TAIWANESE JOURNAL OF MATHEMATICS
Vol. 14, No. 2, pp. 479-499, April 2010
This paper is available online at http://www.tjm.nsysu.edu.tw/

THE CAUCHY PROBLEM FOR A GENERALIZED KORTEWEG-DE VRIES EQUATION IN HOMOGENEOUS SOBOLEV SPACES

Ruying Xue and Sufen Hu

Abstract. Considered in this article is the Cauchy problem of a generalized Korteweg-de Vries equation

$$\begin{cases} u_t + u_{xxx} + uu_x + |D_x|^{2\alpha} u = 0, & t \in \mathbb{R}^+,\, x \in \mathbb{R}, \\ u(x,0) = \varphi(x) \end{cases}$$

with $0 \le \alpha \le 1$. The local well-posedness of the Cauchy problem in the homogeneous Sobolev space $\dot{H}^s(\mathbb{R})$ for $s \in (\frac{\alpha-3}{2(2-\alpha)}, 0]$ is proved. In addition, the mapping that associated to appropriate initial-data the corresponding solution is analytic as a function between appropriate Banach spaces.

1. Introduction and Statement of the Result

Consider the Cauchy problem of the generalized Korteweg-de Vries equation

$$\begin{cases} u_t + u_{xxx} + uu_x + |D_x|^{2\alpha} u = 0, & t \in \mathbb{R}^+,\, x \in \mathbb{R} \\ u(0) = \varphi(x), \end{cases} \tag{1.1}$$

where, $0 \le \alpha \le 1$, $|D_x|^{2\alpha}$ is the Fourier multiplier associated with the symbol $|\xi|^{2\alpha}$.

Equation (1.1) has been derived as a model for the propagation of weakly nonlinear dispersive long waves in some physical contexts when dissipative effects occur. When $\alpha = 0$, (1.1) is the Korteweg-de Vries equation. The best known local well-posedness of the Korteweg-de Vries equation has been derived by Kenig, Ponce and Vega (see [5, 6]). They proved that the Cauchy problem for the KdV

Received June 28, 2007, accepted June 16, 2008.
Communicated by J. C. Yao.
2000 *Mathematics Subject Classification*: 35Q58, 35Q60.
Key words and phrases: Korteweg-de Vries equation, Well-posedness.
Supported by NSFC (10571158) and Zhejiang Provincial NSF of China (Y605076).

equation is locally well-posed in $H^s(\mathbb{R})$ for $s > -\frac{3}{4}$, and that the flow-map for the KdV equation is not locally uniformly continuous in $H^s(\mathbb{R})$ for $s < -\frac{3}{4}$. For the Cauchy problem of the dissipative Burgers equation

$$u_t - u_{xx} + uu_x = 0,$$

it is known that the local well-posedness in $H^s(\mathbb{R})$ holds for $s \geq -\frac{1}{2}$ (see [1]), and some non-uniqueness phenomena occur for $s < -\frac{1}{2}$ (see [3]). When $\alpha = 1$, (1.1) is the Korteweg-de Vries-Burgers equation. In [8], Molinet and Ribaud proved that the KdV-Burgers equation is globally well-posed in $H^s(\mathbb{R})$ for $s > -1$ and ill-posed in homogeneous Sobolev space $\dot{H}^s(\mathbb{R})$ for $s < -1$. In the same article they pointed out that the Cauchy problem (1.1) is ill-posed in the homogenous Sobolev space $\dot{H}^s(\mathbb{R})$ for $s < \frac{\alpha-3}{2(2-\alpha)}$, and conjectured that the Cauchy problem (1.1) is well-posed in $H^s(\mathbb{R})$ for $s > \frac{\alpha-3}{2(2-\alpha)}$. The aim of this article is to answer this open problem partially. We prove in this article that the Cauchy problem (1.1) is locally well-posed in homogenous Sobolev spaces $\dot{H}^s(\mathbb{R})$ for $s \in (\frac{\alpha-3}{2(2-\alpha)}, 0]$. In a future publication, we shall prove that the Cauchy problem (1.1) is globally well-posed in Sobolev spaces $H^s(\mathbb{R})$ for $s > -\min\{\frac{3+2\alpha}{4}, 1\}$.

To state our main result we introduce some definitions and notations. Let $\dot{H}^s(\mathbb{R})$ be the usual homogenous Sobolev space. Denote by $< \cdot >= (1+|\cdot|^2)^{\frac{1}{2}}$, and define

$$\dot{X}_\alpha^{b,s} = \{u \in \mathcal{S}'(\mathbb{R}^2) : \| u \|_{\dot{X}_\alpha^{b,s}} < +\infty\},$$

$$\dot{X}_{\alpha,T}^{b,s} = \{u : \exists v \in \dot{X}_\alpha^{b,s} \text{ satisfying } u = v \text{ in} \, \mathbb{R} \times [0,T]\},$$

with

$$\| u \|_{\dot{X}_\alpha^{b,s}} = \| < i(\tau - \xi^3) + |\xi|^{2\alpha} >^b |\xi|^s \hat{u}(\xi,\tau)\|_{L^2(\mathbb{R}^2)},$$

$$\| u \|_{\dot{X}_{\alpha,T}^{b,s}} = \inf\{\| v \|_{\dot{X}_\alpha^{b,s}} : v \in \dot{X}_\alpha^{b,s} \text{ satisfying } u = v \text{ in } R \times [0,T]\}.$$

The main result obtained in this article is

Theorem 1.1. *Let $s \in (\frac{\alpha-3}{2(2-\alpha)}, 0]$ and let $\varphi \in \dot{H}^s(\mathbb{R})$. There exists a positive constant $T_0 = T_0(\|\varphi\|_{\dot{H}^s(\mathbb{R})})$ depending only on $\|\varphi\|_{\dot{H}^s(\mathbb{R})}$ such that the Cauchy problem (1.1) possesses a unique solution u satisfying*

$$u \in Z_{T_0} = C([0,T_0], \dot{H}^s(\mathbb{R})) \cap \dot{X}_{\alpha,T_0}^{\frac{1}{2},s} \cap C((0,T_0], L^2(\mathbb{R})).$$

Moreover the mapping that associated to appropriate initial-data the corresponding solution is analytic as a function between appropriate Banach spaces.

Remark. In, [8]. Molinet and Ribaud proved that (1.1) is ill-posed in the homogenous Sobolev space $\dot{H}^s(\mathbb{R})$ for $s < \frac{\alpha-3}{2(2-\alpha)}$. Our result is sharp in some sense.

In this article, we denote by C some large constant which may vary from line to line, denote by $A \lesssim B$ the statement that $A \leq CB$, and similarly denote by $A \ll B$ the statement $A \leq C^{-1}B$. We use $A \sim B$ to denote the statement that $A \lesssim B \lesssim A$. Any summations over capitalized variables such as N_j, L_j , H are presumed to be dyadic, i.e. these variables range over numbers of the form 2^k for $k \in \mathbb{Z}$ or for $k \in \mathbb{N}$. In addition to the usual notation χ_E for characteristic function, we define χ_P for statements P to be 1 if P is true and 0 otherwise, e.g. $\chi_{1\leq|\xi|\leq 2}$.

We adopt the following summation conventions. Any summation of the form $L_{max} \sim \cdot$ is a sum over the three dyadic variables $L_1, L_2, L_3 \gtrsim 1$, thus for instance

$$\sum_{L_{max}\sim H} := \sum_{L_1,L_2,L_3\gtrsim 1; L_{max}\sim H} .$$

Similarly, any summation of the form $N_{max} \sim \cdot$ sum over the three dyadic variables $N_1, N_2, N_3 > 0$, thus for instance

$$\sum_{N_{max}\sim N_{med}\sim N} := \sum_{N_1,N_2,N_3>0; N_{max}\sim N_{med}\sim N} .$$

The rest of this article is organized as follows. In section 2 we give some linear estimates. In section 3 we prove the crucial bilinear estimates by using Tao's $[k; Z]$-multiplier method introduced in [9]. The local well-posedness is given in Section 4.

2. Linear Estimates

Let $U(\cdot)$ be the free evolution of the KdV equation defined by $U(t) = e^{itP(D_x)}$, where $P(D_x)$ is the Fourier multiplier with the symbol $P(\xi) = \xi^3$. Obviously $U(\cdot)$ is a unitary group in $\dot{H}^s(\mathbb{R}), s \in \mathbb{R}$. Since $\mathcal{F}(U(-t)u)(\tau, \xi) = \mathcal{F}(u)(\tau + \xi^3, \xi)$, one can rewrite the norm of $\dot{X}^{b,s}_\alpha$ as

$$\| u \|_{\dot{X}^{b,s}_\alpha} = \| < i\tau + |\xi|^{2\alpha} >^b |\xi|^s \mathcal{F}(U(-t)u)(\tau, \xi) \|_{L^2(\mathbb{R}^2)} .$$

Let $W(\cdot)$ be the semigroup associated with the free evolution of (1.1) defined by

$$\mathcal{F}_x(W(t)\varphi)(\xi) = e^{it\xi^3 - t|\xi|^{2\alpha}} \hat{\varphi}(\xi), \quad \varphi \in \mathcal{S}'(\mathbb{R}), t \geq 0,$$

and we extend $W(\cdot)$ to a linear operator defined on the whole real axis by setting

$$\mathcal{F}_x(W(t)\varphi)(\xi) = e^{it\xi^3 - |t||\xi|^{2\alpha}} \hat{\varphi}(\xi), \quad \varphi \in \mathcal{S}'(\mathbb{R}), t \in \mathbb{R}.$$

Denote by ψ the time cut-off function satisfying

$$\psi \in C_0^\infty(\mathbb{R}),\ supp\psi \subset [-2,2],\ \psi \equiv 1 \text{ on} [-1,1],$$

and let $\psi_T(\cdot) = \psi(\cdot/T)$ for a given $T > 0$.

