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ON EXISTENCE AND APPROXIMATION OF SOLUTIONS OF SECOND ORDER ABSTRACT CAUCHY PROBLEM

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Dedicated to the Memory of Professor Sen-Yen Shaw

Abstract. Let A be the generator of a nondegenerate local α-times integrated C-cosine function $C(\cdot)$ on a Banach space X for some $\alpha \geq 0$, $f \in L^1_{loc}([0,T_0),X) \cap C((0,T_0),X)$, and $x, y \in X$. We first show that the abstract Cauchy problem : $ACP(A, Cf, Cx, Cy)$ $\quad u''(t) = Au(t) + Cf(t)$, $u(0) = Cx$ and $u'(0) = Cy$, has a strong solution is equivalent to the function $v(\cdot) = C(\cdot)x + j_0 * C(\cdot)y + j_0 * C * f(\cdot) \in C^{\alpha+1}([0,T_0),X)$ and $D^{\alpha+1}v(\cdot) \in C^1((0,T_0),X)$, and then use it to prove some new existence and approximation theorems concerning strong solutions of $ACP(A, Cz+j_{\alpha-1}*Cg, Cx, Cy)$ and mild solutions of $ACP(A, Cx+j_1Cy+j_2Cz+j_{\alpha-1}*Cg, 0, 0)$ (for $\alpha \geq 2$) in $C^2([0,T_0),X)$ when $C(\cdot)$ is locally Lipschitz continuous, and vectors x, y and z satisfy some suitable regularity assumptions. Here $0 < T_0 \leq \infty$ is fixed.

1. Introduction

Let X be a Banach space over $\mathbb{F}$ with norm $\|\cdot\|$, and let $B(X)$ denote the family of all bounded linear operators from X into itself. We consider the following second order abstract Cauchy problem:

$$(1.1) \qquad ACP(A, f, x, y) \qquad \begin{cases} u''(t) = Au(t) + f(t) \text{ for } 0 < t < T_0, \\ u(0) = x \text{ and } u'(0) = y, \end{cases}$$

where $0 < T_0 \leq \infty$ and $x, y \in X$ are given, $A : D(A) \subset X \to X$ is a closed linear operator in X with domain $D(A)$ and range $R(A)$, and f is an X-valued function defined on $(0,T_0)$. A function $u : [0,T_0) \to X$ is called a strong solution of $ACP(A,f,x,y)$ if $u \in C^2((0,T_0),X) \cap C^1([0,T_0),X) \cap C((0,T_0),[D(A)])$ and

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satisfies $ACP(A, f, x, y)$. Here $[D(A)]$ denotes the Banach space $D(A)$ equipped with the graph norm $|x|_A = \|x\| + \|Ax\|$. For each $\alpha > 0$ and $C \in B(X)$, a family $C(\cdot)(=\{C(t)|0 \le t < T_0\})$ in $B(X)$ is called a local α-times integrated C-cosine function on X if

$$\begin{aligned}&C(\cdot) \text{ is strongly continuous, that is, for each } x \in X,\\ &C(\cdot)x : [0, T_0) \to X \text{ is continuous,}\end{aligned} \tag{1.2}$$

$$C(\cdot)C = CC(\cdot), \text{ that is, } C(t)C = CC(t) \text{ on } X \text{ for all } 0 \le t < T_0, \tag{1.3}$$

$$\begin{aligned}2C(t)C(s)x &= \frac{1}{\Gamma(\alpha)}\{[\int_0^{t+s} - \int_0^t - \int_0^s](t+s-r)^{\alpha-1}C(r)Cxdr\\ &+ \int_{|t-s|}^t (s-t+r)^{\alpha-1}C(r)Cxdr+\\ &\int_{|t-s|}^s (t-s+r)^{\alpha-1}C(r)Cxdr+\\ &\int_0^{|t-s|}(|t-s|+r)^{\alpha-1}C(r)Cxdr\}\end{aligned} \tag{1.4}$$

for all $x \in X$ and $0 \le t, s, t+s < T_0$ (see [9]); or called a local (0-times integrated) C-cosine function on X if it satisfies (1.2), (1.3), and

$$2C(t)C(s)x = C(t+s)Cx + C(|t-s|)Cx \tag{1.5}$$

for all $x \in X$ and $0 \le t, s, t+s < T_0$ (see [7]). Here $\Gamma(\cdot)$ denotes the Gamma function. Moreover, we say that $C(\cdot)$ is

(i) locally Lipschitz continuous, if for each $0 < t_0 < T_0$ there exists a $K_{t_0} > 0$ such that

$$\|C(t+h) - C(t)\| \le K_{t_0}h \text{ for all } 0 \le t, h \le t+h \le t_0; \tag{1.6}$$

(ii) nondegenerate, if $x = 0$ whenever $C(t)x = 0$ for all $0 \le t < T_0$. In this case, its (integral) generator $A : D(A) \subset X \to X$ is a closed linear operator in X defined by $D(A) = \{x \in X|$ there exists a $y_x \in X$ such that $C(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}Cx = \int_0^t\int_0^s C(r)y_xdrds$ for $0 \le t < T_0\}$ and $Ax = y_x$ for all $x \in D(A)$.

In general, a local α-times integrated C-cosine function on X is also called an α-times integrated C-cosine function on X if $T_0 = \infty$ (see [1,2,6-14,16,17]), an α-times integrated C-cosine function may not be exponentially bounded (see

[9]), and the generator of a local α-times integrated C-cosine function may not be densely defined (see [2, 16]). Moreover, a local α-times integrated C-cosine function is not necessarily extendable to the half line $[0,\infty)$ except for $C = I$ the case of cosine function(that is, $C = I$ and $T_0 = \infty$)(see [1,3,4]). Here I denotes the identity operator on X. The concept of α-times integrated C-cosine functions has been extensively applied to discuss the existence of (strong, mild or weak) solutions of $ACP(A, f, x, y)$ when $\alpha \in \mathbb{N} \cup \{0\}$(see [3, 7, 12, 14, 16]) or $C = I$(see [2, 4, 13] and their references). Some equivalence conditions between the existence of an α-times integrated C-cosine function and the unique existence of (strong or weak) solutions of $ACP(A, f, x, y)$ are also discussed as in [9-11]. Several examples concerning α-times integrated cosine functions with densely defined generators are given as in [2, 6, 17]. All consequences of this paper are motivated by the aforementioned results as in [1, 6, 8, 12, 15] for which the concept of α-times integarted C-semigroups is used to obtain some existence and approximation theorems concerning (strong or mild) solutions of the following first order abstract Cauchy problem:

$$(1.7) \qquad ACP(A, f, x) \qquad \begin{cases} u'(t) = Au(t) + f(t) \text{ for } 0 < t < T_0, \\ u(0) = x. \end{cases}$$

