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SOLUTIONS OF A CLASS OF N-TH ORDER ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS VIA FRACTIONAL CALCULUS†*

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Abstract. Solutions of the n-th order linear ordinary differential equations

$$(z+b)^l \prod_{k=1}^{n-l}(z+a_k)\varphi_n + \sum_{k=1}^{n} \varphi_{n-k}\{C_k^\lambda\{Q(z)\}_k + C_{k-1}^\lambda\{G(z)\}_{k-1}\} = f$$
$$(z \neq -a_k\ (k=1,2,\ldots,n-l)\ z \neq -b;\ a_i \neq a_j \neq b\ if\ i \neq j;\ n>l,\ l \geq 2)$$

and the partial differential equations

$$(z+b)^l \prod_{k=1}^{n-l}(z+a_k)\cdot\frac{\partial^n \mu}{\partial z^n} + \sum_{k=1}^{n-1}\frac{\partial^{n-k}\mu}{\partial z^{n-k}}\{C_k^\lambda\{Q(z)\}_k + C_{k-1}^\lambda\{G(z)\}_{k-1}\}$$
$$+\alpha\mu(z,t) = M\frac{\partial^2\mu}{\partial t^2} + N\frac{\partial \mu}{\partial t}$$

$(z \neq -a_k\ (k=1,2,\ldots,n-l)\ z \neq -b;\ a_i \neq a_j \neq b\ if\ i \neq j;\ n>l,\ l \geq 2)$

are discussed.

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0. Introduction

We have discussed all the solutions of certain third order differential equations (ordinary or partial) with three regular singular points in the previous paper [1]. In this paper we carry on the same idea to deal with the non-homogeneous n-th order differential equations (ordinary of partial) with n regular points.

0-1. Definition

Let $D = \{\underset{-}{D}, \underset{+}{D}\}$, $C = \{\underset{-}{C}, \underset{+}{C}\}$, $\underset{-}{C}$ be a curve along the cut joining two points z and $-\infty+i\mathrm{Im}(z)$, $\underset{+}{C}$ be a curve along the cut joining two points z and $\infty+i\mathrm{Im}(z)$, $\underset{-}{D}$ be a domain surrounded by $\underset{-}{C}$, $\underset{+}{D}$ be a domain surrounded by $\underset{+}{C}$. (Here D contains the points over the curve C.)
Moreover, let $f = f(z)$ be a regular function in D $(z \in D)$,

$$f_v = (f)_v =_C (f)_v = \frac{\Gamma(v+1)}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{v+1}} d\zeta \ (v \notin \mathbb{Z}^-)$$

$$(f)_{-m} = \lim_{v \to -m} (f)_v \ \ (m \in \mathbb{Z}^+),$$

where

$$-\pi \le \arg(\zeta - z) \le \pi \text{ for } \underset{-}{C},\ 0 \le \arg(\zeta - z) \le 2\pi \text{ for } \underset{+}{C},$$

$$\zeta \neq z,\ ,z \in C,\ v \in \mathbb{R},\ \Gamma : \text{Gamma function.}$$

Then $(f)_v$ is the fractional differintegration of arbitrary order v (derivatives of order v for $v > 0$, and integrals of order $-v$ for $v < 0$), with respect to z, of the function f, if $|(f)_v| < \infty$.

0-2. The set $\Im$

We call the function $f = f(z)$ such that $|f_v| < \infty$ in D as a fractional differintegrable function by arbitrary order v and denote the set of them with notation $\Im = \{f||f_v| < \infty, v \in \mathbb{R}\}$. Then we have

$$|f_v| < \infty \Longleftrightarrow f \in \Im \quad (\text{ in } D).$$

In order to discuss the solutions of ordinary and partial differential equations, we need the following lemmas and properties [1].

Lemma 1. (Linearity property) *Let U(z) and V(z) be analytic and one-valued functions. We have then*

(i) $(U \cdot a)_v = aU_v$;

(ii) $(U \cdot a + V \cdot b)_v = aU_v + bV_v$,
where U_v and V_v exist, a and b are constants, $z \in \mathbb{C}$, $v \in \mathbb{R}$.

Lemma 2. (Index law) *If $f(z)$ is an analytic and one-valued function, then*

$$(f_\mu)_v = f_{\mu+v} = (f_v)_\mu \quad for\ f_\mu,\ f_v \neq 0,$$

where $\mu,\ v \in \mathbb{R},\ z \in \mathbb{C}$ *and* $\left|\dfrac{\Gamma(\mu+v+1)}{\Gamma(\mu+1)\Gamma(v+1)}\right| < \infty$.

Lemma 3. (Generalized Leibniz's Rule) *Let U(z) and V(z) be analytic and one valued functions. If U_v and V_v exist, then*

$$(U \cdot V)_v = \sum_{n=0}^{\infty} \frac{\Gamma(v+1)}{\Gamma(v-n+1)\Gamma(n+1)} \cdot U_{v-n} \cdot V_n, \quad where\ v \in \mathbb{R}.$$

Remark. $\mid \Gamma(-k) \mid = \infty$ for $k \in \mathbb{Z}^+ \bigcup \{0\}$.

For properties we have

Property 1. $(e^{\alpha z})_v = \alpha^v \cdot e^{\alpha z}$, $\alpha \neq 0$, $z \in \mathbb{C}$, $v \in \mathbb{R}$.

Property 2. $(e^{-\alpha z})_v = e^{-i\pi v} \cdot \alpha^v \cdot e^{-\alpha z}$, $\alpha \neq 0$, $z \in \mathbb{C}$, $v \in \mathbb{R}$.

