n(x)}{n},$

(*iv*) *for each $r \in \mathbb{N}_0$, the following recurrence formula holds:*

$$\begin{aligned} D_n(\varphi^{r+1}; x) = {} & \frac{u_n(x)}{nu_n'(x)} \left\{D_n'(\varphi^r; x) + rD_n(\varphi^{r-1}; x)\right\} \\ & +(u_n(x) - x) D_n(\varphi^r; x), \end{aligned} \tag{2.2}$$

where $D(\varphi^{-1}; x) := 1$.

Proof. (i), (ii) and (iii) immediately follow from Lemma 3.1 of [3]. So, we only prove (iv). By (1.1), we can directly show that

$$D_n'(\varphi^r; x) = \frac{d}{dx} \left\{ \sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right)^r p_{n,k}(x) \right\}$$

$$= -r\sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^{r-1}p_{n,k}(x)+\sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^{r}p'_{n,k}(x),$$
$$= -rD_n(\varphi^{r-1};x)+\sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^{r}p'_{n,k}(x)$$

and hence

$$\frac{u_n(x)}{nu'_n(x)}\left\{D'_n(\varphi^r;x)+rD_n(\varphi^{r-1};x)\right\}=\sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^{r}\frac{u_n(x)}{nu'_n(x)}p'_{n,k}(x).$$

Now using (2.1), we get

$$\frac{u_n(x)}{nu'_n(x)}\left\{D'_n(\varphi^r;x)+rD_n(\varphi^{r-1};x)\right\}=D_n(\varphi^{r+1};x)-(u_n(x)-x)D_n(\varphi^r;x),$$

which completes the proof. ∎

Now we use the test functions $e_r(y)=y^r$ for $r\in\mathbb{N}_0$. Then, we obtain the following result.

Lemma 2.3. *Let* $\{D_n\}$ *be given by* (1.1) *and* (1.4). *Assume that* $u'_n(x)$ *exists and* $u'_n\neq 0$ *on* $[0,\infty)$. *Then, we have, for each* $x\in[0,\infty)$ *and* $n\in\mathbb{N}$, *that*

(i) $D_n(e_0;x)=e_0(x)$,

(ii) $D_n(e_1;x)=u_n(x)$,

(iii) $D_n(e_2;x)=u_n^2(x)+\dfrac{u_n(x)}{n}$,

(iv) *for each* $r\in\mathbb{N}$, *the following recurrence formula holds:*

$$D_n(e_{r+1};x)=\frac{u_n(x)}{nu'_n(x)}D'_n(e_r;x)+u_n(x)D_n(e_r;x). \tag{2.3}$$

Proof. As in the proof of Lemma 2.2, it is enough to prove (iv). Since

$$D'_n(e_r;x)=\sum_{k=0}^{\infty}\left(\frac{k}{n}\right)^{r}p'_{n,k}(x),$$

it follows from (2.1) that

$$\begin{aligned}\frac{u_n(x)}{nu'_n(x)}D'_n(e_r;x) &= \sum_{k=0}^{\infty}\left(\frac{k}{n}\right)^{r}\left(\frac{k}{n}-x\right)p_{n,k}(x)\\ &\quad -(u_n(x)-x)D_n(e_r;x)\\ &= D_n(e_{r+1};x)-xD_n(e_r;x)\\ &\quad -(u_n(x)-x)D_n(e_r;x),\end{aligned}$$

which gives (2.3). ∎

Furthermore, using Lemmas 2.1-2.3, one can get the next result by an induction.

Lemma 2.4. *Let $\{D_n\}$ be given by (1.1) and (1.4). Assume that $u_n'(x)$ exists and $u_n' \neq 0$ on $[0,\infty)$. Then, we have, for each $x \in [0,\infty)$ and $r, n \in \mathbb{N}$, that*

$$
\begin{aligned}
D_n(e_r;x) &= \sum_{j=1}^{r} b_{r,j} u_n^j(x) n^{j-r} : \\
&= u_n^r(x) + \frac{r(r-1)}{2n} u_n^{r-1}(x) + \ldots + n^{1-r} u_n(x),
\end{aligned}
\tag{2.4}
$$

where $b_{j,r}$'s are positive coefficients.