In section 2, we first show that $ACP(A, Cf, Cx, Cy)$ has a unique strong solution is equivalent to $v(\cdot) = C(\cdot)x + j_0 * C(\cdot)y + j_0 * C * f(\cdot) \in C^{\alpha+1}([0, T_0), X)$ and $D^{\alpha+1}v(\cdot) \in C^1((0, T_0), X)$ when A generates a nondegenerate local α-times integrated C-cosine function $C(\cdot)$ on X for some $\alpha \geq 0$, $f \in L^1_{loc}([0, T_0), X) \cap C((0, T_0), X)$, and $x, y \in X$. Here $j_\beta(t) = t^\beta/\Gamma(\beta+1)$ for $\beta > -1$ and $t > 0$, and $j_{-1}(t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{for } t \neq 0. \end{cases}$ In this case, $u = D^\alpha v$ (the αth order derivative of v) on $[0, T_0)$. Then, assuming $C(\cdot)$ is locally Lipschitz continuous, $g \in L^1_{loc}([0, T_0), X)$ and $z \in X$, we show that $ACP(A, Cz + j_{\alpha-1} * Cg, Cx, Cy)$ has a unique strong solution u in $C^2((0, T_0), X)$ (resp., in $C^2([0, T_0), X)$) when $x \in D(A)$,

$$(1.8) \qquad y \in \begin{cases} D(A^l) \text{ and } A^l y \in C^1 (\text{ resp., } A^l y \in D(A)) \text{ if } [\alpha] \text{ is even} \\ D(A^{l+1}) \text{ if } [\alpha] \text{ is odd,} \end{cases}$$

and

$$(1.9) \quad w (= Ax + z) \in \begin{cases} D(A^l) \text{ if } [\alpha] \text{ is even} \\ D(A^l) \text{ and } A^l w \in C^1 (\text{ resp., } A^l w \in D(A)) \text{ if } [\alpha] \text{ is odd.} \end{cases}$$

Moreover, $ACP(A, Cx + j_1 Cy + j_2 Cz + j_{\alpha-1} * Cg, 0, 0)$ has a unique strong solution u in $C^2((0, T_0), X)$ (resp., in $C^2([0, T_0), X)$) when $0 \leq \alpha < 1$; or $x \in D(A)$ and either $1 \leq \alpha < 2$, or $\alpha \geq 2$ with

$$(1.10)\qquad y \in \begin{cases} D(A^{l-1}) \text{ and } A^{l-1}y \in C^1 (\text{ resp., } A^{l-1}y \in D(A)) \text{ if } [\alpha] \text{ is even} \\ D(A^l) \text{ if } [\alpha] \text{ is odd} \end{cases}$$

and

$$(1.11)\qquad w(= Ax + z) \in \begin{cases} D(A^{l-1}) \text{ if } [\alpha] \text{ is even} \\ D(A^{l-1}) \text{ and } A^{l-1}w \in C^1 (\text{ resp., } A^{l-1}w \in D(A)) \\ \quad \text{if } [\alpha] \text{ is odd.} \end{cases}$$

Here $[\alpha]$ denotes the largest integer that is less than or equal to α, $l = [\frac{\alpha}{2}]$, and $C^1 = \{x \in X | C(\cdot)x \text{ is continuously differentiable on } (0, T_0)\}$. In particular, $ACP(A, Cz + j_{\alpha-3} * Cg, Cx, Cy)$ has a unique mild solution in $C^2((0, T_0), X)$ (resp., in $C^2([0, T_0), X))$ when $\alpha \geq 2$, and both (1.10) and (1.11) are satisfied. Applying these results we can also deduce some new approximation theorems in section 3 concerning the unique strong solution of $ACP(A, f, x, y)$.

2. Existence Theorems

From now on, we always write $[\alpha]$ to denote the largest integer that is less that or equal to the real number α, and $f * g(\cdot) = \int_0^{\cdot} f(\cdot - s)g(s)ds$ on $[0, t_0]$ for all $0 < t_0 < T_0$, $f \in L^1([0, t_0])$ the set of all $\mathbb{F}$-valued Lebesgue integrable functions on $[0, t_0]$, and $g \in L^1([0, t_0], X)$ the set of all Bochner integrable functions from $[0, t_0]$ into the Banach space X over $\mathbb{F}$.

Definition 2.1. Let $\alpha > 0$, $k = [\alpha] + 1$, and I be a subinterval of $[0, T_0)$ containing $\{0\}$. A function $v : I \to X$ is said to be α-times continuously differentiable on $[0, T_0)$, if $v = v(0) + j_{\alpha-k} * u$ on I for some $u \in C^{k-1}(I, X)$. In this case, we write $v \in C^\alpha(I, X)$, and the $(k-1)$th order derivative $u^{(k-1)}$ of u on I is called the αth order derivative of v on I and denoted by $D^\alpha v$ on I or $D^\alpha v : I \to X$. Here $C^k(I, X)$ denotes the set of all k-times continuously differentiable functions from I into X, and $C^0(I, X) = C(I, X)$ the set of all continuous functions from I into X.

Next we note some basic properties concerning nondegenerate local α-times integrated C-cosine functions which are frequently applied in the following and have been deduced as in [9] for the case $T_0 = \infty$, and so their proofs are omitted.

Proposition 2.2. *Let $\alpha \geq 0$, and A be the generator of a nondegenerate local α-times integrated C-cosine function $C(\cdot)$ on X. Then*

$$(2.1)\qquad C \text{ is injective and } C^{-1}AC = A,$$

$$(2.2)\qquad C(t)x \in D(A) \text{ and } AC(t)x = C(t)Ax$$

for all $x \in D(A)$ *and* $0 \leq t < T_0$,

$$\int_0^t S(r)xdr \in D(A) \text{ and } A\int_0^t S(r)xdr = C(t)x - j_\alpha(t)Cx \tag{2.3}$$

for all $x \in X$ *and* $0 \leq t < T_0$*, where* $S(r)x = \int_0^r C(s)xds$,

(2.4) $C(0)=C$ on X if $\alpha=0$, and $C(0)=0$ the zero operator on X if $\alpha>0$.

The next lemma is a direct consequence of Definition 2.1, and so its proof is omitted.

Lemma 2.3. *Let* $\alpha \geq 0$, $v \in C^\alpha(I, X)$ *with* $v(0) = 0$ *for some subinterval* I *of* $[0, T_0)$ *containing* $\{0\}$*, and* $k = [\alpha] + 1$*. Then* $j_{k-\alpha-1} * v \in C^k(I, X)$, $v \in C^{\alpha-i}(I, X)$*, and* $D^{\alpha-i}v = (j_{k-\alpha-1} * v)^{(k-i)}$ *on* I *for all integers* $0 \leq i \leq k-1$*. In particular, for each* $x \in X$*, we have* $j_\alpha(\cdot)x \in C^\alpha([0, T_0), X)$ *and* $D^{\alpha-i}j_\alpha(\cdot)x = D^{k-i}j_k(\cdot)x = j_i(\cdot)x$ *on* $[0, T_0)$ *for all integers* $0 \leq i \leq k-1$.

The next theorem is motivated by Arendt [1, Prop. 5.1 and Thm. 5.2] in which the first order Cauchy problem (1.7) is considered.

Theorem 2.4. *Let* A *be the generator of a nondegenerate local* α*-times integrated* C*-cosine function* $C(\cdot)$ *on* X *for some* $\alpha \geq 0$, $f \in L^1_{loc}([0, T_0), X) \cap C((0, T_0), X)$*, and* $x, y \in X$*. Assume that* $v(\cdot) = C(\cdot)x + S(\cdot)y + S * f(\cdot)$ *on* $[0, T_0)$*. Then* $ACP(A, f, x, y)$ *has a strong solution* u *if and only if* $v(t) \in R(C)$ *for all* $0 \leq t < T_0$, $C^{-1}v(\cdot) \in C^{\alpha+1}([0, T_0), X)$ *and* $D^{\alpha+1}C^{-1}v(\cdot) \in C^1((0, T_0), X)$*. Here* $S * f(t) = \int_0^t S(t-s)f(s)ds$ *for* $0 \leq t < T_0$*. In this case, we have* $u = D^\alpha C^{-1}v$*. Moreover,* $C^{-1}v \in C^{\alpha+2}([0, T_0), X)$ *(resp.,* $C^{-1}v \in C^\alpha([0, T_0), [D(A)])$*) if and only if* $u \in C^2([0, T_0), X)$ *(resp.,* $u \in C([0, T_0), [D(A)])$*).*