Property 3. If $\mid \Gamma(v-\alpha)/\Gamma(-\alpha) \mid < \infty$, *then*

$$(z^\alpha)_v = e^{-i\pi v} \cdot \frac{\Gamma(v-\alpha)}{\Gamma(-\alpha)} \cdot z^{\alpha - v}, \quad z \in \mathbb{C},\ v \in \mathbb{R}.$$

1. Solutions of a N-th Order Linear Ordinary Differential Equation

With the help of above lemmas, we have the following one of our main results of this paper.

Theorem 1. *If f_λ $(\neq 0)$ exists, then the non-homogeneous n-th order linear ordinary differential equation*

$$\begin{aligned} &L[\varphi(z), a_1, a_2, \ldots, a_{n-l}, b, \lambda, B_1, B_2, \cdots, B_n] \\ &\equiv (z+b)^l \prod_{k=1}^{n-l} (z+a_k) \cdot \varphi_n + \sum_{k=1}^{n} \varphi_{n-k} \{C_k^\lambda \{Q(z)\}_k \\ &\quad + C_{k-1}^\lambda \{G(z)\}_{k-1}\} = f \end{aligned} \tag{1}$$

$$\{z \neq -a_k\ (k = 1, 2, \ldots, n-l)\ z \neq -b\ ;\ a_i \neq a_j \neq b\ if\ i \neq j;$$

$$n > l,\ l \geq 2\}$$

has a particular solution of the form

$$\begin{aligned}\varphi = \Bigg\{ & \left[f_{-\lambda} \prod_{k=1}^{n-l} (z+a_k)^{P_k - 1} (z+b)^{-l} \exp\left(\left[\frac{R(z)}{(z+b)^l} \right]_{-1} \right) \right]_{-1} \cdot \\ & \prod_{k=1}^{n-l} (z+a_k)^{-P_k} \exp\left(- \left[\frac{R(z)}{(z+b)^l} \right]_{-1} \right) \Bigg\}_{\lambda - n + 1}, \end{aligned} \tag{2}$$

where

$$\begin{cases} Q(z) = \sum_{k=0}^{n} A_k z^{n-k} \equiv (z+b)^l \prod_{k=1}^{n-l} (z+a_k) \text{ with } A_0 = 1 \\ \qquad \text{and } n > l,\ l \geq 2; \\ G(z) = \sum_{k=1}^{n} B_k z^{n-k}, \text{ and } R(z) \text{ satisfies the relation} \\ \dfrac{G(z)}{Q(z)} = \sum_{k=1}^{n-l} \dfrac{P_k}{z+a_k} + \dfrac{R(z)}{(z+b)^l} \text{ with } R(-b) = \dfrac{G(-b)}{\prod_{k=1}^{n-l} (a_k - b)} \\ \text{and } P_i = \dfrac{G(-a_i)}{(b-a_i)^l \prod_{\substack{k=1 \\ k \neq i}}^{n-l} (a_k - a_i)} \ (i = 1, 2, \dots, n-l) \\ a_1, a_2, \dots, a_{n-l}, b, B_1, B_2, \dots, B_n \text{ are arbitrary given constants and} \\ \lambda \in \mathbb{R}. \text{ All the regular singular points } a_k\ (k = 1, 2, \dots, n-l) \\ \text{and } b \text{ are distinct.} \\ \varphi_0 = \varphi,\ C_0^n = 1 \text{ and } C_r^n = \dfrac{\Gamma(n+1)}{\Gamma(n+1-r)\Gamma(r+1)}. \end{cases} \tag{3}$$

Remark 1. Equation (1) has the l-th order regular singular point at $z = -b$.

Remark 2. When $l = 1$, equation (1) is reduced to the one which is discussed in our previous paper [10].

Proof. Let $\varphi = W_\lambda$, it yields $\varphi_1 = W_{1+\lambda}$, $\varphi_2 = W_{2+\lambda}, \dots, \varphi_n = W_{n+\lambda}$,

$$\sum_{k=0}^{n} A_k \cdot z^{n-k} \equiv (z+b)^l \prod_{k=1}^{n-l} (z+a_k) \equiv Q(z)$$

$$\text{with } A_0 = 1 \text{ and } G(z) = \sum_{k=1}^{n} B_k z^{n-k}.$$

We consider the following function

$$
\begin{aligned}
&\left\{W_n \cdot (z+b)^l \prod_{k=1}^{n-l}(z+a_k)\right\}_\lambda + \{W_{n-1}G(z)\}_\lambda \\
(4)\qquad &= \left\{W_n \cdot \left(\sum_{k=0}^{n} A_k \cdot z^{n-k}\right)\right\}_\lambda + \left\{W_{n-1} \cdot \left(\sum_{k=1}^{n} B_k z^{n-k}\right)\right\}_\lambda \\
&= \sum_{k=0}^{n} A_k \cdot (W_n \cdot z^{n-k})_\lambda + \sum_{k=1}^{n} B_k \cdot (W_{n-1} \cdot z^{n-k})_\lambda.
\end{aligned}
$$

By Generalized Leibniz's Rule

$$
(5)\qquad (W_n \cdot z^{n-k})_\lambda = W_{n+\lambda} \cdot z^{n-k} + \sum_{j=1}^{\infty} C_j^\lambda (W_\lambda)_{n-j}(z^{n-k})_j.
$$

So

$$
\begin{aligned}
\sum_{k=0}^{n} A_k (W_n \cdot z^{n-k})_\lambda &= W_{n+\lambda} \cdot \left\{\sum_{k=0}^{n} A_k \cdot z^{n-k}\right\} \\
(6)\qquad &\quad + \sum_{k=0}^{n}\sum_{j=1}^{\infty} C_j^\lambda \cdot (W_\lambda)_{n-j} \cdot A_k \cdot (z^{n-k})_j \\
&= W_{n+\lambda} \cdot Q(z) + \sum_{k=0}^{n}\sum_{j=1}^{\infty} C_j^\lambda \cdot (W_\lambda)_{n-j} \cdot A_k \cdot (z^{n-k})_j.
\end{aligned}
$$