Proposition 2.1. *For $s \in \mathbb{R}$, we have*

$$\| \psi(t)W(t)\varphi \|_{\dot{X}_\alpha^{\frac{1}{2},s}} \lesssim \| \varphi \|_{\dot{H}^s}, \quad \forall \varphi \in \dot{H}^s(\mathbb{R}).$$

Proof. Set $g_\xi = \psi(t)e^{-|t||\xi|^{2\alpha}}$. For $b \in \{0, \frac{1}{2}\}$ we have

$$\| g_\xi \|_{H_t^b} \leq \| < \tau >^b \hat{\psi} \|_{L^1} \| e^{-|t||\xi|^{2\alpha}} \|_{L_t^2} + \| \hat{\psi} \|_{L^1} \| e^{-|t||\xi|^{2\alpha}} \|_{\dot{H}_t^b}.$$

Note that $\| < \tau >^b \hat{\psi} \|_{L^1} \leq C$ because of $\psi \in C_0^\infty(\mathbb{R})$, and

$$\| e^{-|t||\xi|^{2\alpha}} \|_{\dot{H}_t^b} \sim (| \xi |^{2\alpha})^{b-\frac{1}{2}} \| e^{-|t|} \|_{\dot{H}_t^b}.$$

Thus,

$$\| g_\xi \|_{H_t^b} \lesssim (| \xi |^{-\alpha} + | \xi |^{2\alpha b-\alpha}) \leq C | \xi |^{2\alpha(b-\frac{1}{2})} \quad \text{for} \quad | \xi | \geq 1, \tag{2.1}$$

and

$$\begin{aligned} \| g_\xi \|_{H_t^b} &\leq \sum_{n=0}^{\infty} \frac{| \xi |^{2\alpha n}}{n!} \| \psi(t)t^n \|_{H_t^b} \\ &\leq \sum_{n=0}^{\infty} \frac{| \xi |^{2\alpha n}}{n!} \| \psi(t)t^n \|_{H_t^1} \lesssim 1 \text{ for} \quad | \xi | \leq 1, \end{aligned} \tag{2.2}$$

which imply

$$\| g_\xi \|_{H_t^b} \lesssim < \xi >^{\alpha(2b-1)}, \quad b = 0 \text{ or } \frac{1}{2}. \tag{2.3}$$

Then one deduces from (2.3) that

$$\begin{aligned} \| \psi(t)W(t)\varphi \|_{\dot{X}_\alpha^{\frac{1}{2},s}} &\lesssim \left\| |\xi|^s \hat{\varphi}(\xi) \| < \tau >^{\frac{1}{2}} \mathcal{F}_t(\psi(t)e^{-|t||\xi|^{2\alpha}})(\tau) \|_{L_\tau^2} \right\|_{L_\xi^2} \\ &\quad + \left\| |\xi|^{s+\alpha} \hat{\varphi}(\xi) \| \psi(t)e^{-|t||\xi|^{2\alpha}} \|_{L_t^2} \right\|_{L_\xi^2} \\ &\lesssim \left\| |\xi|^s \hat{\varphi}(\xi) \| g_\xi(t) \|_{H_t^{\frac{1}{2}}} \right\|_{L_\xi^2} + \left\| |\xi|^{s+\alpha} \hat{\varphi}(\xi) \| g_\xi(t) \|_{H_t^0} \right\|_{L_\xi^2} \\ &\lesssim \| |\xi|^s \hat{\varphi}(\xi) \|_{L_\xi^2} + C \| |\xi|^s \hat{\varphi}(\xi) \|_{L_\xi^2} \lesssim \| \varphi \|_{\dot{H}^s}. \end{aligned}$$

∎

The following proposition comes from Proposition 2.2 in [8].

Proposition 2.2. *For given $\omega \in \mathcal{S}(\mathbb{R}^2)$ one denotes K_ξ by*

$$K_\xi(t) = \psi(t) \int_{\mathbb{R}} \frac{e^{it\tau} - e^{-|t||\xi|^{2\alpha}}}{i\tau + \mid \xi \mid^{2\alpha}} \hat{\omega}(\tau) d\tau.$$

Then

$$\begin{aligned} &\left\| < i\tau + \mid \xi \mid^{2\alpha} >^{\frac{1}{2}} \mathcal{F}_t(K_\xi)(\tau) \right\|^2_{L^2_\tau(\mathbb{R})} \\ &\lesssim \left[\left(\int_{\mathbb{R}} \frac{\mid \hat{\omega}(\tau) \mid}{< i\tau + \mid \xi \mid^{2\alpha} >} d\tau \right)^2 + \int_{\mathbb{R}} \frac{\mid \hat{\omega}(\tau) \mid^2}{< i\tau + \mid \xi \mid^{2\alpha} >} d\tau \right] \end{aligned} \tag{2.4}$$

holds for any $\xi \in \mathbb{R}$.

Proposition 2.3. *Given $s \in \mathbb{R}$ and $\delta \in (0, \frac{1}{2})$, one has*

$$\begin{aligned} &\left\| \chi_{R^+}(t) \psi(t) \int_0^t W(t-t') v(t') dt' \right\|_{\dot{X}_\alpha^{\frac{1}{2},s}} \\ &\lesssim \|v\|_{\dot{X}_\alpha^{-\frac{1}{2},s}} + \left(\int_{\mathbb{R}} |\xi|^{2s} \left(\int_{\mathbb{R}} \frac{\mid \hat{v}(\tau) \mid}{< i\tau + \mid \xi \mid^{2\alpha c} >} d\tau \right)^2 d\xi \right)^{\frac{1}{2}} \\ &\textit{for any } v \in \mathcal{S}(\mathbb{R}^2), \end{aligned} \tag{2.5}$$

$$\begin{aligned} &\left\| \chi_{R^+}(t) \psi(t) \int_0^t W(t-t') v(t') dt' \right\|_{\dot{X}_\alpha^{\frac{1}{2},s}} \\ &\lesssim \| v \|_{\dot{X}_\alpha^{-\frac{1}{2}+\delta,s}} \quad \textit{for any } v \in \dot{X}_\alpha^{-\frac{1}{2}+\delta,s}. \end{aligned} \tag{2.6}$$

Proof. Assume that $v \in \mathcal{S}(\mathbb{R}^2)$. Taking that for x-Fourier transform we get

$$\begin{aligned} &\chi_{R^+}(t) \psi(t) \int_0^t W(t-t') v(t') dt' \\ &= U(t) \chi_{R^+}(t) \psi(t) \int_{\mathbb{R}} e^{ix\xi} \int_0^t e^{-(t-t')|\xi|^{2\alpha}} \mathcal{F}_x(U(-t')v(t')) dt' d\xi \\ &= U(t) \chi_{R^+}(t) \psi(t) \int_{\mathbb{R}^2} e^{ix\xi} \hat{\omega}(\tau, \xi) e^{-t|\xi|^{2\alpha}} \int_0^t e^{t'|\xi|^{2\alpha}} e^{it'\tau} dt' d\xi d\tau \\ &= U(t) \chi_{R^+}(t) \psi(t) \int_{\mathbb{R}^2} e^{ix\xi} \hat{\omega}(\tau, \xi) \frac{e^{it\tau} - e^{-t|\xi|^{2\alpha}}}{i\tau + \mid \xi \mid^{2\alpha}} d\xi d\tau \\ &= U(t) \chi_{R^+}(t) \int_{\mathbb{R}} e^{ix\xi} K_\xi(t) d\xi, \end{aligned}$$

where we denote by $\omega(t') = U(-t')v(t')$. By using Proposition 2.2, one deduces

$$\left\|\chi_{R^+}(t)\psi(t)\int_0^t W(t-t')v(t')dt'\right\|_{\dot{X}_\alpha^{\frac{1}{2},s}} \leq \left\|< i\tau+|\xi|^{2\alpha} >^{\frac{1}{2}} |\xi|^s \mathcal{F}_t(K_\xi(t))\right\|_{L^2(\mathbb{R}^2)}$$

$$\lesssim \left(\int_{\mathbb{R}} |\xi|^{2s}\left(\int_{\mathbb{R}} \frac{|\hat{\omega}(\tau)|}{< i\tau+|\xi|^{2\alpha}>} d\tau\right)^2 d\xi\right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}} |\xi|^{2s}\int_{\mathbb{R}} \frac{|\hat{\omega}(\tau)|^2}{< i\tau+|\xi|^{2\alpha}>} d\tau d\xi\right)^{\frac{1}{2}}$$

$$\lesssim \left(\int_{\mathbb{R}} |\xi|^{2s}\left(\int_{\mathbb{R}} \frac{|\hat{v}(\tau)|}{< i\tau+|\xi|^{2\alpha}>} \| e^{-it\xi^3} \|_{L^\infty} d\tau\right)^2 d\xi\right)^{\frac{1}{2}} + \| v \|_{\dot{X}_\alpha^{-\frac{1}{2},s}}$$

$$\lesssim \| v \|_{\dot{X}_\alpha^{-\frac{1}{2},s}} + \left(\int_{\mathbb{R}} |\xi|^{2s}\left(\int_{\mathbb{R}} \frac{|\hat{v}(\tau)|}{< i\tau+|\xi|^{2\alpha}>} d\tau\right)^2 d\xi\right)^{\frac{1}{2}},$$

which completes the proof of (2.5). For $\delta \in (0, \frac{1}{2})$,it is obvious that

$$\| v \|_{\dot{X}_\alpha^{-\frac{1}{2},s}} \leq \| v \|_{\dot{X}_\alpha^{-\frac{1}{2}+\delta,s}} .$$