Proof. We consider only the case $\alpha>0$, for the case $\alpha=0$ can be treated similarly. Now if u is a strong solution of $ACP(A, f, x, y)$. For each $0 \leq t < T_0$, we set $w(\cdot) = C(t-\cdot)u(\cdot)$ on $[0, t]$. Since $u \in C^1([0, T_0), X)$, we have $\frac{d}{ds}w(s)|_{s=s_0} = \frac{d}{ds}C(t-s)u(s_0)|_{s=s_0} + C(t-s_0)u'(s)|_{s=s_0} = -j_{\alpha-1}(t-s_0)Cu(s_0) - S(t-s_0)Au(s_0) + C(t-s_0)u'(s_0)$ for all $0 \leq s_0 \leq t$. Since $u \in C^2((0, T_0), X) \cap C((0, T_0), [D(A)])$, we also have $u'(s) - u'(s_0) = \int_{s_0}^s Au(r)dr + \int_{s_0}^s f(r)dr = A\int_{s_0}^s u(r)dr + \int_{s_0}^s f(r)dr$ for all $0 \leq s_0 \leq s \leq t$, and so

$$\begin{aligned}
&\frac{d}{ds}w(s)\\
=&\frac{d}{ds}C(t-s)u(s)\\
=&-j_{\alpha-1}(t-s)Cu(s) - S(t-s)Au(s) + C(t-s)[y + A\int_0^s u(r)dr + \int_0^s f(r)dr]\\
=&-j_{\alpha-1}(t-s)Cu(s) - S(t-s)Au(s) + C(t-s)[y + j_0 * Au(s) + j_0 * f(s)]
\end{aligned}$$

for all $0 \le s \le t$. Hence

$$\begin{aligned}
&C(t)x\\
=&-\int_0^t \frac{d}{ds}w(s)ds\\
=&\int_0^t j_{\alpha-1}(t-s)Cu(s)ds + \int_0^t S(t-s)Au(s)ds-\\
&\left[\int_0^t C(t-s)yds + \int_0^t C(t-s)j_0 * Au(s)ds + \int_0^t C(t-s)j_0 * f(s)ds\right]\\
=&Cj_{\alpha-1} * u(t) + S * Au(t) - [S(t)y + S * Au(t) + S * f(t)]\\
=&Cj_{\alpha-1} * u(t) - S(t)y - S * f(t).
\end{aligned}$$

Consequently, $v(t) = Cj_{\alpha-1} * u(t) \in R(C)$ for all $0 \le t < T_0$, $C^{-1}v(\cdot) = j_{\alpha-1} * u(\cdot) \in C^{\alpha+1}([0,T_0),X)$, $D^{\alpha+1}C^{-1}v(\cdot) \in C^1((0,T_0),X)$, and $u = D^{\alpha}C^{-1}v$. Conversely, if $v(t) \in R(C)$ for all $0 \le t < T_0$, $C^{-1}v(\cdot) \in C^{\alpha+1}([0,T_0),X)$, and $D^{\alpha+1}C^{-1}v(\cdot) \in C^1((0,T_0),X)$. By (2.3) and (2.4) with $\alpha > 0$, we have $v(0) = 0$, $j_1 * v(t) \in D(A)$ and

$$\begin{aligned}
&Aj_1 * v(t)\\
=&C(t)x - j_{\alpha}(t)Cx + S(t)y - j_{\alpha+1}(t)Cy + S * f(t) - j_{\alpha+1} * Cf(t)\\
=&v(t) - C[j_{\alpha}(t)x + j_{\alpha+1}(t)y + j_{\alpha+1} * f(t)]
\end{aligned}$$

for all $0 \le t < T_0$, and so $ACj_1 * C^{-1}v(t) = Aj_1 * v(t) \in R(C)$ and

$$\begin{aligned}
Aj_1 * C^{-1}v(t) =&C^{-1}ACj_1 * C^{-1}v(t)\\
=&C^{-1}v(t) - [j_{\alpha}(t)x + j_{\alpha+1}(t)y + j_{\alpha+1} * f(t)]
\end{aligned}$$

for all $0 \le t < T_0$. Now if $k = [\alpha] + 1$. By Lemma 2.3, we have $D^{\alpha+1-i}j_{\alpha}(t) = \frac{d}{dt}D^{\alpha-i}j_{\alpha}(t) = \frac{d}{dt}j_i(t)$, $D^{\alpha+1-i}j_{\alpha+1}(t) = j_i(t)$ and $D^{\alpha+1}j_{\alpha}(t) = 0 = D^{\alpha+2}j_{\alpha+1}(t)$ for all integers $0 \le i \le k$ and all $0 \le t < T_0$. Combining this, and the closedness of A with the fact $j_{k-\alpha-1} * C^{-1}v(\cdot) \in C^{k+2}((0,T_0),X) \cap C^{k+1}([0,T_0),X)$, we have $Aj_1 * j_{k-\alpha-1} * C^{-1}v(t) = j_{k-\alpha-1} * C^{-1}v(t) - [j_k(t)x + j_{k+1}(t)y + j_{k+1} * f(t)]$ for all $0 \le t < T_0$, $AD^i(j_{k-\alpha-1} * C^{-1}v)(\cdot) = D^{i+2}(j_{k-\alpha-1} * C^{-1}v)(\cdot) - [j_{k-(i+2)}(\cdot)x + j_{k-(i+1)}(\cdot)y + j_{k-(i+1)} * f(\cdot)]$ on $[0,T_0)$ for all integers $0 \le i \le k-2$ if $k \ge 2$, $AD^{k-1}j_{k-\alpha-1} * C^{-1}v(\cdot) = D^{k+1}j_{k-\alpha-1} * C^{-1}v(\cdot) - (y + j_0 * f(\cdot))$ on $[0,T_0)$ and $AD^k j_{k-\alpha-1} * C^{-1}v(t) = D^{k+2}(j_{k-\alpha-1} * C^{-1}v)(t) - f(t)$ for all $0 \le t < T_0$. Combining these facts with induction, we also have $D^i j_{k-\alpha-1} * C^{-1}v(0) = 0$ for all integers $0 \leq i \leq k-1$ and $D^k j_{k-\alpha-1} * C^{-1}v(0) - x = 0 = D^{k+1} j_{k-\alpha-1} * C^{-1}v(0) - y$. Consequently, $D^k j_{k-\alpha-1} * C^{-1}v(\cdot) = D^\alpha C^{-1}v(\cdot)$ is a strong solution of $ACP(A, f, x, y)$. ∎

By slightly modifying the proof of Theorem 2.4, the next corollary is also attained.