Since $(z^{n-k})_j = 0$ for $j > n-k$ and $\{Q(z)\}_j = \sum_{k=0}^{n} A_k \cdot (z^{n-k})_j$. (6) becomes

$$
\begin{aligned}
\sum_{k=0}^{n} A_k \cdot (W_n \cdot z^{n-k})_\lambda &= W_{n+\lambda} \cdot Q(z) + \sum_{j=1}^{n} C_j^\lambda (W_\lambda)_{n-j} \cdot \sum_{k=0}^{n} A_k (z^{n-k})_j \\
&= \varphi_n \cdot Q(z) + \sum_{j=1}^{n} C_j^\lambda \{Q(z)\}_j \cdot \varphi_{n-j}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
(W_{n-1} \cdot z^{n-k})_\lambda &= (W_\lambda)_{n-1} \cdot z^{n-k} + \sum_{j=1}^{\infty} C_j^\lambda (W_\lambda)_{n-1-j} \cdot (z^{n-k})_j \\
\sum_{k=1}^{n} B_k (W_{n-1} z^{n-k})_\lambda &= (W_\lambda)_{n-1} \left(\sum_{k=1}^{n} B_k z^{n-k}\right) \\
(7)\qquad &\quad + \sum_{k=1}^{n}\sum_{j=1}^{\infty} C_j^\lambda (W_\lambda)_{n-1-j} B_k (z^{n-k})_j \\
&= (W_\lambda)_{n-1} G(z) + \sum_{k=1}^{n}\sum_{j=1}^{\infty} C_j^\lambda (W_\lambda)_{n-1-j} B_k (z^{n-k})_j.
\end{aligned}
$$

Note that

$$G(z) = \sum_{k=1}^{n} B_k \cdot z^{n-k} \quad \text{and} \quad (z^{n-k})_j = 0 \text{ for } j > n-k.$$

Since

$$\{G(z)\}_j = \sum_{k=1}^{n} B_k \cdot (z^{n-k})_j \ ,$$

(7) becomes

$$\begin{aligned}
&\sum_{k=1}^{n} B_k \cdot (W_{n-1} \cdot z^{n-k})_\lambda \\
&\quad = (W_\lambda)_{n-1} \cdot G(z) + \sum_{j=1}^{n} C_j^\lambda \{G(z)\}_j (W_\lambda)_{n-1-j} \\
&\quad = \sum_{j=0}^{n} C_j^\lambda \{G(z)\}_j (W_\lambda)_{n-1-j} (C_0^\lambda = 1,\ \{G(z)\}_0 = G(z)) \\
&\quad = \sum_{j=0}^{n-1} C_j^\lambda \{G(z)\}_j \varphi_{n-1-j} \quad (\{G(z)\}_j = 0 \ if \ j = n).
\end{aligned} \tag{8}$$

Substituting (6) and (8) into (4), we have

$$\begin{aligned}
&\left\{ W_n \cdot (z+b)^l \prod_{k=1}^{n-l} (z+a_k) + W_{n-1} G(z) \right\}_\lambda \\
&\quad = \varphi_n \cdot Q(z) + \sum_{j=1}^{n} C_j^\lambda \{Q(z)\}_j \cdot \varphi_{n-j} + \sum_{j=0}^{n-1} C_j^\lambda \{G(z)\}_j \cdot \varphi_{n-1-j} \\
&\quad = \varphi_n \cdot Q(z) + \sum_{j=1}^{n} \varphi_{n-j} \cdot \{ C_j^\lambda \{Q(z)\}_j + C_{j-1}^\lambda \{G(z)\}_{j-1} \} = f,
\end{aligned} \tag{9}$$

or equivalently

$$W_n \cdot (z+b)^l \prod_{k=1}^{n-l} (z+a_k) + W_{n-1} G(z) = f_{-\lambda}. \tag{10}$$

The equation (10) has a sloution of the form

$$W_{n-1} \cdot \exp\left(\left[\frac{G(z)}{Q(z)} \right]_{-1} \right) = \left(\exp\left(\left[\frac{G(z)}{Q(z)} \right]_{-1} \right) \cdot \frac{f_{-\lambda}}{Q(z)} \right)_{-1}, \tag{11}$$

where $Q(z) = (z+b)^l \prod_{k=1}^{n-l} (z+a_k)$.

Let
$$\frac{G(z)}{Q(z)} = \frac{G(z)}{(z+a_1)\cdots(z+a_{n-l})(z+b)^l} = \frac{P_1}{(z+a_1)} + \cdots + \frac{P_{n-l}}{(z+a_{n-l})} + \frac{R(z)}{(z+b)^l},$$
then we obain

$$(12)\quad \begin{cases} P_i = \dfrac{G(-a_i)}{(b-a_i)^l \prod\limits_{\substack{k=1\\k\neq i}}^{n-l}(a_k-a_i)} \qquad (i=1,2,\ldots,n-l), \\ \text{and } R(z) \text{ satisfies} \\ \dfrac{G(z)}{Q(z)} = \sum\limits_{k=1}^{n-l}\dfrac{P_k}{z+a_k} + \dfrac{R(z)}{(z+b)^l} \text{ with } R(-b) = \dfrac{G(-b)}{\prod\limits_{k=1}^{n-l}(a_k-b)}, \\ \text{where} \qquad G(z) = \sum\limits_{k=1}^{n} B_k z^{n-k}. \end{cases}$$