Lemma 2.5. *Let $\{D_n\}$ be given by (1.1) and (1.4). Assume that $u_n'(x)$ exists and $u_n' \neq 0$ on $[0,\infty)$. Then, for each $p \in \mathbb{N}_0$, there exists a constant M_p such that*

$$
\mu_p(x) D_n\left(\frac{1}{\mu_p};x\right) \leq M_p \tag{2.5}
$$

holds for all $x \in [0,\infty)$ and $n \in \mathbb{N}$, where μ_p is given by (1.5). Moreover, for all $f \in C_p[0,\infty)$, we have

$$
\|D_n(f)\|_p \leq M_p \|f\|_p . \tag{2.6}
$$

Proof. By (1.5) and (2.4), we get

$$
\begin{aligned}
\mu_p(x) D_n\left(\frac{1}{\mu_p};x\right) &= \mu_p(x)\left\{D_n(e_0;x) + D_n(e_p;x)\right\} \\
&= \mu_p(x)\left\{1 + u_n^p(x) + \frac{p(p-1)}{2n} u_n^{p-1}(x) + \ldots + n^{1-p} u_n(x)\right\} \\
&\leq \mu_p(x)\left\{1 + x^p + \frac{p(p-1)}{2n} x^{p-1} + \ldots + \frac{1}{n^{p-1}} x\right\}
\end{aligned}
$$

Then, using (1.4), we obtain that

$$
\mu_p(x) D_n\left(\frac{1}{\mu_p};x\right) \leq \mu_p(x)\left\{1 + x^p + \frac{p(p-1)}{2n} x^{p-1} + \ldots + \frac{1}{n^{p-1}} x\right\} \tag{2.7}
$$

Now, we can find a constant C_p depending on p such that

$$
\frac{1}{1+x^p} \leq C_p,\ \frac{x^p}{1+x^p} \leq C_p,\ \frac{p(p-1)x^{p-1}}{2n(1+x^p)} \leq C_p, \ldots, \frac{x}{n^{p-1}(1+x^p)} \leq C_p
$$

holds for every $x \in [0, \infty)$, $p \in \mathbb{N}_0$ and $n \in \mathbb{N}$. So, letting $M_p := (p+1)C_p$, we get from (2.7) that

$$\mu_p(x) D_n \left(\frac{1}{\mu_p}; x \right) \leq M_p,$$

which gives (2.5). On the other hand, for all $f \in C_p[0, \infty)$ and every $x \in [0, \infty)$, it follows from (2.5) that

$$\begin{aligned} \mu_p(x) \left| D_n(f; x) \right| &\leq \mu_p(x) e^{-nu_n(x)} \sum_{k=0}^{\infty} \frac{\mu\left(\frac{k}{n}\right) \left| f\left(\frac{k}{n}\right) \right|}{\mu\left(\frac{k}{n}\right)} \frac{(nu_n(x))^k}{k!} \\ &\leq \|f\|_p \, \mu_p(x) D_n \left(\frac{1}{\mu_p}; x \right) \\ &\leq M_p \, \|f\|_p \, . \end{aligned}$$

Now taking supremum over $x \in [0, \infty)$, the last inequality implies (2.6). ■

Remark. We easily see from (2.6) that our operators D_n map $C_p[0, \infty)$ into itself.

Lemma 2.6. *Let* $\{D_n\}$ *be given by* (1.1) *and* (1.4). *Assume that* $u_n'(x)$ *exists and* $u_n' \neq 0$ *on* $[0, \infty)$. *Then, for each* $p \in \mathbb{N}_0$, *there exists a constant* M_p *such that, for all* $x \in [0, \infty)$ *and* $n \in \mathbb{N}$, *we have*

$$\mu_p(x) D_n \left(\frac{\varphi^2}{\mu_p}; x \right) \leq M_p \left\{ (u_n(x) - x)^2 + \frac{u_n(x)}{n} \right\}, \tag{2.8}$$

where $\varphi(y) = y - x$ *as stated before.*

Proof. For $p = 0$, it is (iii) of Lemma 2.2. Now consider the case $p = 1$. By (2.2), (1.4), and (iii) of Lemma 2.2, we see that

$$\begin{aligned} D_n(\varphi^3; x) &= \frac{u_n(x)}{nu_n'(x)} \left\{ D_n'(\varphi^2; x) + 2D_n(\varphi; x) \right\} + (u_n(x) - x) D_n(\varphi^2; x) \\ &= \frac{u_n(x)}{nu_n'(x)} \left\{ 2(u_n(x) - x) \left(u_n'(x) - 1 \right) + \frac{u_n'(x)}{n} + 2(u_n(x) - x) \right\} \\ &\quad + (u_n(x) - x) \left\{ (u_n(x) - x)^2 + \frac{u_n(x)}{n} \right\} \\ &= \frac{u_n(x)}{n^2} + 3(u_n(x) - x) \frac{u_n(x)}{n} + (u_n(x) - x)^3 \\ &\leq \frac{u_n(x)}{n^2}, \end{aligned}$$