Corollary 2.5. *Let A be the generator of a nondegenerate local α-times integrated C-cosine function $C(\cdot)$ on X for some $\alpha \geq 0$, $f \in L^1_{loc}([0, T_0), X) \cap C((0, T_0), X)$, and $x, y \in X$. Assume that $v(\cdot) = C(\cdot)x + S(\cdot)y + S * f(\cdot)$ on $[0, T_0)$. Then $ACP(A, Cf, Cx, Cy)$ has a unique strong solution u if and only if $v \in C^{\alpha+1}([0, T_0), X)$ and $D^{\alpha+1}v \in C^1((0, T_0), X)$. In this case, we have $u = D^\alpha v$ on $[0, T_0)$. Moreover, $v \in C^{\alpha+2}([0, T_0), X)$ (resp., $v \in C^\alpha([0, T_0), [D(A)])$) if and only if $u \in C^2([0, T_0), X)$ (resp., $u \in C([0, T_0), [D(A)])$).*

Proposition 2.6. *Let $\alpha \geq 1$, and $C(\cdot)$ be a nondegenerate locally Lipschitz continuous local α-times integrated C-cosine function on X with generator A. Then A_1 the part of A in $X_1 (= \overline{D(A)})$ generates a nondegenerate local $(\alpha - 1)$-times integrated C_1-cosine function $C_1(\cdot)$ on X_1. Here C_1 denotes the part of C in X_1 and $C_1(t)x = \frac{d}{dt}C(t)x$ for all $x \in X_1$ and $0 \leq t < T_0$.*

Proof. It is easy to see that $A_1 : D(A_1) \subset X_1 \to X_1$ is a closed linear operator satisfying $C_1^{-1}A_1C_1 = A_1$. Since $\{x \in X | C(\cdot)x$ is continuously differentiable on $[0, T_0)\}$ is a closed subspace of X containing $D(A)$, we have $\frac{d}{dt}C(t)x \in \overline{D(A)}$ for all $x \in \overline{D(A)}$ and $0 \leq t < T_0$. Applying the closedness of A and (2.3), we also have $C_1(t)x - j_{\alpha-1}(t)C_1x = \frac{d}{dt}C(t)x - j_{\alpha-1}(t)Cx = AS(t)x = A_1 \int_0^t \int_0^s C_1(r)x\,dr\,ds$ for all $x \in \overline{D(A)}$ and $0 \leq t < T_0$. It follows from the uniqueness of solutions of $ACP(A, j_{\alpha-1}(\cdot)Cx, 0, 0)$ that $S(\cdot)x = j_1 * C_1(\cdot)x$ is the unique strong solution of $ACP(A_1, j_{\alpha-1}C_1x, 0, 0)$ in $C^2([0, T_0), X_1) \cap C([0, T_0), [D(A_1)])$ for all $x \in \overline{D(A)}$. We conclude from [9, Thm.2.3 or 11, Thm.2.5] that $C_1(\cdot)$ is a nondegenerate local $(\alpha - 1)$-times integrated C_1-cosine function on X_1 with generator A_1. ∎

Remark 2.7. Let A be the generator of a nondegenerate locally Lipschitz continuous local α-times integrated C-cosine function $C(\cdot)$ on X for some $0 < \alpha < 1$. Then A is also the generator of a nondegenerate norm continuous local C-cosine function $\widetilde{C}(\cdot)$ on X which is defined by $\widetilde{C}(t)x = \frac{d}{dt}C * j_{-\alpha}(t)x$ for all $x \in X$ and $0 \leq t < T_0$. In particular, $A \in B(X)$ if $C = I$.

Proposition 2.8. *Let $\alpha \geq 1$, and $C(\cdot)$ be a nondegenerate locally Lipschitz continuous local α-times integrated C-cosine function on X with generator A. Then for each $0 < \theta < 1$ there exists a nondegenerate local $(\alpha - 1 + \theta)$-times integrated*

C-cosine function $\widetilde{C}(\cdot)$ on X with generator A such that for each $0 < t_0 < T_0$, we have

$$\|\widetilde{C}(t+h) - \widetilde{C}(t)\| \leq K_{t_0} h^{\theta} \tag{2.5}$$

for all $0 \leq t, h \leq t+h \leq t_0$, where K_{t_0} is given as in (1.6).

Proof. Clearly, $-1 < \theta - 1 < 0$. It follows that $C * j_{\theta-1}(\cdot)x \in C^1([0,T_0), X)$ for all $x \in X$, and $C*j_{\theta-1}(\cdot)$ is a local $(\alpha+\theta)$-times integrated C-cosine function on X with generator A. Now let $\widetilde{C}(t) : X \to X$ be defined by $\widetilde{C}(t)x = \frac{d}{dt} C*j_{\theta-1}(t)x$ for all $x \in X$. Just as in the proof of Proposition 2.6, we can show that $\widetilde{C}(\cdot)$ is a local $(\alpha - 1 + \theta)$-times integrated C-cosine function on X with generator A and (2.5) is satisfied. ■

Theorem 2.9. *Let $\alpha \geq 0$, $l = [\frac{\alpha}{2}]$, and $C(\cdot)$ be a nondegenerate locally Lipschitz continuous local α-times integrated C-cosine function on X with generator A. Assume that $x \in D(A)$, $y, z \in X$, and $g \in L^1_{loc}([0,T_0), X)$. Then $ACP(A, Cz + j_{\alpha-1} * Cg, Cx, Cy)$ has a unique strong solution u in $C^2([0,T_0), X)$(resp., in $C^2((0,T_0), X))$ when*

$$y \in \begin{cases} D(A^l) \text{ and } A^l y \in D(A) (\text{ resp., } A^l y \in C^1) \text{ if } [\alpha] \text{ is even} \\ D(A^{l+1}) \text{ if } [\alpha] \text{ is odd} \end{cases} \tag{2.7}$$

and

$$w (= Ax + z) \in \begin{cases} D(A^l) \text{ if } [\alpha] \text{ is even} \\ D(A^l) \text{ and } A^l w \in D(A) (\text{resp., } A^l w \in C^1) \text{ if } [\alpha] \text{ is odd}\ . \end{cases} \tag{2.8}$$

In fact, $ACP(A, Cz + j_{\alpha-1} * Cg, Cx, Cy)$ has a unique strong solution u in $C^2([0,T_0), X)$ when $\alpha \in \mathbb{N} \cup \{0\}$, $x \in D(A)$, and either α is even with $A^l y \in \overline{D(A)}$ and $w \in D(A^l)$; or α is odd with $y \in D(A^{l+1})$ and $A^l w \in \overline{D(A)}$.

Proof. Indeed, if we set $k = [\alpha]$ and $f = z + j_{\alpha-1} * g$ on $[0, T_0)$, then $k = \begin{cases} 2l \text{ if } [\alpha] \text{ is even} \\ 2l+1 \text{ if } [\alpha] \text{ is odd.} \end{cases}$ By (2.3), we have $\widetilde{C}(\cdot)x + \widetilde{S}(\cdot)y + \widetilde{S} * f(\cdot) = j_k(\cdot)Cx + \widetilde{S}(\cdot)y + j_0 * \widetilde{S}(\cdot)w + j_{\alpha-1} * \widetilde{S} * g(\cdot)$ on $[0, T_0)$. Here $\widetilde{C}(\cdot)$ denotes the local $[\alpha]$-times integrated C-cosine function on X with generator A which is given as in either Remark 2.7 when $0 \leq \alpha < 1$ or Proposition 2.8 when $\alpha \geq 1$ with $\theta = [\alpha] - (\alpha - 1)$, and $\widetilde{S}(\cdot) = j_0 * \widetilde{C}(\cdot)$. Applying Corollary 2.5, we need only to show that $v(\cdot) = \widetilde{C}(\cdot)x + \widetilde{S}(\cdot)y + \widetilde{S} * f(\cdot) \in C^{k+2}([0, T_0), X)$(resp., $v(\cdot) \in C^{k+2}((0, T_0), X))$. Now if $k = 0$, then $l = 0$, and so $j_0 * \widetilde{S}(\cdot)w$, $j_{\alpha-1} *$