Thus (11) becomes
$$W_{n-1}(z) = \left\{ f_{-\lambda} \prod_{k=1}^{n-l}(z+a_k)^{P_k-1}(z+b)^{-l}\exp\left(\left[\frac{R(z)}{(z+b)^l}\right]_{-1}\right)\right\}_{-1} \cdot \prod_{k=1}^{n-l}(z+a_k)^{-P_k}\exp\left(-\left[\frac{R(z)}{(z+b)^l}\right]_{-1}\right),$$
or equivalently

$$(13)\quad \begin{aligned} \varphi = \Bigg[&\left\{ f_{-\lambda} \prod_{k=1}^{n-l}(z+a_k)^{P_k-1}(z+b)^{-l}\exp\left(\left[\frac{R(z)}{(z+b)^l}\right]_{-1}\right)\right\}_{-1} \\ &\cdot \prod_{k=1}^{n-l}(z+a_k)^{-P_k}\exp\left(-\left[\frac{R(z)}{(z+b)^l}\right]_{-1}\right)\Bigg]_{\lambda-n+1}, \\ &\text{where } (z \neq -a_k\,(k=1,2,\ldots,n-l)\ z\neq -b;\ a_i \neq a_j \neq b \\ &dif\ i\neq j;\ n>l). \end{aligned}$$

Conversely, if (13) holds, since $\varphi = W_\lambda$, we have

$$
\begin{aligned}
W_n = & \Bigg[\Bigg\{ f_{-\lambda} \prod_{k=1}^{n-l} (z+a_k)^{P_k-1} (z+b)^{-l} \exp\Bigg(\bigg[\frac{R(z)}{(z+b)^l} \bigg]_{-1} \Bigg) \Bigg\}_{-1} \\
& \cdot \prod_{k=1}^{n-l} (z+a_k)^{-P_k} \exp\Bigg(- \bigg[\frac{R(z)}{(z+b)^l} \bigg]_{-1} \Bigg) \Bigg]_1 \\
= & f_{-\lambda} \prod_{k=1}^{n-l} (z+a_k)^{-1} (z+b)^{-l} + \Bigg\{ f_{-\lambda} \prod_{k=1}^{n-l} (z+a_k)^{P_k-1} (z+b)^{-l} \\
& \cdot \exp\Bigg(\bigg[\frac{R(z)}{(z+b)^l} \bigg]_{-1} \Bigg) \Bigg\}_{-1} \\
& \cdot \Bigg\{ \prod_{k=1}^{n-l} (z+a_k)^{-P_k} \exp\Bigg(- \bigg[\frac{R(z)}{(z+b)^l} \bigg]_{-1} \Bigg) \Bigg\}_1 .
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
W_{n-1} = & \Bigg\{ f_{-\lambda} \prod_{k=1}^{n-l} (z+a_k)^{P_k-1} (z+b)^{-l} \exp\Bigg(\bigg[\frac{R(z)}{(z+b)^l} \bigg]_{-1} \Bigg) \Bigg\}_{-1} \cdot \\
& \prod_{k=1}^{n-l} (z+a_k)^{-P_k} \exp\Bigg(- \bigg[\frac{R(z)}{(z+b)^l} \bigg]_{-1} \Bigg) .
\end{aligned} \tag{15}
$$

Substituting (14) and (15) into L.H.S of (10) yields

$$
\begin{aligned}
& W_n (z+b)^l \prod_{k=1}^{n-l} (z+a_k) + W_{n-1} G(z) \\
& = \Bigg\{ f_{-\lambda} \prod_{k=1}^{n-l} (z+a_k)^{-1} (z+b)^{-l} \\
& \quad + \Bigg\{ f_{-\lambda} \prod_{k=1}^{n-l} (z+a_k)^{P_k-1} (z+b)^{-l} \exp\Bigg(\bigg[\frac{R(z)}{(z+b)^l} \bigg]_{-1} \Bigg) \Bigg\}_{-1} \\
& \quad \cdot \Bigg\{ \prod_{k=1}^{n-l} (z+a_k)^{-P_k} \exp\Bigg(- \bigg[\frac{R(z)}{(z+b)^l} \bigg]_{-1} \Bigg) \Bigg\}_1 \Bigg\} \cdot (z+b)^l \prod_{k=1}^{n-l} (z+a_k) \\
& \quad + \Bigg\{ f_{-\lambda} \cdot \prod_{k=1}^{n-l} (z+a_k)^{P_k-1} (z+b)^{-l} \exp\Bigg(\bigg[\frac{R(z)}{(z+b)^l} \bigg]_{-1} \Bigg) \Bigg\}_{-1} \\
& \quad \cdot \prod_{k=1}^{n-l} (z+a_k)^{-P_k} \exp\Bigg(- \bigg[\frac{R(z)}{(z+b)^l} \bigg]_{-1} \Bigg) G(z)
\end{aligned}
$$

$$
\begin{aligned}
&= f_{-\lambda} + \Big\{ f_{-\lambda} \prod_{k=1}^{n-l} (z+a_k)^{P_k - 1} (z+b)^{-l} \exp\Big(\Big[\frac{R(z)}{(z+b)^l} \Big]_{-1} \Big) \Big\}_{-1} \\
&\quad \cdot \Big\{ \Big[\prod_{k=1}^{n-l} (z+a_k)^{-P_k} \cdot \exp\Big(- \Big(\frac{R(z)}{(z+b)^l} \Big)_{-1} \Big) \Big]_1 \Big\} (z+b)^l \prod_{k=1}^{n-l} (z+a_k) \\
&\quad + \prod_{k=1}^{n-l} (z+a_k)^{-P_k} \exp\Big(- \Big[\frac{R(z)}{(z+b)^l} \Big]_{-1} \Big) G(z) \Big\} \\
&= f_{-\lambda}.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\left\{ \left[\prod_{k=1}^{n-l} (z+a_k)^{-P_k} \exp\left(- \left(\frac{R(z)}{(z+b)^l} \right)_{-1} \right) \right]_1 \right\} (z+b)^l \prod_{k=1}^{n-l} (z+a_k) \\
&\qquad + \prod_{k=1}^{n-l} (z+a_k)^{-P_k} \exp\left(- \left[\frac{R(z)}{(z+b)^l} \right]_{-1} \right) G(z) = 0,
\end{aligned}
$$

this completes the proof of Theorem 1.