and hence

$$\begin{aligned}
\mu_1(x)D_n\left(\frac{\varphi^2}{\mu_1};x\right) &= \mu_1(x)\left\{(1+x)\,D_n(\varphi^2;x)+D_n(\varphi^3;x)\right\}\\
&\le (u_n(x)-x)^2+\frac{u_n(x)}{n}+\frac{u_n(x)}{(1+x)n^2}\\
&\le (u_n(x)-x)^2+\frac{2u_n(x)}{n}\\
&\le 2\left\{(u_n(x)-x)^2+\frac{u_n(x)}{n}\right\},
\end{aligned}$$

which shows (2.8) with $M_1=2$ for $p=1$. Finally, assume that $p\ge 2$. Then, we obtain from (2.4) that

$$\begin{aligned}
D_n\left(\varphi^2\cdot e_p;x\right) &= D_n\left(e_{p+2};x\right)-2xD_n\left(e_{p+1};x\right)+x^2D_n\left(e_p;x\right)\\
&= u_n^{p+2}(x)+\frac{(p+2)\,(p+1)}{2n}u_n^{p+1}(x)+\ldots+\frac{u_n(x)}{n^{p+1}}\\
&\quad -2x\left(u_n^{p+1}(x)+\frac{p(p+1)}{2n}u_n^{p}(x)+\ldots+\frac{u_n(x)}{n^{p}}\right)\\
&\quad +x^2\left(u_n^{p}(x)+\frac{p(p-1)}{2n}u_n^{p-1}(x)+\ldots+\frac{u_n(x)}{n^{p-1}}\right)\\
&= (u_n(x)-x)^2u_n^p(x)+\frac{u_n(x)}{n}\left\{\frac{(p+2)\,(p+1)}{2}u_n^p(x)\right.\\
&\quad \left.-\frac{p(p+1)}{2}xu_n^{p-1}(x)+\frac{p(p-1)}{2}x^2u_n^{p-2}(x)+\ldots+\frac{1}{n^{p-2}}\right\}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\mu_p(x)D_n\left(\frac{\varphi^2}{\mu_p};x\right) &= \mu_p(x)(u_n(x)-x)^2\left(1+u_n^p(x)\right)\\
&\quad +\frac{\mu_p(x)u_n(x)}{n}\left\{\frac{1}{1+x^p}+\frac{(p+2)\,(p+1)}{2}u_n^p(x)\right.\\
&\quad \left.-\frac{p(p+1)}{2}xu_n^{p-1}(x)+\frac{p(p-1)}{2}x^2u_n^{p-2}(x)+\ldots+\frac{1}{n^{p-2}}\right\}\\
&\le (u_n(x)-x)^2+\frac{u_n(x)}{n}\left\{1+\frac{(p+2)\,(p+1)}{2}\frac{x^p}{1+x^p}\right.\\
&\quad \left.+\frac{p(p+1)}{2}\frac{x^p}{1+x^p}+\frac{|p(p-1)|}{2}\frac{x^p}{1+x^p}+\ldots+\frac{1}{n^{p-2}}\right\}.
\end{aligned}$$

Now we can find a positive constant M_p depending on p such that

$$\mu_p(x)D_n\left(\frac{\varphi^2}{\mu_p};x\right)\le M_p\left\{(u_n(x)-x)^2+\frac{u_n(x)}{n}\right\}$$

holds. Lemma is proved. ■

Now, for $p \in \mathbb{N}$, consider the space

$$C_p^2[0,\infty) := \left\{ f \in C_p[0,\infty) : f'' \in C_p[0,\infty) \right\}$$

Lemma 2.7. *Let* $\{D_n\}$ *be given by* (1.1) *and* (1.4), *and let* $g \in C_p^2[0,\infty)$. *Assume that* $u_n'(x)$ *exists and* $u_n' \neq 0$ *on* $[0,\infty)$. *If* $\Omega_n(f;x) := D_n(f;x) - f(u_n(x)) + f(x)$, *there exists a constant* M_p *such that, for all* $x \in [0,\infty)$ *and* $n \in \mathbb{N}$, *we have*

$$\mu_p(x) \left| \Omega_n\left(g;x\right) - g(x) \right| \le M_p \left\| g'' \right\|_p \left\{ (u_n(x)-x)^2 + \frac{u_n(x)}{n} \right\}. \tag{2.9}$$