$\widetilde{S} * g = S * g \in C^2([0, T_0), X)$, and $\widetilde{S}(\cdot)y \in C^2([0, T_0), X)$ for $y \in D(A)$ (resp., $\widetilde{S}(\cdot)y \in C^2((0, T_0), C) \cap C^1([0, T_0), X)$ for $y \in C^1$). Hence $v \in C^{k+2}([0, T_0), X)$ for $y \in D(A)$ (resp., $v \in C^{k+2}((0, T_0), X) \cap C^{k+1}([0, T_0), X)$ for $y \in C^1$). Next if $k \geq 1$, then

$$\frac{d^k}{dt^k} j_{\alpha-1} * \widetilde{S} * g(t) = j_{\alpha-k-1} * \widetilde{S} * g(t) = S * g(t), \tag{2.9}$$

$$\frac{d^k}{dt^k} \widetilde{S}(t)y = \frac{d^{k-1}}{dt^{k-1}} \widetilde{C}(t)y = \begin{cases} \widetilde{C}(t)y & \text{if } k = 1 \\ \dfrac{d}{dt} \dfrac{d^{2(l-1)}}{dt^{2(l-1)}} \widetilde{C}(t)y & \text{if } k = 2l \geq 2. \\ \dfrac{d^{2l}}{dt^{2l}} \widetilde{C}(t)y & \text{if } k = 2l+1 \geq 3, \end{cases} \tag{2.10}$$

and

$$\begin{aligned} &\frac{d^k}{dt^k} j_0 * \widetilde{S}(t)w \\ &= \begin{cases} \widetilde{S}(t)w & \text{if } k = 1 \\ \dfrac{d^{k-2}}{dt^{k-2}} \widetilde{C}(t)w = \dfrac{d^{2(l-1)}}{dt^{2(l-1)}} \widetilde{C}(t)w & \text{if } k = 2l \geq 2 \\ \dfrac{d}{dt} \dfrac{d^{k-1}}{dt^{k-1}} j_0 * \widetilde{S}(t)w = \dfrac{d}{dt} \dfrac{d^{2(l-1)}}{dt^{2(l-1)}} \widetilde{C}(t)w & \text{if } k = 2l+1 \geq 3. \end{cases} \end{aligned} \tag{2.11}$$

By induction, we have

$$\begin{aligned} \frac{d^{2m}}{dt^{2m}} \widetilde{C}(t)v &= \sum_{i=0}^{m-1} j_{k-2(i+1)}(t) C A^{m-1-i} v + \widetilde{C}(t) A^m v \\ &= \sum_{i=1}^{m} j_{k-2i}(t) C A^{m-i} v + \widetilde{C}(t) A^m v \end{aligned} \tag{2.12}$$

for all $m \in \mathbb{N}$ and $v \in D(A^m)$, and so

$$(\widetilde{C}(\cdot)v)^{(2m)} \in \begin{cases} C^2([0, T_0), X) \text{ if } v \in D(A^{m+1}) \\ C^1([0, T_0), X) \text{ if } \alpha \in \mathbb{N}, v \in D(A^m) \text{ and } A^m v \in \overline{D(A)} \\ C^1((0, T_0), X) \text{ if } v \in D(A^m) \text{ and } A^m v \in C^1 \end{cases} \tag{2.13}$$

for all $m \in \mathbb{N} \cup \{0\}$, where $(\widetilde{C}(\cdot)v)^{(0)} = \widetilde{C}(\cdot)v$ and $A^0 = I$. Hence

$$(2.14)\qquad \begin{aligned}&(\widetilde{S}(\cdot)y)^{(k+1)}\\ &=\begin{cases}(\widetilde{C}(\cdot)y)' \in C^1([0,T_0),X) \\ \qquad \text{if } k=1 \text{ and } y\in D(A) \\ (\widetilde{C}(\cdot)y)^{(2l)} \in C^1((0,T_0),X)\cap C([0,T_0),X) \\ \qquad \text{if } k=2l\geq 2, y\in D(A^l) \text{ and } A^l y\in C^1 \\ (\widetilde{C}(\cdot)y)^{(2l+1)} \in C^1([0,T_0),X) \\ \qquad \text{if } k=2l+1\geq 3 \text{ and } y\in D(A^{l+1})\end{cases}\end{aligned}$$

and

$$(2.15)\qquad \begin{aligned}&(j_0*\widetilde{S}(\cdot)w)^{(k+1)}\\ &=\begin{cases}\widetilde{C}(\cdot)w \in C^1((0,T_0),X)\cap C([0,T_0),X) \\ \qquad \text{if } k=1 \text{ and } w\in C^1 \\ (\widetilde{C}(\cdot)w)^{(2l-1)} \in C^1([0,T_0),X) \\ \qquad \text{if } k=2l\geq 2 \text{ and } w\in D(A^l) \\ (\widetilde{C}(\cdot)w)^{(2l)} \in C^1((0,T_0),X)\cap C([0,T_0),X) \\ \qquad \text{if } k=2l+1\geq 3, w\in D(A^l) \text{ and } A^l w\in C^1.\end{cases}\end{aligned}$$

Consequently, $v(\cdot)\in C^{(k+2)}([0,T_0),X)$ (resp., $v(\cdot)\in C^{(k+2)}((0,T_0),X)\cap C^{k+1}([0,T_0),X))$ and $u=D^k v$ is a strong solution of $ACP(A,Cz+j_{\alpha-1}*Cg,Cx,Cy)$ in $C^2([0,T_0),X)$ (resp., in $C^2((0,T_0),X)$) when (2.7) and (2.8) both are satisfied. The uniqueness of strong solutions of $ACP(A,Cz+j_{\alpha-1}*Cg,Cx,Cy)$ follows from the uniqueness of strong solutions of $ACP(A,0,0,0)$ (see [9, Thm. 2.3] or [11, Thm. 2.4]). In this case, we have

$$(2.16)\qquad \begin{aligned}&u(\cdot)=S*g(\cdot)+Cx\\ &+\begin{cases}\widetilde{S}(\cdot)y+j_0*\widetilde{S}(\cdot)w \text{ if } k=0\\ \widetilde{C}(\cdot)y+\widetilde{S}(\cdot)w \text{ if } k=1\\ \sum_{i=0}^{l-1} j_{k-(2i+1)}(\cdot)CA^{l-1-i}y+\widetilde{S}(\cdot)A^l y\\ \quad +\sum_{i=1}^{l-1} j_{k-2i}(\cdot)CA^{l-1-i}w+\widetilde{C}(\cdot)A^{l-1}w \text{ if } k=2l\geq 2\\ \sum_{i=1}^{l} j_{k-2i}(\cdot)CA^{l-i}y+\widetilde{C}(\cdot)A^l y\\ \quad +\sum_{i=0}^{l-1} j_{k-(2i+1)}(\cdot)CA^{l-1-i}w+\widetilde{S}(\cdot)A^l w \text{ if } k=2l+1\geq 3,\end{cases}\end{aligned}$$

$$u'(\cdot) = C * g(\cdot) + \tag{2.17}$$
$$\begin{cases} \widetilde{C}(\cdot)y + \widetilde{S}(\cdot)w & \text{if } k=0 \\ Cy + \widetilde{S}(\cdot)Ay + \widetilde{C}(\cdot)w & \text{if } k=1 \\ \sum\limits_{i=1}^{l} j_{k-2i}(\cdot)CA^{l-i}y + \widetilde{C}(\cdot)A^l y & \\ \quad + \sum\limits_{i=0}^{l-1} j_{k-(2i+1)}A^{l-1-i}w + \widetilde{S}(\cdot)A^l w & \text{if } k=2l \geq 2 \\ \sum\limits_{i=0}^{l} j_{k-(2i+1)}(\cdot)CA^{l-i}y + \widetilde{S}(\cdot)A^{l+1}y & \\ \quad + \sum\limits_{i=1}^{l} j_{k-2i}(\cdot)CA^{l-i}w + \widetilde{C}(\cdot)A^l w & \text{if } k=2l+1 \geq 3 \end{cases}$$