From Theorem 1, we obtain the following.

Corollary 1. *The homogeneous n-th order linear ordinary differential equation*

$$
(16) (z+b)^l \prod_{k=1}^{n-l} (z+a_k) \cdot \varphi_n + \sum_{k=1}^{n} \varphi_{n-k} \{ C_k^{\lambda} \{Q(z)\}_k + C_{k-1}^{\lambda} \{G(z)\}_{k-1} \} = 0
$$

has a particular solution of the form

$$
\varphi(z) = K \left[\prod_{k=1}^{n-l} (z+a_k)^{-P_k} \exp\left(- \left[\frac{R(z)}{(z+b)^l} \right]_{-1} \right) \right]_{\lambda-n+1}, \tag{17}
$$

where

$$
Q(z) = \sum_{k=0}^{n} A_k z^{n-k} \equiv (z+b)^l \prod_{k=1}^{n-l} (z+a_k) \qquad \textit{with } A_0 = 1 \textit{ and } n > l,\ l \geq 2;
$$

$$
G(z) = \sum_{k=1}^{n} B_k z^{n-k}, \quad \textit{and } R(z) \textit{ satisfies the relation}
$$

$$
\frac{G(z)}{Q(z)} = \sum_{k=1}^{n-l} \frac{P_k}{z+a_k} + \frac{R(z)}{(z+b)^l} \qquad \textit{with } R(-b) = \frac{G(-b)}{\prod_{k=1}^{n-l} (a_k - b)},
$$

$$\text{and } P_i = \frac{G(-a_i)}{(b-a_i)^l \prod_{\substack{k=1\\k\neq i}}^{n-l}(a_k-a_i)} \qquad (i=1,2,\ldots,n-l),$$

$a_1, a_2, \ldots, a_{n-l}, b, B_1, B_2, \ldots, B_n$ are arbitrary given constants, K is an arbitrary constant. All the regular singular points a_k $(k=1,2,\ldots,n-l)$ and b are distinct.

$$\varphi_0 = \varphi,\ C_0^n = 1 \text{ and } C_r^n = \frac{\Gamma(n+1)}{\Gamma(n+1-r)\Gamma(r+1)}.$$

2. Solutions of a N-th Order Partial Differential Equation

Theroem 2. *A partial differential equation of the n-th order*

$$(z+b)^l \prod_{k=1}^{n-l}(z+a_k)\cdot\frac{\partial^n\mu}{\partial z^n} + \sum_{k=1}^{n-1}\frac{\partial^{n-k}\mu}{\partial z^{n-k}}\{C_k^\lambda\{Q(z)\}_k + C_{k-1}^\lambda\{G(z)\}_{k-1}\} + \alpha\mu(z,t) = M\frac{\partial^2\mu}{\partial t^2} + N\frac{\partial\mu}{\partial t} \tag{18}$$
$$(z\neq -a_k\ (k=1,2,\ldots,n-l)\ z\neq -b\ ;\ a_i\neq a_j\neq b\ \ if\ i\neq j\ ;\ n>l,\ l\geq 2)$$

has solutions of the forms
(a) $M\neq 0$

$$\mu(z,t) = K\left[\prod_{k=1}^{n-l}(z+a_k)^{-P_k}\exp\left(-\left[\frac{R(z)}{(z+b)^l}\right]_{-1}\right)\right]_{\lambda-n+1} \cdot\exp\left\{\frac{-N\pm\sqrt{N^2+4M(\alpha-\delta)}}{2M}t\right\}, \tag{19}$$

(b) $M=0$ *and* $N\neq 0$

$$\mu(z,t) = K\left[\prod_{k=1}^{n-l}(z+a_k)^{-P_k}\exp\left(-\left[\frac{R(z)}{(z+b)^l}\right]_{-1}\right)\right]_{\lambda-n+1} \cdot\exp\left\{\frac{\alpha-\delta}{N}t\right\}, \tag{20}$$

where $\delta = \alpha - M\beta^2 - N\beta = C_n^\lambda\{Q(z)\}_n + C_{n-1}^\lambda\{G(z)\}_{n-1}$, $B_i (i=1,\ldots,n)$, $a_k(k=1,\ldots,n-l), b, \alpha, M, N, \lambda$ are given constants, K is an arbitrary con-

stant,

$$\varphi_0 = \varphi,\ C_0^n = 1,\ C_r^n = \frac{\Gamma(n+1)}{\Gamma(n+1-r)\Gamma(r+1)},$$
$$\frac{G(z)}{Q(z)} = \sum_{k=1}^{n-l} \frac{P_k}{z+a_k} + \frac{R(z)}{(z+b)^l},$$
$$P_i = \frac{G(-a_i)}{(b-a_i)^l \prod\limits_{\substack{k=1\\k\neq i}}^{n-l}(a_k - a_i)}\ (i = 1, 2, \cdots, n-l)\ for\ G(z) = \sum_{k=1}^{n} B_k z^{n-k},$$

and a_k, b are distinct.