By the $H\ddot{o}lder$ inequality,

$$\int_{\mathbb{R}} \frac{|\hat{v}(\tau)|}{< i\tau+|\xi|^{2\alpha}>} d\tau \lesssim \left\| |\hat{v}(\tau)| < i\tau+|\xi|^{2\alpha}>^{-\frac{1}{2}+\delta}\right\|_{L^2(\mathbb{R})},$$

and then one gets

$$\left(\int_{\mathbb{R}} |\xi|^{2s}\left(\int_{\mathbb{R}} \frac{|\hat{v}(\tau)|}{< i\tau+|\xi|^{2\alpha}>} d\tau\right)^2 d\xi\right)^{\frac{1}{2}} \lesssim \| v \|_{\dot{X}_\alpha^{-\frac{1}{2}+\delta,s}} .$$

Proposition 2.4. *Let $s \in \mathbb{R}$ and $\delta > 0$. For $f \in \dot{X}_\alpha^{-\frac{1}{2}+\delta,s}$ one has*

$$t \mapsto \int_0^t W(t-t')f(t')dt' \in C(\mathbb{R}^+, \dot{H}^{s+2\delta}). \tag{2.7}$$

Moreover, if $\{f_n\}$ is a sequence with $f_n \to 0$ in $\dot{X}_\alpha^{-\frac{1}{2}+\delta,s}$ as $n \to \infty$, then

$$\left\|\int_0^t W(t-t')f_n(t')dt'\right\|_{L^\infty(\mathbb{R}^+,\dot{H}^{s+2\delta})} \to 0, \ \ as \ \ n \to \infty. \tag{2.8}$$

Proof. The proof is similar to that of Proposition 2.4 in [8], we omit it. ■

3. A Bilinear Estimate

In this section we give a bilinear estimate by using Tao's $[k;Z]$-multiplier method introduced in [9]. Let Z be any abelian additive group with an invariant measure $d\eta$. For any integer $k \geq 2$, we denote by $\Gamma_k(Z)$ the hyperplane

$$\Gamma_k(Z) = \{(\eta_1, \cdots, \eta_k) \in Z^k : \eta_1 + \cdots + \eta_k = 0\},$$

endowing with the obvious measure

$$\int_{\Gamma_k(Z)} f := \int_{Z^{k-1}} f(\eta_1, \cdots, \eta_{k-1}, -\eta_1 - \cdots - \eta_{k-1}) d\eta_1 \cdots d\eta_{k-1}.$$

We define a $[k;Z]$-multiplier to be any function $m : \Gamma_k(Z) \to \mathbb{C}$. If m is a $[k;Z]$-multiplier, we define $\|m\|_{[k;Z]}$ to be the best constant such that the inequality

$$\left| \int_{\Gamma_k(Z)} m(\eta) \Pi_{j=1}^k f_j(\eta_j) \right| \leq \|m\|_{[k;Z]} \Pi_{j=1}^k \|f_j\|_{L^2(Z)},$$

holds for all test functions f_j on Z.

In the paper, we choose $Z = \mathbb{R} \times \mathbb{R}$, $k = 3$ and $\eta = (\tau, \xi)$. For $N_1, N_2, N_3 > 0$, we define the quantities $N_{max} \geq N_{med} \geq N_{min}$ to be the maximum, median and minimum of N_1, N_2, N_3 respectively. Similarly define $L_{max} \geq L_{med} \geq L_{min}$ whenever $L_1, L_2, L_3 \geq 1$. Define

$$h_j(\xi_j) = i\xi_j^3 - |\xi_j|^{2\alpha}, \lambda_j = i\tau_j - h_j(\xi_j),\ j = 1, 2, 3,$$

and

$$h(\xi) = h_1(\xi_1) + h_2(\xi_2) + h_3(\xi_3).$$

We shall take homogenous dyadic decomposition of the variable $|\xi_j| \sim N_j > 0$, and take non-homogenous dyadic decomposition of the variable $|\lambda_j| \sim L_j \geq 1$ as well as the function $|h(\xi)| \sim H \geq 1$ (here the notations $|\lambda_j| \sim 1$ and $|h(\xi)| \sim 1$ mean $|\lambda_j| \leq 1$, $|h(\xi)| \leq 1$, respectively). Define

$$X_{N_1,N_2,N_3;H;L_1,L_2,L_3} := \chi_{|h(\xi)| \sim H} \Pi_{j=1}^3 \chi_{|\xi_j| \sim N_j} \chi_{|\lambda_j| \sim L_j}.$$

Lemma 3.1. *Let $N_1, N_2, N_3 > 0$, $L_1, L_2, L_3 \gtrsim 1$ and $H \gtrsim 1$ satisfy*

$$N_{max} \sim N_{med},\ L_{max} \sim \max\{H, L_{med}\},\ H \sim \max\{N_{max}^2 N_{min}, N_{max}^{2\alpha}\}. \tag{3.1}$$

(1) *In the high modulation case $L_{max} \sim L_{med} \gg H$ we have*

$$\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim L_{min}^{\frac{1}{2}} N_{min}^{\frac{1}{2}}. \tag{3.2}$$

(2) *In the low modulation case $L_{max} \sim H$,*

(a) *if* $N_{max} \sim N_{med} \sim N_{min}$*, we have*

$$\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim L_{min}^{\frac{1}{2}} \min\{N_{max}^{-\frac{1}{4}} L_{med}^{\frac{1}{4}}, L_{med}^{\frac{1}{4\alpha}}\}; \tag{3.3}$$

(b) *if* $N_2 \sim N_3 \gg N_1$ *and* $H \sim L_1 \geq L_2, L_3$*, we have, for any* $\beta \in (0, 2]$*,*

$$\begin{aligned} &\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}\times\mathbb{R}]} \\ &\lesssim L_{min}^{\frac{1}{2}} \min\{N_1^{\frac{1}{2}}, L_{med}^{\frac{1}{4\alpha}}, N_2^{\frac{\beta-2}{2\beta}} N_1^{-\frac{1}{2\beta}} L_{med}^{\frac{1}{2\beta}}\}; \end{aligned} \tag{3.4}$$

similarly for permutations;

(c) *in all other cases, we have*

$$\begin{aligned} &\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}\times\mathbb{R}]} \\ &\lesssim L_{min}^{\frac{1}{2}} \min\{N_{max}^{-1} L_{med}^{\frac{1}{2}}, L_{med}^{\frac{1}{4\alpha}}, N_{min}^{\frac{1}{2}}\}. \end{aligned} \tag{3.5}$$

Proof. We consider the high modulation case $L_{max} \sim L_{med} \gg H$. By using the comparison principle (Lemma 3.1 in [9]), we have (without loss of generality we assume $L_1 \geq L_2 \geq L_3$ and $N_1 \geq N_2 \geq N_3$)

$$\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim \left\|\chi_{|\lambda_3|\sim L_3}\chi_{|\xi_3|\sim N_3}\right\|_{[3,\mathbb{R}\times\mathbb{R}]}. \tag{3.6}$$

By Lemma 3.14 and Lemma 3.6 in [9],

$$\left\|\chi_{|\lambda_3|\sim L_3}\chi_{|\xi_3|\sim N_3}\right\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim \left\|\|\chi_{|\lambda_3|\sim L_3}\|_{[3,\mathbb{R}]}\chi_{|\xi_3|\sim N_3}\right\|_{[3,\mathbb{R}]} \lesssim L_3^{\frac{1}{2}} N_3^{\frac{1}{2}}. \tag{3.7}$$

Although we derived (3.7) assuming $L_1 \geq L_2 \geq L_3$ and $N_1 \geq N_2 \geq N_3$, it is clear from symmetry that

$$\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim L_{min}^{\frac{1}{2}} N_{min}^{\frac{1}{2}}. \tag{3.8}$$