on $[0, T_0)$, and

$$u''(\cdot) = (C * g)'(\cdot) + \tag{2.18}$$
$$\begin{cases} (\widetilde{C}(\cdot)y)' + \widetilde{C}(\cdot)w & \text{if } k=0 \\ \widetilde{C}(\cdot)Ay + Cw + \widetilde{S}(\cdot)Aw & \text{if } k=1 \\ \sum\limits_{i=1}^{l} j_{k-(2i+1)}(\cdot)CA^{l-i}y + (\widetilde{C}(\cdot)A^l y)' & \\ \quad + \sum\limits_{i=1}^{l} j_{k-2i}(\cdot)CA^{l-i}w + \widetilde{C}(\cdot)A^l w & \text{if } k=2l \geq 2 \\ \sum\limits_{i=1}^{l+1} j_{k-2i}(\cdot)CA^{l+1-i}y + \widetilde{C}(\cdot)A^{l+1}y & \\ \quad + \sum\limits_{i=1}^{l} j_{k-(2i+1)}(\cdot)CA^{l-i}w + (\widetilde{C}(\cdot)A^l w)' & \text{if } k=2l+1 \geq 3 \end{cases}$$

on $[0, T_0)$ (resp., on $(0, T_0)$). ∎

By slightly modifying the proof of Theorem 2.9, the next theorem is also attained.

Theorem 2.10. *Let $\alpha \geq 0$, $l = [\frac{\alpha}{2}]$, and $C(\cdot)$ be a nondegenerate locally Lipschitz continuous local α-times integrated C-cosine function on X with generator A. Assume that $x, y, z \in X$ and $g \in L^1_{loc}([0, T_0), X)$. Then $ACP(A, Cx + j_1 Cy + j_2 Cz + j_{\alpha-1} * Cg, 0, 0)$ has a unique strong solution u in $C^2([0, T_0), X)$ (resp., in $C^2((0, T_0), X)$) when $0 \leq \alpha < 1$; or $x \in D(A)$ and either $1 \leq \alpha < 2$, or $\alpha \geq 2$ with*

$$y \in \begin{cases} D(A^{l-1}) \text{ and } A^{l-1}y \in D(A) (\text{ resp., } A^{l-1}y \in C^1) & \text{if } [\alpha] \text{ is even} \\ D(A^l) & \text{if } [\alpha] \text{ is odd} \end{cases} \tag{2.19}$$

and

$$w(=Ax+z)\in \tag{2.20}$$
$$\begin{cases} D(A^{l-1}) & \text{if } [\alpha] \text{ is even} \\ D(A^{l-1}) \text{ and } A^{l-1}w\in D(A)(\text{ resp., } A^{l-1}w\in C^1) & \text{if } [\alpha] \text{ is odd.} \end{cases}$$

In fact, $ACP(A, Cx+j_1Cy+j_2Cz+j_{\alpha-1}*Cg, 0, 0)$ *has a unique strong solution in* $C^2([0,T_0),X)$ *when* $\alpha\in\mathbb{N}\setminus\{1\}$, $x\in D(A)$, *and either* α *is even with* $A^{l-1}y\in\overline{D(A)}$ *and* $w\in D(A^{l-1})$; *or* α *is odd with* $y\in D(A^l)$ *and* $A^{l-1}w\in\overline{D(A)}$.

Remark 2.11. If $\alpha\geq 2$, and u is the unique strong solution of $ACP(A, Cx+j_1Cy+j_2Cz+j_{\alpha-1}*Cg, 0, 0)$ in $C^2([0,T_0),X)$. Then u'' is the unique mild solution of $ACP(A, Cz+j_{\alpha-3}*Cg, Cx, Cy)$ in $C([0,T_0),X)$. That is, u'' is the unique continuous function v from $[0,T_0)$ into X which satisfies the integral equation $v=Aj_1*v+Cx+j_1(\cdot)Cy+j_1*(Cz+j_{\alpha-3}*Cg)$ on $[0,T_0)$.

Corollary 2.12. *Let* $\alpha\geq 0$, $l=[\frac{\alpha}{2}]$, *and* $C(\cdot)$ *be a nondegenerate locally Lipschitz continuous local* α*-times integrated* C*-cosine function on* X *with generator* A. *Assume that* $x\in D(A)$, $y,z\in X$, *and* $g\in L^1_{loc}([0,T_0),X)$. *Then* $ACP(A, Cz+j_{\alpha-1}*Cg, Cx, Cy)$ *has a unique strong solution* u *in* $C^2([0,T_0),X)$ *when* $y\in D(A^{l+1})$ *and*

$$w(=Ax+z)\in\begin{cases} D(A^l) \text{ if } [\alpha] \text{ is even} \\ D(A^{l+1}) \text{ if } [\alpha] \text{ is odd .}\end{cases}$$

Corollary 2.13. *Let* $\alpha\in\mathbb{N}\cup\{0\}$, $l=[\frac{\alpha}{2}]$, *and* $C(\cdot)$ *be a nondegenerate locally Lipschitz continuous local* α*-times integrated* C*-cosine function on* X *with generator* A. *Assume that* $x\in D(A)$, $y,z\in X$, *and* $g\in L^1_{loc}([0,T_0),X)$. *Then* $ACP(A, Cz+j_{\alpha-1}*Cg, Cx, Cy)$ *has a unique strong solution* u *in* $C^2([0,T_0),X)$ *when either* α *is even with* $A^ly\in\overline{D(A)}$ *and* $w\in D(A^l)$; *or* α *is odd with* $y\in D(A^{l+1})$ *and* $A^lw\in\overline{D(A)}$.

Corollary 2.14. *Let* $\alpha\in\mathbb{N}\cup\{0\}$, $l=[\frac{\alpha+1}{2}]$, *and* $C(\cdot)$ *be a nondegenerate local* α*-times integrated* C*-cosine function on* X *with densely defined generator* A. *Assume that* $x\in D(A)$, $y,z\in X$, *and* $g\in L^1_{loc}([0,T_0),X)$. *Then* $ACP(A, Cz+j_\alpha*Cg, Cx, Cy)$ *has a unique strong solution* u *in* $C^2([0,T_0),X)$ *when* $w\in D(A^l)$, *and either* α *is even with* $y\in D(A^{l+1})$; *or* α *is odd with* $y\in D(A^l)$.