Proof. Let $\mu(z,t) = \varphi(z)\cdot e^{\beta t}$ $(\beta \neq 0)$ be a solution of (18).

$$\frac{\partial\mu}{\partial t} = \varphi\cdot\beta\cdot e^{\beta t}, \quad \frac{\partial^2\mu}{\partial t^2} = \varphi\cdot\beta^2\cdot e^{\beta t},$$

$$\frac{\partial\mu}{\partial z} = \varphi_1\cdot e^{\beta t}, \quad \frac{\partial^2\mu}{\partial z^2} = \varphi_2\cdot e^{\beta t},\ \ldots\ , \quad \frac{\partial^n\mu}{\partial z^n} = \varphi_n\cdot e^{\beta t}.$$

Then (18) becomes

$$\varphi_n(z+b)^l \prod_{k=1}^{n-l}(z+a_k) + \sum_{k=1}^{n-1}\varphi_{n-k}\cdot\Big\{C_k^\lambda\{Q(z)\}_k + C_{k-1}^\lambda\{G(z)\}_{k-1}\Big\} + \varphi(\alpha - M\beta^2 - N\beta) = 0. \tag{21}$$

Choose β such that

$$\delta \equiv \alpha - M\beta^2 - N\beta = C_n^\lambda\{Q(z)\}_n + C_{n-1}^\lambda\{G(z)\}_{n-1}, \tag{22}$$

that is

$$\beta = \begin{cases} \dfrac{-N \pm \sqrt{N^2 + 4M(\alpha-\delta)}}{2M} & \text{for } M \neq 0 \\ \dfrac{\alpha-\delta}{N} & \text{for } M = 0 \text{ and } N \neq 0, \end{cases} \tag{23}$$

then (21) becomes

$$(z+b)^l \prod_{k=1}^{n-l}(z+a_k)\cdot\varphi_n + \sum_{k=1}^{n}\varphi_{n-k}\{C_k^\lambda\{Q(z)\}_k + C_{k-1}^\lambda\{G(z)\}_{k-1}\} = 0.$$

By Corollary 1, its solution is given by

$$\varphi(z,t) = K\left[\prod_{k=1}^{n-l}(z+a_k)^{-P_k}\exp\left(-\left[\frac{R(z)}{(z+b)^l}\right]_{-1}\right)\right]_{\lambda-n+1}, \tag{24}$$

where

$Q(z) = \sum_{k=0}^{n} A_k z^{n-k} \equiv (z+b)^l \prod_{k=1}^{n-l}(z+a_k)$ with $A_0 = 1$ and $n > l$,

$G(z) = \sum_{k=1}^{n} B_k z^{n-k}$ and $R(z)$ satisfies the relation,

$\dfrac{G(z)}{Q(z)} = \sum_{k=1}^{n-l}\dfrac{P_k}{z+a_k} + \dfrac{R(z)}{(z+b)^l}$ with $R(-b) = \dfrac{G(-b)}{\prod_{k=1}^{n-l}(a_k-b)}$,

and $P_i = \dfrac{G(-a_i)}{(b-a_i)^l\prod_{\substack{k=1\\k\neq i}}^{n-l}(a_k-a_i)}$ $(i = 1, 2, \cdots, n-l)$,

$a_1, a_2, \cdots, a_{n-l}, b, B_1, B_2, \cdots$ and B_n are arbitrary given constants, K is arbitrary constant. All the regular singular points a_k $(k = 1, 2, \cdots, n-l)$ and b are distinct.

$\varphi_0 = \varphi$, $C_0^n = 1$ and $C_r^n = \dfrac{\Gamma(n+1)}{\Gamma(n+1-r)\Gamma(r+1)}$.

Thus for $M \neq 0$, the solution of (18) is given by

$$\begin{aligned}\mu(z,t) = &K\left[\prod_{k=1}^{n-l}(z+a_k)^{-P_k}\exp\left(-\left[\frac{R(z)}{(z+b)^l}\right]_{-1}\right)\right]_{\lambda-n+1}\cdot\\ &\exp\left\{\tfrac{-N\pm\sqrt{N^2+4M(\alpha-\delta)}}{2M}t\right\}.\end{aligned} \tag{25}$$

Moreover, for $M = 0$ and $N \neq 0$, the solution of (18) is given by

$$\begin{aligned}\mu(z,t) = &K\left[\prod_{k=1}^{n-l}(z+a_k)^{-P_k}\exp\left(-\left[\frac{R(z)}{(z+b)^l}\right]_{-1}\right)\right]_{\lambda-n+1}\cdot\\ &\exp\left\{\tfrac{\alpha-\delta}{N}t\right\},\end{aligned} \tag{26}$$

where

$$\delta = \alpha - M\beta^2 - N\beta = C_n^\lambda\{Q(z)\}_n + C_{n-1}^\lambda\{G(z)\}_{n-1},$$

$B_i (i = 1, \ldots, n)$, a_k $(k = 1, \ldots, n-l)$, b, α, M, N, λ are given constants, K is an arbitrary constant, $\varphi_0 = \varphi$, $C_0^n = 1$, $C_r^n = \dfrac{\Gamma(n+1)}{\Gamma(n+1-r)\Gamma(r+1)}$,

$\dfrac{G(z)}{Q(z)} = \displaystyle\sum_{k=1}^{n-l} \frac{P_k}{z+a_k} + \frac{R(z)}{(z+b)^l}$,

$P_i = \dfrac{G(-a_i)}{(b-a_i)^l \displaystyle\prod_{\substack{k=1\\k\neq i}}^{n-l}(a_k - a_i)}$ $(i = 1, 2, \ldots, n-l)$ for $G(z) = \displaystyle\sum_{k=1}^{n} B_k z^{n-k}$,

and a_k, b are distinct.