We first consider the low modulation case $H \sim L_{max}$. Suppose for the moment that $N_1 \geq N_2 \geq N_3$. The ξ_3 variable is currently localized to the annulus $\{|\xi_3| \sim N_3\}$. By a finite partition of unity we can restrict it further to a ball $\{|\xi_3 - \xi_3^0| \ll N_3\}$ for some $|\xi_3^0| \sim N_3$. Then by Box Localization (Lemma 3.13 in [9]) we may localize ξ_1, ξ_2 similarly to regions $\{|\xi_1 - \xi_1^0| \ll N_3\}$ and $\{|\xi_2 - \xi_2^0| \ll N_3\}$ where $|\xi_j^0| \sim N_j$. We may assume that $|\xi_1^0 + \xi_2^0 + \xi_3^0| \ll N_3$ since we have $\xi_1 + \xi_2 + \xi_3 = 0$. We summarize this symmetrically as

$$\begin{aligned} &\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}\times\mathbb{R}]} \\ &\lesssim \left\|\chi_{|h(\xi)|\sim H}\Pi_{j=1}^3 \chi_{|\lambda_j|\sim L_j}\chi_{|\xi_j - \xi_j^0|\ll N_{min}}\right\|_{[3,\mathbb{R}\times\mathbb{R}]}, \end{aligned} \tag{3.9}$$

for some $\xi_1^0, \xi_2^0, \xi_3^0$ satisfying

$$|\xi_j^0| \sim N_j, |\xi_1^0 + \xi_2^0 + \xi_3^0| \ll N_{min}.$$

Without loss of generality, we assume $L_1 \geq L_2 \geq L_3$. By Lemma 3.6 , Lemma 3.1 and Corollary 3.10 in [9] we get

$$\begin{aligned} &\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}\times\mathbb{R}]} \\ \lesssim\ & \left\|\chi_{|h(\xi)|\sim H}\Pi_{j=2}^3\chi_{|\xi_j-\xi_j^0|\ll N_{min}}\chi_{|\lambda_j|\sim L_j}\right\|_{[3,\mathbb{R}\times\mathbb{R}]} \\ \lesssim\ & \big|\{(\tau_2,\xi_2) : |\xi_2-\xi_2^0| \ll N_{min}, |i\tau_2 - h_2(\xi_2)| \sim L_2, \\ & \qquad |\xi-\xi_2-\xi_3^0| \ll N_{min}, |i(\tau-\tau_2) - h_3(\xi-\xi_2)| \sim L_3\}\big|^{\frac{1}{2}} \end{aligned} \tag{3.10}$$

for some $(\tau, \xi) \in \mathbb{R}\times\mathbb{R}$. For fixed ξ_2, the set of possible τ_2 ranges in an interval of length $O(\min\{L_2, L_3\})$, and vanishes unless

$$|i\tau - h_2(\xi_2) - h_3(\xi-\xi_2)| = O(\max\{L_2, L_3\}).$$

Then we get, for some $(\tau, \xi) \in \mathbb{R}\times\mathbb{R}$,

$$\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim L_3^{\frac{1}{2}}|\{\xi_2 : |\xi_2-\xi_2^0| \ll N_{min},$$

$$|\xi-\xi_2-\xi_3^0| \ll N_{min}, |i\tau - h_2(\xi_2) - h_3(\xi-\xi_2)| = O(L_2)\}|^{\frac{1}{2}}.$$

Note that the inequality $|\xi-\xi_2-\xi_3^0| \ll N_{min}$ implies $|\xi-\xi_1^0| \ll N_{min}$. Then we have

$$\begin{aligned} &\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}\times\mathbb{R}]} \\ \lesssim\ & L_3^{\frac{1}{2}}|\{\xi_2 : |\xi_2-\xi_2^0| \ll N_{min}, |\xi-\xi_1^0| \ll N_{min}, \\ & \qquad |i\tau - h_2(\xi_2) - h_3(\xi-\xi_2)| = O(L_2)\}|^{\frac{1}{2}}. \end{aligned} \tag{3.11}$$

To compute the right-hand side of the expression (3.11) we use the identity

$$|i\tau - h_2(\xi_2) - h_3(\xi-\xi_2)| = \left|i\tau - 3i\xi(\xi_2-\frac{\xi}{2})^2 + i\frac{\xi^3}{4} + (|\xi_2|^{2\alpha} + |\xi_2-\xi|^{2\alpha})\right| = O(L_2),$$

which implies

$$3\xi(\xi_2-\frac{\xi}{2})^2 + \frac{\xi^3}{4} = \tau + O(L_2) \tag{3.12}$$

and

$$|\xi_2|^{2\alpha} + |\xi_2-\xi|^{2\alpha} = O(L_2). \tag{3.13}$$

We need only consider three cases: $N_1 \sim N_2 \sim N_3$, $N_1 \sim N_2 \gg N_3$, and $N_2 \sim N_3 \gg N_1$. (The case $N_1 \sim N_3 \gg N_2$ then follows by symmetry).

If $N_1 \sim N_2 \sim N_3$, by $|\xi - \xi_1^0| \ll N_{min}$, we deduce $|\xi| \sim N_1$. We see from (3.12) that ξ_2 variable is contained in the union of two intervals of length $O(N_1^{\frac{1}{2}} L_2^{\frac{1}{2}})$ at worst, and from (3.13) that $|\xi_2| \leq L_2^{\frac{1}{2\alpha}}$, and (3.3) follows from (3.11).

If $N_1 \sim N_2 \gg N_3$, by $|\xi - \xi_1^0| \ll N_{min}$, $|\xi_2 - \xi_2^0 - \frac{\xi - \xi_1^0}{2} - \xi_3^0| \ll N_{min}$ and

$$\left|\xi_2 - \frac{\xi}{2}\right| = \left|\xi_2 - \xi_2^0 - \frac{\xi - \xi_1^0}{2} - \xi_3^0 - \frac{\xi_1^0}{2}\right|$$

we get $|\xi| \sim N_1$ and $|\xi_2 - \frac{\xi}{2}| \sim N_1$. We see from (3.12) that ξ_2 variable is contained in the union of two intervals of length $O(N_1^{-2} L_2)$ at worst, and from (3.13) that $|\xi_2| \leq L_2^{\frac{1}{2\alpha}}$, and (3.5) follows from (3.11).

If $N_2 \sim N_3 \gg N_1$, then we must have $|\xi| \sim N_1$ and $|\xi_2 - \frac{\xi}{2}| \sim N_2$. For a given $\beta \in (0, 2]$, we have $|\xi||\xi_2 - \frac{\xi}{2}|^{2-\beta} \sim N_1 N_2^{2-\beta}$. We see from (3.12) that ξ_2 variable is contained in the union of two intervals of length $O(N_1^{-\frac{1}{\beta}} N_2^{\frac{\beta-2}{\beta}} L_2^{\frac{1}{\beta}})$ at worst, and from (3.13) that $|\xi_2| \leq L_2^{\frac{1}{2\alpha}}$. (3.4) follows from (3.11) and the fact that $|\xi_2 - \xi_2^0| \ll N_3$ for some $|\xi_2^0| \ll N_3$. ∎

Lemma 3.2. *For a given* $\rho \in [0, \frac{3-\alpha}{2(2-\alpha)})$ *and for any* $\delta > 0$ *small we have*

$$\left\| \frac{|\xi_1|^\rho |\xi_2|^\rho |\xi_3|^{1-\rho}}{< \lambda_1 >^{\frac{1}{2}} < \lambda_2 >^{\frac{1}{2}} < \lambda_3 >^{\frac{1}{2}-\delta}} \right\|_{[3, \mathbb{R}\times\mathbb{R}]} \lesssim 1. \tag{3.14}$$

Proof. By using the comparison principle in [9] and by taking non-homogenous dyadic decomposition of the variable $|\lambda_j| \sim L_j \geq 1$, we get

$$\begin{aligned}
&\left\| \frac{|\xi_1|^\rho |\xi_2|^\rho |\xi_3|^{1-\rho} \chi_{|\xi_1| \lesssim 1} \chi_{|\xi_2| \lesssim 1} \chi_{|\xi_3| \lesssim 1}}{< \lambda_1 >^{\frac{1}{2}} < \lambda_2 >^{\frac{1}{2}} < \lambda_3 >^{\frac{1}{2}-\delta}} \right\|_{[3, \mathbb{R}\times\mathbb{R}]} \\
\lesssim &\left\| \frac{\chi_{|\xi_1| \lesssim 1} \chi_{|\xi_2| \lesssim 1} \chi_{|\xi_3| \lesssim 1}}{< \lambda_1 >^{\frac{1}{2}} < \lambda_2 >^{\frac{1}{2}} < \lambda_3 >^{\frac{1}{2}-\delta}} \right\|_{[3, \mathbb{R}\times\mathbb{R}]} \\
\lesssim &\sum_{L_1, L_2, L_3 \gtrsim 1} \frac{\left\| \Pi_{j=1}^3 \chi_{|\xi_j| \lesssim 1} \chi_{|\lambda_j| \sim L_j} \right\|_{[3, \mathbb{R}\times\mathbb{R}]}}{< L_1 >^{\frac{1}{2}} < L_2 >^{\frac{1}{2}} < L_3 >^{\frac{1}{2}-\delta}}
\end{aligned} \tag{3.15}$$

$$\lesssim \sum_{L_1,L_2,L_3\gtrsim 1} \frac{L_{min}^{\frac{1}{2}}}{< L_1 >^{\frac{1}{2}}< L_2 >^{\frac{1}{2}}< L_3 >^{\frac{1}{2}-\delta}}$$

$$\lesssim \sum_{L_{min},L_{med},L_{max}\gtrsim 1} \frac{1}{< L_{med} >^{\frac{1}{2}}< L_{max} >^{\frac{1}{2}-\delta}} \lesssim 1,$$

here we have used the estimate (without loss of generality we assume $L_1 \lesssim L_2 \lesssim L_3$)

$$\left\|\Pi_{j=1}^3 \chi_{|\xi_j|\lesssim 1}\chi_{|\lambda_1|\sim L_j}\right\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim \left\|\chi_{|\xi_1|\lesssim 1}\left\|\chi_{|i\tau_1-h_1(\xi)|\sim L_1}\right\|_{[3,\mathbb{R}]}\right\|_{[3,\mathbb{R}]} \lesssim L_1^{\frac{1}{2}} = L_{min}^{\frac{1}{2}}.$$