Corollary 2.15. *Let* $\alpha\geq 0$, $l=[\frac{\alpha}{2}]$, *and* $C(\cdot)$ *be a nondegenerate locally Lipschitz continuous local* α*-times integrated* C*-cosine function on* X *with generator* A. *Assume that* $x,y,z\in X$ *and* $g\in L^1_{loc}([0,T_0),X)$. *Then* $ACP(A, Cx+j_1Cy+j_2Cz+j_{\alpha-1}*Cg, 0, 0)$ *has a unique strong solution* u *in* $C^2([0,T_0),X)$ *when* $0\leq\alpha<1$; *or* $x\in D(A)$ *and either* $1\leq\alpha<2$, *or* $\alpha\geq 2$ *with* $y\in D(A^l)$ *and*

$$w(=Ax+z) \in \begin{cases} D(A^{l-1}) \text{ if } \alpha \text{ is even} \\ D(A^l) \text{ if } \alpha \text{ is odd} . \end{cases}$$

Corollary 2.16. *Let $\alpha \in \mathbb{N}$, $l = [\frac{\alpha}{2}]$, and $C(\cdot)$ be a nondegenerate locally Lipschitz continuous local α-times integrated C-cosine function on X with generator A. Assume that $x \in D(A)$, $y, z \in X$, and $g \in L^1_{loc}([0, T_0), X)$. Then $ACP(A, Cx + j_1Cy + j_2Cz + j_{\alpha-1} * Cg, 0, 0)$ has a unique strong solution u in $C^2([0, T_0), X)$ when $\alpha = 1$, or $\alpha \geq 2$ and either α is even with $A^{l-1}y \in \overline{D(A)}$ and $w \in D(A^{l-1})$; or α is odd with $y \in D(A^l)$ and $A^{l-1}w \in \overline{D(A)}$.*

Corollary 2.17. *Let $\alpha \in \mathbb{N} \cup \{0\}$, $l = [\frac{\alpha+1}{2}]$, and $C(\cdot)$ be a nondegenerate local α-times integrated C-cosine function on X with densely defined generator A. Assume that $x \in D(A)$, $y, z \in X$, and $g \in L^1_{loc}([0, T_0), X)$. Then $ACP(A, Cx + j_1Cy + j_2Cz + j_\alpha * Cg, 0, 0)$ has a unique strong solution u in $C^2([0, T_0), X)$ when $\alpha = 0$; or $\alpha \geq 1$ with $w(= Ax + z) \in D(A^{l-1})$ and*

$$y \in \begin{cases} D(A^l) \text{ if } \alpha \text{ is even} \\ D(A^{l-1}) \text{ if } \alpha \text{ is odd} . \end{cases}$$

3. Approximation Theorems

Definition 3.1. A sequence of local α-times integrated C-cosine functions $\{C_m(\cdot)\}_{m=1}^\infty$ on X is said to be uniformly locally Lipschitz continuous, if for each $0 < t_0 < T_0$ there exists a $K_{t_0} > 0$ such that

$$\|C_m(t+h) - C_m(t)\| \leq K_{t_0} h \tag{3.1}$$

for all $0 \leq t, h \leq t + h \leq t_0$ and $m \in \mathbb{N}$.

We first apply Theorem 2.9 to obtain an approximation theorem concerning strong solutions of $ACP(A, Cz + j_{\alpha-1} * Cg, Cx, Cy)$ in $C^2([0, T_0), X)$.

Theorem 3.2. *Let $\alpha > 0$, the hypotheses of Corollary 2.12 hold for $C(\cdot)$, A, x, y, z and $w(= Ax + z)$, and also for $C_m(\cdot)$, A_m, x_m, y_m, z_m and $w_m(= A_m x_m + z_m)$ in place of $C(\cdot)$, A, x, y, z and w, respectively. Assume that*

(i) $\{C_m(\cdot)\}_{m=1}^\infty$ is uniformly locally Lipschitz continuous, and $\lim_{m\to\infty} C_m(\cdot)v = C(\cdot)v$ uniformly on compact subsets of $[0, T_0)$ for all $v \in X$,

(ii) $x_m \to x$, and $A^i_m y_m \to A^i y$ in X for all integers $0 \leq i \leq l + 1$,

(iii) $A^i_m w_m \to A^i w$ in X for all integers $0 \leq i \leq l$ if $[\alpha]$ is even; or $A^i_m w_m \to A^i w$ in X for all integers $0 \leq i \leq l + 1$ if $[\alpha]$ is odd,

(*iv*) $g_m \to g$ *in* $L^1_{loc}([0,T_0),X)$. *That is,* $\|g_m - g\|_{L^1([0,t_0],X)}(= \int_0^{t_0} \|g_m(s) - g(s)\|ds) \to 0$ *in* $\mathbb{R}$ *for all* $0 < t_0 < T_0$.

Then the strong solution u_m *of* $ACP(A_m, Cz_m + j_{\alpha+1} * Cg_m, Cx_m, Cy_m)$ *converges to the strong solution* u *of* $ACP(A, Cz+j_{\alpha+1}*Cg, Cx, Cy)$ *in* $C^2([0,T_0),X)$. *That is,* $u_m \to u$, $u'_m \to u'$ *and* $u''_m \to u''$ *uniformly on compact subsets of* $[0,T_0)$.

Proof. Indeed, if $k = [\alpha]$, and $\widetilde{C}_m(\cdot)$ denotes the local k-times integrated C-cosine function on X with generator A_m. By (2.6), we have $\widetilde{C}_m(t)v = \frac{d}{dt}j_{k-\alpha} * C_m(t)v$ for all $v \in X$ and $0 \le t < T_0$. Combining (2.16)-(2.18), we also have

$$u_m(\cdot) = S_m * g_m(\cdot) + Cx_m +$$

$$\tag{3.2} \begin{cases} \widetilde{S}_m(\cdot)y_m + j_0 * \widetilde{S}_m(\cdot)w_m & \text{if } k=0 \\ \widetilde{C}_m(\cdot)y_m + \widetilde{S}_m(\cdot)w_m & \text{if } k=1 \\ \sum\limits_{i=0}^{l-1} j_{k-(2i+1)}(\cdot)CA_m^{l-1-i}y_m + \widetilde{S}_m(\cdot)A_m^l y_m & \\ \quad + \sum\limits_{i=1}^{l-1} j_{k-2i}(\cdot)CA_m^{l-1-i}w_m + \widetilde{C}_m(\cdot)A_m^{l-1}w_m & \text{if } k=2l\ge 2 \\ \sum\limits_{i=1}^{l} j_{k-2i}(\cdot)CA_m^{l-i}y_m + \widetilde{C}_m(\cdot)A_m^l y_m & \\ \quad + \sum\limits_{i=0}^{l-1} j_{k-(2i+1)}(\cdot)CA_m^{l-1-i}w_m + \widetilde{S}_m(\cdot)A_m^l w_m & \text{if } k=2l+1\ge 3, \end{cases}$$

$$u'_m(\cdot) = C_m * g_m(\cdot) +$$

$$\tag{3.3} \begin{cases} \widetilde{C}_m(\cdot)y_m + \widetilde{S}_m(\cdot)w_m & \text{if } k=0 \\ Cy_m + \widetilde{S}_m(\cdot)A_m y_m + \widetilde{C}_m(\cdot) & \text{if } k=1 \\ \sum\limits_{i=1}^{l} j_{k-2i}(\cdot)CA_m^{l-i}y_m + \widetilde{C}_m(\cdot)A_m^l y_m & \\ \quad + \sum\limits_{i=0}^{l-1} j_{k-(2i+1)}(\cdot)CA_m^{l-1-i}w_m + \widetilde{S}_m(\cdot)A_m^l w_m & \text{if } k=2l\ge 2 \\ \sum\limits_{i=0}^{l} j_{k-(2i+1)}(\cdot)CA_m^{l-i}y_m + \widetilde{S}_m(\cdot)A_m^{l+1} y_m & \\ \quad + \sum\limits_{i=0}^{l} j_{k-(2i+1)}(\cdot)CA_m^{l-i}w_m + \widetilde{C}_m(\cdot)A_m^l w_m & \text{if } k=2l+1\ge 3, \end{cases}$$