Conversely, for $M \neq 0$, we shall show that (25) satisfies (18). Let

$$\varphi(z) = K\left[\prod_{k=1}^{n-l}(z+a_k)^{-P_k} \exp\left(-\left[\frac{R(z)}{(z+b)^l}\right]_{-1}\right)\right]_{\lambda-n+1},$$
$$\beta = \frac{-N \pm \sqrt{N^2 + 4M(\alpha - \delta)}}{2M}.$$

Then (25) becomes $\mu(z,t) = \varphi(z) \cdot e^{\beta t}$ $(\beta \neq 0)$. Since

$$\frac{\partial \mu}{\partial t} = \varphi \cdot \beta \cdot e^{\beta t}\ , \frac{\partial^2 \mu}{\partial t^2} = \varphi \cdot \beta^2 \cdot e^{\beta t},$$

$$\frac{\partial \mu}{\partial z} = \varphi_1 \cdot e^{\beta t}\ , \quad \frac{\partial^2 \mu}{\partial z^2} = \varphi_2 \cdot e^{\beta t}, \ \ldots, \quad \frac{\partial^n \mu}{\partial z^n} = \varphi_n \cdot e^{\beta t}.$$

The L.H.S of (5.1)

$$= e^{\beta t}\Bigg[\varphi_n \cdot (z+b)^l \prod_{k=1}^{n-l}(z+a_k) + \sum_{k=1}^{n-l} \varphi_{n-k} \cdot \{C_k^\lambda \{Q(z)\}_k$$
$$+ C_{k-1}^\lambda \{G(z)\}_{k-1}\} + \alpha\varphi(z)\Bigg]$$
$$= e^{\beta t}[-\varphi \cdot (C_n^\lambda \{Q(z)\}_n + C_{n-1}^\lambda \{G(z)\}_{n-1}) + \alpha\varphi] \qquad \text{(By Corollary 1)}$$
$$= e^{\beta t}(\alpha - \delta)\varphi \qquad \text{(By (22))}$$
$$= e^{\beta t}\varphi[M^2\beta^2 + N\beta] = M\frac{\partial^2 \mu}{\partial t^2} + N\frac{\partial \mu}{\partial t}.$$

Thus the solution of the form (8) satisfies (2). The proof of (9) is obvious.

3. Examples

Example 1. The nonhomogeneous fourth order linear ordinary differential equation of the form

$$\begin{aligned} z^2(z+1)(z+2)\varphi_4 &+(16z^3+36z^2+16z)\varphi_3+(72z^2+108z+24)\varphi_2 \\ &+(96z+72)\varphi_1+24\varphi=120z \qquad (z\neq 0,-1,-2) \end{aligned} \tag{27}$$

has a particular solution

$$\varphi=\frac{z^3}{(z+1)(z+2)}.$$

Let $b=0$, $a_1=1$, $a_2=2$ and $f=120z$, $Q(z)=z^2(z+1)(z+2)=z^4+3z^3+2z^2$. Comparing (27) with Theorem 1, we have

$$\begin{cases} 4\lambda+B_1=16, \quad \lambda(6\lambda-6+3B_1)=72, \quad \lambda(\lambda-1)(4\lambda-8+3B_1)=96, \\ 9\lambda+B_2=36, \quad \lambda(9\lambda-9+2B_2)=108, \quad \lambda(\lambda-1)(3\lambda-6+B_2)=72, \\ 4\lambda+B_3=16, \quad \lambda(2\lambda-2+B_3)=24, \quad \lambda(\lambda-1)(\lambda-2)(\lambda+1)=24, \\ \qquad B_4=0. \end{cases}$$

For their common solution, we get $\lambda=3$, $B_1=4$, $B_2=9$, $B_3=4$, $B_4=0\Rightarrow$ $P_1=1$, $P_2=1$, $P_3=2$, $P_4=0$. Thus from (2), the particular solution is given by

$$\begin{aligned} \varphi &= \left[(z+1)^{-1}(z+2)^{-1}z^{-2}e^0\int(z+1)^0(z+2)^0z^{2-2}e^{-0/z}(120z)_{-3}dz\right]_0 \\ &=(z+1)^{-1}(z+2)^{-1}z^{-2}\int 5z^4dz \\ &=\frac{z^3}{(z+1)(z+2)}. \end{aligned}$$

Example 2. The nonhomogeneous fifth order linear ordinary differential equation of the form

$$\begin{aligned} &z^2(z-1)(z+1)(z+2)\varphi_5+[24z^4+38z^3-14z^2-18z]\varphi_4 \\ &\quad+[184z^3+216z^2-52z-32]\varphi_3+[528z^2+408z-48]\varphi_2 \\ &\quad+[504z+192]\varphi_1+96\varphi=96 \qquad (z\neq 1,0,-1,-2) \end{aligned} \tag{28}$$

has a particular solution

$$\varphi=\frac{z^3}{(z-1)(z+1)(z+2)}.$$

Let $b=0,\ a_1=-1,\ a_2=1,\ a_3=2$ and $f=96$, and $Q(z)=z^2(z-1)(z+1)(z+2)=z^5+2z^4-z^3-2z^2$. Comparing (28) with Theorem 1, we have

$$\begin{cases} 5\lambda+B_1=24, & \lambda(10\lambda-10+4B_1)=184,\\ 8\lambda+B_2=38, & \lambda(12\lambda-12+3B_2)=216,\\ -3\lambda+B_3=-14, & \lambda(-3\lambda+3+2B_3)=-52,\\ -4\lambda+B_4=-18, & \lambda(-2\lambda+2+B_4)=-32,\\ B_5=0, & \lambda(\lambda-1)(10\lambda-20+6B_1)=528,\\ \lambda(\lambda-1)(8\lambda-16+3B_2)=408, & \lambda(\lambda-1)(-\lambda+2+B_3)=-48,\\ \lambda(\lambda-1)(\lambda-2)(5\lambda-15+4B_1)=504, & \lambda(\lambda-1)(\lambda-2)(2\lambda-8+B_2)=192,\\ \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4+B_1)=96. & \end{cases}$$

For their common solution, we get $\lambda=4,\ B_1=4,\ B_2=6,\ B_3=-2,\ B_4=-2,\ B_5=0\Rightarrow P_1=1,\ P_2=1,\ P_3=1,\ P_4=1,\ P_5=0.$
Thus from (2), the particular solution is given by

$$\begin{aligned}\varphi &= \left[(z-1)^{-1}(z+1)^{-1}(z+2)^{-1}z^{-1}\int z^{-1}(96)_{-4}dz\right]_0\\ &= (z-1)^{-1}(z+1)^{-1}(z+2)^{-1}z^{-1}\int 4z^3dz\\ &= \frac{z^3}{(z-1)(z+1)(z+2)}.\end{aligned}$$