What remains is to estimate the term

$$\left\|\frac{|\xi_1|^\rho|\xi_2|^\rho|\xi_3|^{1-\rho}\chi_{\max\{|\xi_1|,|\xi_2|,|\xi_3|\}\gtrsim 1}}{< \lambda_1 >^{\frac{1}{2}}< \lambda_2 >^{\frac{1}{2}}< \lambda_3 >^{\frac{1}{2}-\delta}}\right\|_{[3,\mathbb{R}\times\mathbb{R}]}. \tag{3.16}$$

By taking the homogenous dyadic decomposition of the variable $|\xi_j| \sim N_j > 0$, by taking the non-homogenous dyadic decomposition of the variable $|\lambda_j| \sim L_j \geq 1$, and the function $|h(\xi)| \sim H \geq 1$ (here the notation $|\lambda_j| \sim L_j = 1$, $|h(\xi)| \sim H = 1$ means $|\lambda_j| \leq 1$, $|h(\xi)| \leq 1$, respectively), we have

$$\begin{aligned}&\left\|\frac{|\xi_1|^\rho|\xi_2|^\rho|\xi_3|^{1-\rho}\chi_{\max\{|\xi_1|,|\xi_2|,|\xi_3|\}\gtrsim 1}}{< \lambda_1 >^{\frac{1}{2}}< \lambda_2 >^{\frac{1}{2}}< \lambda_3 >^{\frac{1}{2}-\delta}}\right\|_{[3,\mathbb{R}\times\mathbb{R}]}\\ \lesssim &\left\|\sum_{N_1,N_2,N_3>0,N_{max}\gtrsim 1}\ \sum_{L_1,L_2,L_3\geq 1}\ \sum_{H\geq 1} \frac{N_3^{1-\rho}N_1^\rho N_2^\rho}{L_1^{\frac{1}{2}}L_2^{\frac{1}{2}}L_3^{\frac{1}{2}-\delta}}\right.\\ &\left. X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\right\|_{[3,\mathbb{R}\times\mathbb{R}]}\end{aligned} \tag{3.17}$$

where $X_{N_1,N_2,N_3;H;L_1,L_2,L_3}$ is the multiplier

$$X_{N_1,N_2,N_3;H;L_1,L_2,L_3} := \chi_{|h(\xi)|\sim H}\Pi_{j=1}^3\chi_{|\xi_j|\sim N_j}\chi_{|\lambda_j|\sim L_j}.$$

From the identities $\xi_1+\xi_2+\xi_3 = 0$ and $\tau_1+\tau_2+\tau_3 = 0$ we see that

$$h(\xi) = -\lambda_1 - \lambda_2 - \lambda_3 = 3i\xi_1\xi_2\xi_3 - (|\xi_1|^{2\alpha} + |\xi_2|^{2\alpha} + |\xi_3|^{2\alpha}).$$

Then the multiplier $X_{N_1,N_2,N_3;H;L_1,L_2,L_3}$ vanishes unless

$$N_{max} \sim N_{med},\ L_{max} \sim \max\{H, L_{med}\},\ H \sim \max\{N_{max}^2 N_{min}, N_{max}^{2\alpha}\}. \tag{3.18}$$

Thus we may implicitly assume (3.18) in the summations. By applying Schur's test (Lemma 3.11 in [9]),

$$\begin{aligned}(3.17) \lesssim \sup_{N\gtrsim 1} &\left\| \sum_{N_{max}\sim N_{med}\sim N}\ \sum_{H\geq 1}\ \sum_{L_{max}\sim\max\{H,L_{med}\}}\right.\\ &\left.\frac{N_3^{1-\rho}N_1^\rho N_2^\rho}{L_1^{\frac{1}{2}}L_2^{\frac{1}{2}}L_3^{\frac{1}{2}-\delta}} X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\right\|_{[3,\mathbb{R}\times\mathbb{R}]}.\end{aligned} \tag{3.19}$$

In light of (3.18) and the comparison principle in [9], we thus see that at least one of the inequalities

$$(3.19) \lesssim \sup_{N\gtrsim 1} \sum_{N_{max}\sim N_{med}\sim N} \sum_{L_{max}\gtrsim L_{med}\gtrsim L_{min}} \frac{N_3^{1-\rho}N_1^{\rho}N_2^{\rho}}{L_1^{\frac{1}{2}}L_2^{\frac{1}{2}}L_3^{\frac{1}{2}-\delta}} \|X_{N_1,N_2,N_3;L_{max};L_1,L_2,L_3}\|_{[3,R\times R]}, \tag{3.20}$$

or

$$(3.19) \lesssim \sup_{N\gtrsim 1} \sum_{N_{max}\sim N_{med}\sim N} \sum_{L_{max}\sim L_{med}} \sum_{H\ll L_{max}} \frac{N_3^{1-\rho}N_1^{\rho}N_2^{\rho}}{L_1^{\frac{1}{2}}L_2^{\frac{1}{2}}L_3^{\frac{1}{2}-\delta}} \|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}\times\mathbb{R}]} \tag{3.21}$$

holds. It is sufficient to prove $(3.20) \lesssim 1$ and $(3.21) \lesssim 1$.

The proof of $(3.21) \lesssim 1$. Note that the inequality $N_{max}^2 N_{min} \geq N_{max}^{2\alpha}$ implies $N_{min} \geq N_{max}^{2\alpha-2}$. By using the estimate (1) in Lemma 3.1, we get from (3.17) and (3.18),

$$\begin{aligned}
(3.21) &\lesssim \sup_{N\gtrsim 1} \sum_{N_{max}\sim N_{med}\sim N} \sum_{L_{max}\sim L_{med}} \sum_{\max\{N_{max}^2 N_{min}, N_{max}^{2\alpha}\}\sim H\ll L_{max}} \frac{N_3^{1-\rho}N_1^{\rho}N_2^{\rho}}{L_1^{\frac{1}{2}}L_2^{\frac{1}{2}}L_3^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} N_{min}^{\frac{1}{2}} \\
&\lesssim \sup_{N\gtrsim 1} \sum_{N_{max}\sim N_{med}\sim N, N_{min}\geq N_{max}^{2\alpha-2}} \sum_{L_{max}\sim L_{med}} \sum_{N_{max}^2 N_{min}\sim H\ll L_{max}} \frac{N_{min}^{1-\rho}N^{2\rho}}{L_{min}^{\frac{1}{2}}L_{max}^{1-\delta}} L_{min}^{\frac{1}{2}} N_{min}^{\frac{1}{2}} \\
&\quad + \sup_{N\gtrsim 1} \sum_{N_{max}\sim N_{med}\sim N, N_{min}\leq N_{max}^{2\alpha-2}} \sum_{L_{max}\sim L_{med}} \sum_{N_{max}^{2\alpha}\sim H\ll L_{max}} \frac{N_{min}^{1-\rho}N^{2\rho}}{L_{min}^{\frac{1}{2}}L_{max}^{1-\delta}} L_{min}^{\frac{1}{2}} N_{min}^{\frac{1}{2}} \\
&\lesssim \sup_{N\gtrsim 1} \sum_{N_{max}\sim N_{med}\sim N, N_{min}\geq N^{2\alpha-2}} \sum_{L_{max}\sim L_{med}\gtrsim N^2 N_{min}} \frac{N_{min}^{\frac{1}{2}-\rho+2\delta} N^{2\rho-2+4\delta}}{L_{max}^{\delta}} log_2(L_{max})
\end{aligned} \tag{3.22}$$

$$+ \sup_{N \gtrsim 1} \sum_{N_{max} \sim N_{med} \sim N, N_{min} \le N^{2\alpha-2}} \sum_{L_{max} \sim L_{med} \gtrsim N^{2\alpha}} \frac{N_{min}^{\frac{3}{2}-\rho} N^{2\rho-2\alpha+2\alpha\delta}}{L_{max}^{\delta}} log_2(L_{max})$$

$$\lesssim \sup_{N \gtrsim 1} \sum_{N_{max} \sim N_{med} \sim N, N_{min} \ge N^{2\alpha-2}} N_{min}^{\frac{1}{2}-\rho+2\delta} N^{2\rho-2+4\delta}$$

$$+ \sup_{N \gtrsim 1} \sum_{N_{max} \sim N_{med} \sim N, N_{min} \le N^{2\alpha-2}} N_{min}^{\frac{3}{2}-\rho} N^{2\rho-2\alpha+2\alpha\delta}.$$

When $\frac{1}{2} < \rho \le 1$, we get from (3.22),

$$(3.23) \qquad (3.21) \lesssim \sup_{N \gtrsim 1} N^{(4-2\alpha)\rho+\alpha-3+(8\alpha-4)\delta} + \sup_{N \gtrsim 1} N^{(4-2\alpha)\rho+\alpha-3+2\alpha\delta} \lesssim 1,$$

for $\delta > 0$ small, since $\rho < \frac{3-\alpha}{2(2-\alpha)}$ implies

$$(4-2\alpha)\rho + \alpha - 3 + (8\alpha-4)\delta < 0, \ (4-2\alpha)\rho + \alpha - 3 + 2\alpha\delta < 0$$

for $\delta > 0$ small. When $\rho \le \frac{1}{2}$, we get from (3.22) that

$$(3.24) \qquad (3.21) \lesssim \sup_{N \gtrsim 1} N^{\rho-\frac{3}{2}+6\alpha} + \sup_{N \gtrsim 1} N^{(4-2\alpha)\rho+\alpha-3+2\alpha\delta} \lesssim 1,$$

for $\delta > 0$ small. We complete the proof of the estimate $(3.21) \lesssim 1$.