and

$$(3.4)\quad u''_m(\cdot) = (C_m * g_m)'(\cdot) + \begin{cases} \widetilde{S}_m(\cdot)A_m y_m + \widetilde{C}_m(\cdot)w_m & \text{if } k=0 \\ \widetilde{C}_m(\cdot)A_m y_m + Cw_m + \widetilde{S}(\cdot)A_m w_m & \text{if } k=1 \\ \sum_{i=0}^{l} j_{k-(2i+1)}(\cdot)CA_m^{l-i}y_m + \widetilde{S}_m(\cdot)A_m^{l+1}y_m & \\ \quad + \sum_{i=1}^{l} j_{k-2i}(\cdot)CA_m^{l-i}w_m + \widetilde{C}_m(\cdot)A_m^l w_m & \text{if } k=2l\geq 2 \\ \sum_{i=1}^{l+1} j_{k-2i}(\cdot)CA_m^{l+1-i}y_m + \widetilde{C}_m(\cdot)A_m^{l+1}y_m & \\ \quad + \sum_{i=0}^{l} j_{k-(2i+1)}(\cdot)CA_m^{l-i}w_m + \widetilde{S}_m(\cdot)A_m^{l+1}w_m & \text{if } k=2l+1\geq 3 \end{cases}$$

on $[0, T_0)$. To show that $u_m \to u$ in $C^2([0, T_0), X)$, we shall first show that $C_m * g_m \to C * g$ uniformly on compact subsets of $[0, T_0)$. Indeed, if $0 < t_0 < T_0$ is fixed. Then for each $\phi \in C([0, t_0], X)$, we deduce from the uniform continuity of ϕ on $[0, t_0]$, the uniform boundedness of $\{\|C_m(\cdot\|\}_{m=1}^{\infty}$ on $[0, t_0]$ and (i) that $C_m(t-\cdot)\phi(\cdot) \to C(t-\cdot)\phi(\cdot)$ uniformly on $[0, t]$ for all $0 < t \leq t_0$, and so $C_m * \phi(t) \to C * \phi(t)$ in X for all $0 \leq t \leq t_0$. The uniform Lipschitz continuity of $\{C_m(\cdot)\}_{m=1}^{\infty}$ on $[0, t_0]$ implies that $\{C_m * \phi(\cdot)\}_{m=1}^{\infty}$ is uniformly bounded and equicontinuous on $[0, t_0]$. It follows from the pointwise convergence of $\{C_m * \phi(\cdot)\}_{m=1}^{\infty}$ to $C * \phi(\cdot)$ on $[0, t_0]$ and Arzela-Ascoli's theorem that each subsequence of $\{C_m * \phi(\cdot)\}_{m=1}^{\infty}$ contains a subsequence which converges to $C * \phi(\cdot)$ uniformly on $[0, t_0]$. Hence $C_m * \phi(\cdot) \to C * \phi(\cdot)$ uniformly on $[0, t_0]$ for all $\phi \in C([0, t_0], X)$. Combining this, and the uniform boundedness of $\{\|C_m(\cdot)\|\}_{m=1}^{\infty}$ on $[0, t_0]$ with the denseness of $C([0, t_0], X)$ in $L^1([0, t_0], X)$, we have $C_m * \phi(\cdot) \to C * \phi(\cdot)$ uniformly on $[0, t_0]$ for all $\phi \in L^1([0, t_0], X)$. Consequently, $C_m * \phi(\cdot) \to C * \phi(\cdot)$ uniformly on compact subsets of $[0, T_0)$ for all $\phi \in L^1_{loc}([0, T_0), X)$. In particular, $C_m * g_m(\cdot) = C_m * (g_m - g)(\cdot) + C_m * g(\cdot) \to C * g(\cdot)$ uniformly on compact subsets of $[0, T_0)$. Similarly, we can show that $\widetilde{C}_m(\cdot)v_m \to \widetilde{C}(\cdot)v$ and $\widetilde{S}_m(\cdot)v_m \to \widetilde{S}(\cdot)v$ uniformly on compact subsets of $[0, T_0)$ whenever $v_m \to v$ in X. Here $\widetilde{C}(\cdot)$ denotes the local k-times integrated C-cosine function on X with generator A and $\widetilde{S}(\cdot) = j_0 * \widetilde{C}(\cdot)$. To show that $u_m \to u$ in $C^2([0, T_0), X)$, we observe from (i)-(iii) and (3.1)-(3.4) that it remains to show that $(C_m * g_m)'(\cdot) \to (C * g)'(\cdot)$ uniformly on compact subsets of $[0, T_0)$. Indeed, if $0 < t_0 < T_0$ is fixed. Then for each $\phi \in C^1([0, t_0], X)$, we deduce from the previous argument and (i) that $(C_m * \phi)'(\cdot) = C_m * \phi'(\cdot) + C_m(\cdot)\phi(0) \to C * \phi'(\cdot) + C(\cdot)\phi(0) = (C * \phi)'(\cdot)$ uniformly on $[0, t_0]$. Combining this, and the denseness of $C^1([0, t_0], X)$ in $L^1([0, t_0], X)$ with the fact

$$(3.5)\qquad \|(C_m * \phi)'(t)\| \leq K_{t_0} \int_0^t \|\phi(s)\| ds$$

for all $\phi \in L^1([0,t_0],X)$, $m \in \mathbb{N}$ and $0 \le t \le t_0$, we have $(C_m*\phi)'(\cdot) \to (C*\phi)'(\cdot)$ uniformly on $[0,t_0]$ for all $\phi \in L^1([0,t_0],X)$, where K_{t_0} is given as in (3,1). Consequently, $(C_m*\phi)'(\cdot) \to (C*\phi)'(\cdot)$ uniformly on compact subsets of $[0,T_0)$ for all $\phi \in L^1_{loc}([0,T_0),X)$, which together with (3.5) implies that $(C_m*g_m)'(\cdot) = (C_m*(g_m-g))'(\cdot) + (C_m*g)'(\cdot) \to (C*g)'(\cdot)$ uniformly on compact subsets of $[0,T_0)$. ∎

Similarly, we can apply Theorem 2.10 to obtain the next approximation theorem concerning strong solutions of $ACP(A.Cx + j_1Cy + j_2Cz + j_{\alpha-1}*Cg, 0, 0)$ in $C^2([0,T_0),X)$.

Theorem 3.3. *Let $\alpha > 0$, the hypotheses of Corollary 2.15 hold for $C(\cdot)$, A, x, y, z and $w(= Ax + z)$, and also for $C_m(\cdot)$, A_m, x_m, y_m, z_m and $w_m(= A_mx_m + z_m)$ in place of $C(\cdot)$, A, x, y, z and w, respectively. Assume that*

- (i) *$\{C_m(\cdot)\}_{m=1}^{\infty}$ is uniformly locally Lipschitz continuous, and $\lim_{m\to\infty} C_m(\cdot)v = C(\cdot)v$ uniformly on compact subsets of $[0,T_0)$ for all $v \in X$,*
- (ii) *$x_m \to x$, and $A_m^i y_m \to A^i y$ in X for all integers $0 \le i \le l$,*
- (iii) *$z_m \to z$ in X if $0 \le \alpha < 1$; $A_m^i w_m \to A^i w$ in X for all integers $0 \le i \le (l-1)$ if $\alpha \ge 1$ and $[\alpha]$ is even; or $A_m^i w_m \to A^i w$ in X for all integers $0 \le i \le l$ if $\alpha \ge 1$ and $[\alpha]$ is odd,*
- (iv) *$g_m \to g$ in $L^1_{loc}([0,T_0),X)$.*

Then the strong solution u_m of $ACP(A_m, Cx_m + j_1Cy_m + j_2Cz_m + j_{\alpha-1}*Cg_m, 0, 0)$ converges to the strong solution u of $ACP(A, Cx + j_1Cy + j_2Cz + j_{\alpha-1}*Cg, 0, 0)$ in $C^2([0,T_0),X)$.

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