Proof. By Lemma 2.2 (ii) we have $\Omega_n(\varphi;x) = 0$ with $\varphi(y) = y - x$. Using the expression

$$g(y) - g(x) = (y-x)g'(x) + \int_x^y (y-t)g''(t)dt \quad \text{for} \quad y \in [0,\infty),$$

we get

$$\begin{aligned} &\left| \Omega_n\left(g;x\right) - g(x) \right| \\ &= D_n\left(\left| \int_x^y (y-t)g''(t)dt \right|; x \right) + \left| \int_{u_n(x)}^x (u_n(x)-t)g''(t)dt \right|. \end{aligned} \tag{2.10}$$

Since

$$\left| \int_x^y (y-t)g''(t)dt \right| \le \frac{\|g''\|_p \, \varphi^2(y)}{2} \left(\frac{1}{\mu_p(x)} + \frac{1}{\mu_p(y)} \right)$$

and

$$\left| \int_{u_n(x)}^x (u_n(x)-t)g''(t)dt \right| \le \frac{\|g''\|_p \, (u_n(x)-x)^2}{2\mu_p(x)},$$

we obtain from Lemma 2.2 (iii), (2.8) and (2.10) that

$$\begin{aligned} \mu_p(x) \left| \Omega_n\left(g;x\right) - g(x) \right| \le\ & \frac{\|g''\|_p}{2} \left\{ D_n\left(\varphi^2;x\right) + D_n\left(\frac{\varphi^2}{\mu_p}; x \right) \right\} \\ & + \frac{\|g''\|_p}{2} \left(u_n(x)-x\right)^2 \\ \le\ & M_p \left\| g'' \right\|_p \left\{ (u_n(x)-x)^2 + \frac{u_n(x)}{n} \right\}, \end{aligned}$$

whence the result. ■

3. The Proof of Theorem 1.1

We first consider the modified Steklov means (see [1, 2]) of a function $f \in C_p[0,\infty)$ as follows:

$$f_h(x) := \frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}\left\{2f(x+s+t)-f(x+2(s+t))\right\}dsdt \text{ for } h>0 \text{ and } x \geq 0.$$

In this case, it is clear that

$$f(y)-f_h(y) = \frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}\Delta_{s+t}^2 f(y)dsdt,$$

which guarantees that

$$\|f-f_h\|_p \leq \omega_p^2(f;h). \tag{3.1}$$

Furthermore, we have

$$f_h''(x) = \frac{1}{h^2}\left(8\Delta_{h/2}^2 f(x) - \Delta_h^2 f(x)\right),$$

which implies

$$\left\|f_h''\right\|_p \leq \frac{9}{h^2}\omega_p^2(f;h). \tag{3.2}$$

Then, combining (3.1) with (3.2) we conclude that the Steklov means f_h corresponding to $f \in C_p[0,\infty)$ belongs to $C_p^2[0,\infty)$.

Now we are ready to prove our Theorem 1.1.

Proof of Theorem 1.1. For $x=0$, the proof immediately follows from the fact that $u_n(0)=0$. Now let $p\in\mathbb{N}_0$, $f\in C_p[0,\infty)$ and $x\in(0,\infty)$ be fixed. Assume that, for $h>0$, f_h denotes the Steklov means of f. For any $n\in\mathbb{N}$, the following inequality holds:

$$\begin{aligned}|D_n(f;x)-f(x)| \leq\ & \Omega_n\left(|f(y)-f_h(y)|;x\right)+|f(x)-f_h(x)| \\ & +|\Omega_n(f_h;x)-f_h(x)|+|f(u_n(x))-f(x)|.\end{aligned}$$

Since $f_h\in C_p^2[0,\infty)$, it follows from (2.9) and (3.1) that

$$\mu_p(x)\left|D_n(f;x)-f(x)\right| \le \|f-f_h\|_p\left\{\mu_p(x)\Omega_n\left(\frac{1}{\mu_p};x\right)+1\right\}$$
$$+M_p\left\|f_h''\right\|_p\left\{(u_n(x)-x)^2+\frac{u_n(x)}{n}\right\}$$
$$+\mu_p(x)\left|f(u_n(x))-f(x)\right|.$$

By (2.5), (3.1) and (3.2), the last inequality yields that

$$\mu_p(x)\left|D_n(f;x)-f(x)\right| \le M_p\omega_p^2(f;h)\left\{1+\frac{1}{h^2}\left((u_n(x)-x)^2+\frac{u_n(x)}{n}\right)\right\}$$
$$+\omega_p^1(f;x-u_n(x)).$$

Thus, choosing $h=\sqrt{(u_n(x)-x)^2+\dfrac{u_n(x)}{n}}$, the proof is completed.

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