Proof of $(3.20) \lesssim 1$. We first deal with the contribution where (3.3) holds. In this case we have $N_1 \sim N_2 \sim N_3 \sim N$, $L_{max} \sim N^3$ and $L_{min} \gtrsim N^{2\alpha}$, since we have $L_j \sim |\lambda_j| \ge |\xi_j|^{2\alpha}$. So we get

$$(3.20) \lesssim \sup_{N \gtrsim 1} \sum_{L_{med} \gtrsim N^{2\alpha}, L_{max} \sim N^3} \frac{N^{1+\rho}}{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} \min\{N^{-\frac{1}{4}} L_{med}^{\frac{1}{4}}, L_{med}^{\frac{1}{4\alpha}}\}$$

$$(3.25) \qquad \lesssim \sup_{N \gtrsim 1} \sum_{L_{med} \gtrsim N^{2\alpha}, L_{max} \sim N^3} \frac{N^{\frac{3}{4}+\rho}}{L_{med}^{\frac{1}{4}} L_{max}^{\frac{1}{2}-\delta}} \lesssim \sup_{N \gtrsim 1} \sum_{L_{max} \sim N^3} \frac{N^{-\frac{3}{4}+\rho-\frac{\alpha}{2}+6\delta}}{L_{max}^{\delta}}$$

$$\lesssim \sup_{N \gtrsim 1} N^{-\frac{3}{4}+\rho-\frac{\alpha}{2}+6\delta} \lesssim 1$$

for $\delta > 0$ small, since we have $\rho < \frac{3-\alpha}{2(2-\alpha)} \le \frac{3}{4} + \frac{\alpha}{2}$.

Second, we deal with the cases where (3.4) applies. We do not have perfect symmetry and must consider

Case I. $N \sim N_1 \sim N_2 \gg N_3; H \sim L_3 \gtrsim L_1, L_2,$

Case II. $N \sim N_1 \sim N_3 \gg N_2; H \sim L_2 \gtrsim L_1, L_3,$

separately.

The estimate in Case I. In this case, we have $L_{max} \sim N^2 N_3$ and $L_{med} \gtrsim N^{2\alpha}$, and then $N_3^{\frac{1}{2}} \leq L_{med}^{\frac{1}{2\alpha}}$.

When $L_{med} \geq N_3^{\beta+1} N^{2-\beta} \geq N^{2\alpha}$, and then $N_3 \geq N^{\frac{2\alpha-2+\beta}{\beta+1}}$, we get from (3.4) and (3.20) that

$$
\begin{aligned}
(3.20) &\lesssim \sum_{N \gg N_3 \geq N^{\frac{2\alpha-2+\beta}{\beta+1}}} \sum_{L_{max} \sim N^2 N_3 \gtrsim L_{med} \geq N_3^{\beta+1} N^{2-\beta}} \frac{N_3^{1-\rho} N^{2\rho}}{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} N_3^{\frac{1}{2}} \\
(3.26) &\lesssim \sum_{N \gg N_3 \geq N^{\frac{2\alpha-2+\beta}{\beta+1}}} \sum_{L_{max} \sim N^2 N_3, L_{med} \geq N_3^{\beta+1} N^{2-\beta}} \frac{N_3^{1-\rho+\delta} N^{2\rho-1+2\delta}}{L_{med}^{\frac{1}{2}}} \\
&\lesssim \sum_{N \gg N_3 \geq N^{\frac{2\alpha-2+\beta}{\beta+1}}} \sum_{L_{med} \lesssim L_{max} \sim N^2 N_3} \frac{N_3^{\frac{1}{2}-\frac{\beta}{2}-\rho+\delta(2+\beta)} N^{2\rho-2+\frac{\beta}{2}+\delta(4-\beta)}}{L_{med}^{\delta}} \\
&\lesssim \sum_{N \gg N_3 \geq N^{\frac{2\alpha-2+\beta}{\beta+1}}} N_3^{\frac{1}{2}-\frac{\beta}{2}-\rho+\delta(2+\beta)} N^{2\rho-2+\frac{\beta}{2}+\delta(4-\beta)}.
\end{aligned}
$$

When $\frac{1}{2} - \frac{\beta}{2} - \rho < 0$, we get from (3.26) that

$$
(3.20) \lesssim N^{2\rho-2+\frac{\beta}{2}+\delta(4-\beta)+\frac{2\alpha-2+\beta}{\beta+1}(\frac{1}{2}-\frac{\beta}{2}-\rho+\delta(2+\beta))} \lesssim 1,
$$

for $\delta > 0$ and $\beta > 0$ small, since the inequality $\rho < \frac{3-\alpha}{2(2-\alpha)}$ means

$$
2\rho - 2 + \frac{\beta}{2} + \delta(4-\beta) + \frac{2\alpha-2+\beta}{\beta+1}(\frac{1}{2} - \frac{\beta}{2} - \rho + \delta(2+\beta)) < 0
$$

hold for $\delta > 0$ and $\beta > 0$ small. When $\frac{1}{2} - \frac{\beta}{2} - \rho \geq 0$, we get from (3.26) that

$$
(3.20) \lesssim N^{\rho-\frac{3}{2}+6\delta} \lesssim 1
$$

for $\delta > 0$ small.

When $L_{med} \geq N_3^{\beta+1} N^{2-\beta}$, $L_{med} \gtrsim N^{2\alpha}$ and $N_3^{\beta+1} N^{2-\beta} \leq N^{2\alpha}$, we have $N_3 \leq N^{\frac{2\alpha-2+\beta}{\beta+1}}$. We get from (3.4) and (3.20) that

$$(3.20) \lesssim \sum_{N_3 \le N^{\frac{2\alpha-2+\beta}{\beta+1}}} \sum_{L_3 \sim N^2 N_3, L_{med} \ge N^{2\alpha}} \frac{N_3^{1-\rho} N^{2\rho}}{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} N_3^{\frac{1}{2}}$$

$$\lesssim \sum_{N_3 \le N^{\frac{2\alpha-2+\beta}{\beta+1}}} \sum_{L_3 \sim N^2 N_3, L_{med} \ge N^{2\alpha}} \frac{N_3^{1-\rho+\delta} N^{2\rho-1+2\delta}}{L_{med}^{\frac{1}{2}}} \tag{3.27}$$

$$\lesssim \sum_{N_3 \le N^{\frac{2\alpha-2+\beta}{\beta+1}}} \sum_{L_{min} \le L_{med}} \frac{N_3^{1-\rho+\delta} N^{2\rho-1+2\delta-(1-2\delta)\alpha}}{L_{med}^{\delta}}$$

$$\lesssim \sum_{N_3 \le N^{\frac{2\alpha-2+\beta}{\beta+1}}} N_3^{1-\rho+\delta} N^{2\rho-1+2\delta-(1-2\delta)\alpha} \lesssim 1$$

for $\delta > 0$ and $\beta > 0$ small, since the inequality $\rho < \frac{3-\alpha}{2(2-\alpha)}$ means

$$2\rho - 1 + 2\delta - (1-2\delta)\alpha + \frac{2\alpha-2+\beta}{\beta+1}(1-\rho+\delta) < 0$$

hold for $\delta > 0$ and $\beta > 0$ small.

When $L_{med} \le N_3^{\beta+1} N^{2-\beta}$ and $L_{med} \gtrsim N^{2\alpha}$, we have $N_3 \ge N^{\frac{2\alpha-2+\beta}{\beta+1}}$. We get from (3.4) and (3.20) that

$$(3.20) \lesssim \sum_{N_3 \ge N^{\frac{2\alpha-2+\beta}{\beta+1}}} \sum_{L_{med} \le N_3^{\beta+1} N^{2-\beta}, L_3 \sim N^2 N_3} \frac{N_3^{1-\rho} N^{2\rho}}{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} N_3^{\frac{1}{2\beta}} N^{\frac{\beta-2}{2\beta}} L_{med}^{\frac{1}{2\beta}} \tag{3.28}$$

$$\lesssim \sum_{N_3 \ge N^{\frac{2\alpha-2+\beta}{\beta+1}}} \sum_{L_{max} \sim N^2 N_3} \frac{N_3^{\frac{1}{2}-\rho+(2+\beta)\delta-\frac{1}{2\beta}} N^{2\rho-2+2\delta+\frac{\beta}{2}+(2-\beta)\delta}}{L_{med}^{\beta}}$$

$$\lesssim \sum_{N_3 \ge N^{\frac{2\alpha-2+\beta}{\beta+1}}} N_3^{\frac{1}{2}-\rho+(2+\beta)\delta-\frac{1}{2\beta}} N^{2\rho-2+2\delta+\frac{\beta}{2}+(2-\beta)\delta} \lesssim 1$$

for $\delta > 0$ and $\beta > 0$ small, since the inequality $\rho < \frac{3-\alpha}{2(2-\alpha)}$ means

$$2\rho - 2 + 2\delta + \frac{\beta}{2} + (2-\beta)\delta + \frac{2\alpha-2+\beta}{\beta+1}\left(\frac{1}{2} - \rho + (2+\beta)\delta - \frac{1}{2\beta}\right) < 0$$

hold for $\delta > 0$ and $\beta > 0$ small. We complete the estimate in Case I.