Example 3. The fourth order partial differential equation of the form

$$z^2(z+1)(z+2)\frac{\partial^4\mu}{\partial z^4}+(16z^3+36z^2+16z)\frac{\partial^3\mu}{\partial z^3}+(72z^2+108z+24)\frac{\partial^2\mu}{\partial z^2}$$
$$+(96z+72)\frac{\partial\mu}{\partial z}+16\mu=\frac{\partial^2\mu}{\partial t^2}+6\frac{\partial\mu}{\partial t}\qquad (z\neq 0,-1,-2)$$

has solutions

$$\mu(z,t)=K\cdot(z+1)^{-1}(z+2)^{-1}z^{-2}e^{-2t}$$

or

$$\mu(z,t)=K\cdot(z+1)^{-1}(z+2)^{-1}z^{-2}e^{-4t}.$$

Take $a_1=1,\ a_2=2,\ b=0,\ \alpha=16,\ M=1,\ N=6$ in Theorem 2. It's similar to Example 1, we have $\lambda=3$, $B_1=4$, $B_2=9$, $B_3=4$, $B_4=0$, $P_1=1$, $P_2=1$, $P_3=2$, $P_4=0$. And we obtain $\delta\equiv C_n^\lambda Q_n(z)+C_{n-1}^\lambda G_{n-1}(z)=24$.
From Theorem 2. The solutions are

$$\mu(z,t)=K\cdot[(z+1)^{-1}(z+2)^{-1}z^{-2}z^0]_0\cdot\exp\left\{\frac{-6\pm\sqrt{6^2+4(16-24)}}{2}t\right\}.$$

Thus

$$\begin{aligned} &\mu(z,t) = K(z+1)^{-1}(z+2)^{-1}z^{-2}e^{-2t}, \\ or \quad &\mu(z,t) = K(z+1)^{-1}(z+2)^{-1}z^{-2}e^{-4t}. \end{aligned}$$

Example 4. The fifth order partial differential equation of the form

$$\begin{aligned} &z^2(z-1)(z+1)(z+2)\frac{\partial^5\mu}{\partial z^5} + [24z^4+38z^3-14z^2-18z]\frac{\partial^4\mu}{\partial z^4} \\ &+[184z^3+216z^2-52z-32]\frac{\partial^3\mu}{\partial z^3} + [528z^2+408z-48]\frac{\partial^2\mu}{\partial z^2} \\ &+[504z+192]\frac{\partial\mu}{\partial z} + 90\mu(z,t) = -6\frac{\partial\mu}{\partial t} \qquad (z \neq 1,0,-1,-2) \end{aligned}$$

has solution

$$\mu(z,t) = K \cdot e^t(z-1)^{-1}(z+1)^{-1}(z+2)^{-1}z^{-1}.$$

Take $a_1=-1,\ a_2=1,\ a_3=2,\ b=0,\ \alpha=90,\ M=0,\ N=-6$ in Theorem 2. It's similar to Example 2, we have $\lambda=4,\ B_1=4\ B_2=6,\ B_3=-2,\ B_4=-2,\ B_5=0,\ P_1=1,\ P_2=1,\ P_3=1,\ P_4=1,\ P_5=0$. And we obtain $\delta \equiv C_n^\lambda Q_n(z) + C_{n-1}^\lambda G_{n-1}(z) = 96$. From Theorem 2, the solution is

$$\begin{aligned} \mu(z,t) &= K \cdot [(z-1)^{-1}(z+1)^{-1}(z+2)^{-1}z^{-1}e^{0/z}]_0 \cdot \exp\left\{\tfrac{90-96}{-6}t\right\} \\ &= K \cdot e^t(z-1)^{-1}(z+1)^{-1}(z+2)^{-1}z^{-1}. \end{aligned}$$

References

1. Shih-Tong Tu, Erh-Tsung Chin, and Wen-Chieh Luo, Solutions of a class of third order ordinary and partial differential equations via fractional calculus, *Chung Yuan J.* **24**(3) (1996), 23-31.
2. K, Nishimoto, Fractional Calculus (Volumes I, II, III, IV and V), Descartes Press, Koriyama, Japan 1984, 1987, 1989, 1991, and 1996.
3. K. Nishimoto and S. T. Tu, Fractional calculus method to a generalized linear second order (nonhomogeneous and homogeneous) ordinary differential equation of Fuchs type, *J. Coll. Engng, Nihon Univ.* **B-33** (1992), 27-52.
4. K. Nishimoto, An essence of Nishimoto's fractional calculus (Integrations and Differentiation of Arbitrary Order), (1991) Descartes Press, Koriyama, Japan.
5. Shih-Tong Tu, K. Nishimoto, Shing-Jong Jaw, and Shy-Der Lin, Applications of fractional calculus to ordinary and partial differential equations of the second order, *Hiroshima Math. J.* **23** (1993), 63-77.

6. K. B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, New York, 1974.

7. B. Ross (Editor), Fractional Calculus, Lecture Notes in Math. ♯457, Springer-Verlag, 1975.

8. B. N. Al-Saqabi and S. L. Kalla, Solution of second and third order linear differential equations, *Tamkang J. Math.* **21**(3) (1990), 191-200.

9. Shih-Tong Tu and Shing-Jong Jaw, Solution of a class of third order ordinary and partial differential equations via fractional calculus, *Chung Yuan J.* **24**(1) (1996), 7-12.

10. Shih-Tong Tu and Ding-Kuo Chyan, Solution of some n-th order ordinary and partial differential equations by means of fractional calculus, *J. Fract. Calc.* **10** (1996), 51-65.

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