The estimate in Case II. In this case, we have $L_{max} \sim N^2 N_2$ and $L_{med} \gtrsim N^{2\alpha}$, and then $N_2^{\frac{1}{2}} \le L_{med}^{\frac{1}{2\alpha}}$.

When $\frac{1}{2} \le \rho \le 1$, we get from (3.4) with $\beta = 2$ and (3.20) that

$$\begin{aligned}(3.20) \quad \lesssim & \sum_{N\sim N_1\sim N_3\gg N_2} \sum_{L_1,L_3\le L_2\sim N^2N_2} \\ & \frac{N_3^{1-\rho}N_1^{\rho}N_2^{\rho}}{L_1^{\frac{1}{2}}L_2^{\frac{1}{2}}L_3^{\frac{1}{2}-\delta}} \|X_{N_1,N_2,N_3;L_{max};L_1,L_2,L_3}\|_{[3,\mathbb{R}\times\mathbb{R}]} \\ \lesssim & \sum_{N\sim N_1\sim N_3\gg N_2} \sum_{L_1,L_3\le L_2\sim N^2N_2} \\ & \frac{N_3^{\rho}N_1^{\rho}N_2^{1-\rho}}{L_1^{\frac{1}{2}}L_2^{\frac{1}{2}-\delta}L_3^{\frac{1}{2}}} \|X_{N_1,N_2,N_3;L_{max};L_1,L_2,L_3}\|_{[3,\mathbb{R}\times\mathbb{R}]}.\end{aligned}$$

By symmetry and the estimate obtained in Case I we get

$$\begin{aligned}(3.20) \lesssim & \sum_{N\sim N_1\sim N_3\gg N_2} \sum_{L_1,L_3\le L_2\sim N^2N_2} \\ & \frac{N_3^{1-\rho}N_1^{\rho}N_2^{\rho}}{L_1^{\frac{1}{2}}L_2^{\frac{1}{2}}L_3^{\frac{1}{2}-\delta}} \|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,R\times R]} \lesssim 1.\end{aligned}$$

When $0 \le \rho < \frac{1}{2}$, we have $L_2 \sim N^2 N_2$ and $L_{med} \gtrsim N^{2\alpha}$, and then $N_2^{\frac{1}{2}} \le L_{med}^{\frac{1}{2\alpha}}$. We get from (3.4) and (3.20) that

$$\begin{aligned}(3.20) \lesssim & \sum_{N_1\sim N_3\sim N} \sum_{L_2\sim N^2N_2\gtrsim L_{med}\ge N_2^3\ge N^{2\alpha}} \frac{N_2^{\rho}N}{L_1^{\frac{1}{2}}L_2^{\frac{1}{2}}L_3^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}}N_2^{\frac{1}{2}} \\ & + \sum_{N_1\sim N_3\sim N} \sum_{L_2\sim N^2N_2\gtrsim L_{med}\ge N^{2\alpha}\ge N_2^3} \frac{N_2^{\rho}N}{L_1^{\frac{1}{2}}L_2^{\frac{1}{2}}L_3^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}}N_2^{\frac{1}{2}} \\ & + \sum_{N_1\sim N_3\sim N} \sum_{L_2\sim N^2N_2, N^{2\alpha}\le L_{med}\le N_2^3} \frac{N_2^{\rho}N}{L_1^{\frac{1}{2}}L_2^{\frac{1}{2}}L_3^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}}N_2^{\frac{1}{4}}L_{med}^{\frac{1}{4}} \qquad (3.29) \\ \lesssim & \sum_{N_2\gtrsim N^{\frac{2\alpha}{3}}} \sum_{L_2\sim N^2N_2, L_{med}\ge N_2^3} \\ & \frac{N_2^{\rho}N^{2\delta}}{L_{med}^{\frac{1}{2}}} + \sum_{N_2\lesssim N^{\frac{2\alpha}{3}}} \sum_{L_2\sim N^2N_2, L_{med}\ge N^{2\alpha}} \frac{N_2^{\rho+\delta}N^{2\delta}}{L_{med}^{\frac{1}{2}}}\end{aligned}$$

$$
\begin{aligned}
& + \sum_{N_2 \gtrsim N^{\frac{2\alpha}{3}}} \sum_{L_2 \sim N^2 N_2, L_{med} \leq N_2^3} \frac{N_2^{\frac{1}{4}+\rho} N}{L_{med}^{\frac{1}{4}-\delta}} \\
\lesssim & \sum_{N_2 \gtrsim N^{\frac{2\alpha}{3}}} \left[N_2^{-\frac{3}{2}+\rho+4\delta} + N_2^{-\frac{3}{4}+\rho+4\delta} \right] N^{2\delta} \\
& + \sum_{N_2 \lesssim N^{\frac{2\alpha}{3}}} N_2^{\rho+\delta} N^{-\alpha+2(1+\alpha)\delta} \lesssim 1,
\end{aligned}
$$

for $\delta > 0$ small. We complete the estimate in Case II.

To finish the estimate of (3.20) it remains to deal with the cases where (3.5) holds. When $N_{min} = N_3$, we have $L_3 \ll L_{max}$, and

$$
\begin{aligned}
(3.20) & \lesssim \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N} \sum_{L_3 \ll L_{max} \sim \max\{N^2 N_{min}, N^{2\alpha}\}} \frac{N_{min}^{1-\rho} N^{2\rho}}{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} N^{-1} L_{med}^{\frac{1}{2}} \\
& \lesssim \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N} \sum_{L_{max} \sim \max\{N^2 N_{min}, N^{2\alpha}\}} \frac{N_{min}^{1-\rho} N^{2\rho-1}}{L_{max}^{\frac{1}{2}} L_{min}^{\frac{1}{2}} L_{med}^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} N^{-1} L_{med}^{\frac{1}{2}} \\
& \lesssim \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N} \sum_{L_{max} \sim \max\{N^2 N_{min}, N^{2\alpha}\}} \frac{N_{min}^{1-\rho} N^{2\rho}}{L_{max}^{\frac{1}{2}} L_{max}^{\frac{1}{2}-\delta}} \\
& \lesssim \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \geq N^{2\alpha-2}} \sum_{L_{max} \sim N^2 N_{min}} \frac{N_{min}^{1-\rho} N^{2\rho}}{L_{max}^{\frac{1}{2}} L_{max}^{\frac{1}{2}-\delta}} \\
(3.30) \quad & + \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \leq N^{2\alpha-2}} \sum_{L_{max} \sim N^{2\alpha}} \frac{N_{min}^{1-\rho} N^{2\rho}}{L_{max}^{\frac{1}{2}} L_{max}^{\frac{1}{2}-\delta}} \\
& \lesssim \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \geq N^{2\alpha-2}} \sum_{L_{max} \sim N^2 N_{min}} \frac{N_{min}^{\frac{1}{2}-\rho+2\delta} N^{2\rho-2+4\delta}}{L_{max}^{\delta}} \\
& + \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \leq N^{2\alpha-2}} \sum_{L_{max} \sim N^{2\alpha}} \frac{N_{min}^{1-\rho} N^{2\rho-1-\alpha+2\alpha\delta}}{L_{max}^{\delta}} \\
& \lesssim \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \geq N^{2\alpha-2}} N_{min}^{\frac{1}{2}-\rho+2\delta} N^{2\rho-2+4\delta} \\
& + \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \leq N^{2\alpha-2}} N_{min}^{1-\rho} N^{2\rho-1-\alpha+2\alpha\delta}.
\end{aligned}
$$

When $\frac{1}{2} < \rho \leq 1$, we get from (3.30) that

$$
(3.31) \qquad (3.20) \lesssim \sup_{N \gtrsim 1} N^{\alpha-3+2(2-\alpha)\rho+4\alpha\delta} + \sup_{N \gtrsim 1} N^{\alpha-3+2(2-\alpha)\rho+2\alpha\delta} \lesssim 1,
$$

for $\delta > 0$ small, since the inequality $\rho < \frac{3-\alpha}{2(2-\alpha)}$ means

$$\alpha - 3 + 2(2-\alpha)\rho + 4\alpha\delta < 0$$

and

$$\alpha - 3 + 2(2-\alpha)\rho + 2\alpha\delta < 0$$

hold for $\delta > 0$ small. When $0 \le \rho \le \frac{1}{2}$, we get from (3.30) that

$$(3.20) \lesssim \sup_{N \gtrsim 1} N^{\rho - \frac{3}{2} + 6\delta} + \sup_{N \gtrsim 1} N^{\alpha - 3 + 2(2-\alpha)\rho + 2\alpha\delta} \lesssim 1, \tag{3.32}$$

for $\delta > 0$ small.

When $N_{min} = N_2$, we have $L_2 \ll L_{max}$, and

$$\begin{aligned}
(3.20) \lesssim\ & \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N} \sum_{L_2 \ll L_{max} \sim \max\{N^2 N_{min}, N^{2\alpha}\}} \\
& \frac{N_{min}^{\rho} N}{L_{min}^{\frac{1}{2}} L_{med}^{\frac{1}{2}} L_{max}^{\frac{1}{2}-\delta}} L_{min}^{\frac{1}{2}} N^{-1} L_{med}^{\frac{1}{2}} \\
\lesssim\ & \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N} \sum_{L_2 \ll L_{max} \sim \max\{N^2 N_{min}, N^{2\alpha}\}} \frac{N_{min}^{\rho}}{L_{max}^{\frac{1}{2}-\delta}} \\
\lesssim\ & \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \ge N^{2\alpha-2}} \sum_{L_{max} \sim N^2 N_{min}} \frac{N_{min}^{\rho}}{L_{max}^{\frac{1}{2}-\delta}} \\
& + \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \le N^{2\alpha-2}} \sum_{L_{max} \sim N^{2\alpha}} \frac{N_{min}^{\rho}}{L_{max}^{\frac{1}{2}-\delta}} \\
\lesssim\ & \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \ge N^{2\alpha-2}} \sum_{L_{max} \sim N^2 N_{min}} \frac{N_{min}^{\rho - \frac{1}{2} + 2\delta} N^{-1+2\delta}}{L_{max}^{\delta}} \\
& + \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \le N^{2\alpha-2}} \sum_{L_{max} \sim N^{2\alpha}} \frac{N_{min}^{\rho} N^{\alpha+2\alpha\delta}}{L_{max}^{\delta}} \\
\lesssim\ & \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \ge N^{2\alpha-2}} N_{min}^{\rho - \frac{1}{2} + 2\delta} N^{-1+2\delta} \\
& + \sup_{N \gtrsim 1} \sum_{N_{med} \sim N_{max} \sim N, N_{min} \le N^{2\alpha-2}} N_{min}^{\rho} N^{\alpha+2\alpha\delta}.
\end{aligned} \tag{3.33}$$

When $\frac{1}{2} \le \rho \le 1$, we get from (3.33) that

$$(3.20) \lesssim \sup_{N \gtrsim 1} N^{\rho - \frac{3}{2} + 4\delta} + \sup_{N \gtrsim 1} N^{\alpha - 2 + 2\alpha\delta} \lesssim 1, \tag{3.34}$$

for $\delta > 0$ small. When $\rho < \frac{1}{2}$, we get from (3.33) that

$$(3.35)\qquad (3.20) \lesssim \sup_{N \gtrsim 1} N^{(2\alpha-2)\rho-\alpha+2\delta(2\alpha-1)} + \sup_{N \gtrsim 1} N^{\alpha-2+2\alpha\delta} \lesssim 1,$$

for $\delta > 0$ small.

By symmetry, the same estimate holds when $N_{min} = N_3$. We complete the proof of $(3.20) \lesssim 1$. ∎

Theorem 3.2. *Given* $s \in (\frac{\alpha-3}{2(2-\alpha)}, 0]$, *there exist* $\mu > 0$, $\delta > 0$ *such that for any couple* $(u, v) \in \dot{X}_\alpha^{\frac{1}{2},s}$ *with compact support in* $[-T, T]$,

$$(3.36)\qquad \|\partial_x(uv)\|_{\dot{X}_\alpha^{-\frac{1}{2}+\delta,s}} \lesssim T^\mu \|u\|_{\dot{X}_\alpha^{\frac{1}{2},s}} \|v\|_{\dot{X}_\alpha^{\frac{1}{2},s}}.$$

Proof. By duality, (3.36) is equivalent to, for all $w \in \dot{X}_\alpha^{\frac{1}{2}-\delta,s}$,

$$(3.37)\qquad |< \partial_x(uv), w >| \lesssim T^\mu \|u\|_{\dot{X}_\alpha^{\frac{1}{2},s}} \|v\|_{\dot{X}_\alpha^{\frac{1}{2},s}} \|w\|_{\dot{X}_\alpha^{\frac{1}{2},s-\delta}},$$

which follows from a combination of Lemma 3.2 with Lemma 3.6 in [8]. ∎

4. Proof of the Main Theorem

Let

$$(4.1)\qquad F(u) = \psi(t)\left[W(t)\varphi - \frac{\chi_{\mathbb{R}^+}(t)}{2}\int_0^t W(t-t')\partial_x(\psi_T^2(t')u^2(t'))dt'\right].$$

Clearly, if u is a solution of the integral equation $u = F(u)$, then u is a solution of (1.1) for $0 \le t \le T$. Let $s \in (\frac{\alpha-3}{2(2-\alpha)}, 0]$. For a given $\varphi \in \dot{H}^s(\mathbb{R})$ and for any $u, v \in \dot{X}^{\frac{1}{2},s}$, one deduces from Proposition 2.1 and Theorem 3.2 that

$$(4.2)\qquad \|F(u)\|_{\dot{X}_\alpha^{\frac{1}{2},s}} \lesssim \|\varphi\|_{\dot{H}^s} + T^\mu \|u\|^2_{\dot{X}_\alpha^{\frac{1}{2},s}}.$$

Similarly, using $\partial_x(u^2) - \partial_x(v^2) = \partial_x[(u+v)(u-v)]$,

$$(4.3)\qquad \|F(u) - F(v)\|_{\dot{X}_\alpha^{\frac{1}{2},s}} \lesssim T^\mu \|u-v\|_{\dot{X}_\alpha^{\frac{1}{2},s}}\left[\|u\|_{\dot{X}_\alpha^{\frac{1}{2},s}} + \|v\|_{\dot{X}_\alpha^{\frac{1}{2},s}}\right].$$

(4.2) and (4.3) implies that, for $T \le T_0 \lesssim \|\varphi\|_{\dot{H}^s(\mathbb{R})}^{-1/\mu}$, F is strictly contractive on the ball of radius $\lesssim \|\varphi\|_{\dot{H}^s(\mathbb{R})}$. This proves the existence of a solution $u \in \dot{X}_\alpha^{-\frac{1}{2},s}$ to the Cauchy problem (1.1) on the time interval $[0, T_0]$ with $T_0 = T_0(\|\varphi\|_{\dot{H}^s}) > 0$.

Let $u_1, u_2 \in \dot{X}^{\frac{1}{2},s}_{\alpha,T}$ be two solution of the integral equation (1.1) on the time interval $[0,T]$ with $T \leq T_0$. On account of Proposition 2.4, $u_1, u_2 \in C([0,T]; \dot{H}^s(\mathbb{R}))$. For $0 < \beta < T/2$ to be specified later, we define $\widetilde{u}_i, i = 1, 2.$ by

$$\widetilde{u}_i = \begin{cases} u_i(t), \quad t \in [0,\beta] \\ u_i(2\beta - t), \quad t \in [\beta, 2\beta] \\ \emptyset, \quad else \end{cases}$$

Obviously $t \mapsto \widetilde{u}_i$ is locally continuous in $\dot{X}^{\frac{1}{2},s}_{\alpha,T}$, and $\widetilde{u}_1 - \widetilde{u}_2 \equiv 0$ on $\mathbb{R}\backslash[0, 2\beta]$. By (4.1) and Theorem 3.2, one has

$$\begin{aligned} \| u_1 - u_2 \|_{\dot{X}^{\frac{1}{2},s}_{\alpha,T}} &\lesssim \| \partial_x((\widetilde{u}_1(t') - \widetilde{u}_2(t'))(u_1(t') + u_2(t'))) \|_{\dot{X}^{-\frac{1}{2}+\delta,s}_{\alpha,T}} \\ &\lesssim \beta^\mu \| \widetilde{u}_1 - \widetilde{u}_2 \|_{\dot{X}^{\frac{1}{2},s}_{\alpha,T}} \| u_1 + u_2 \|_{\dot{X}^{\frac{1}{2},s}_{\alpha,T}}, \end{aligned}$$

for some $\mu > 0$. By the construction, we easily get

$$\| \widetilde{u}_1 - \widetilde{u}_2 \|_{X^{\frac{1}{2},s}_{\alpha,T}} \lesssim \| u_1 - u_2 \|_{X^{\frac{1}{2},s}_{\alpha,T}} .$$

Hence,

$$\| u_1 - u_2 \|_{\dot{X}^{\frac{1}{2},s}_{\alpha,T}} \lesssim \beta^\mu (\| u_1 \|_{\dot{X}^{\frac{1}{2},s}_{\alpha,T}} + \| u_2 \|_{\dot{X}^{\frac{1}{2},s}_{\alpha,T}}) \| u_1 - u_2 \|_{\dot{X}^{\frac{1}{2},s}_{\alpha,T}} .$$

Taking $\beta \lesssim [\| u_1 \|_{\dot{X}^{\frac{1}{2},s}_{\alpha,T}} + \| u_2 \|_{\dot{X}^{\frac{1}{2},s}_{\alpha,T}}]^{-\mu}$, we conclude $u_1 \equiv u_2$ on $[0,\beta]$. Iterating this argument, one extends the uniqueness result on the whole time interval $[0, T_0]$.

Note that for $\varphi \in \dot{H}^s(\mathbb{R})$ with $s \leq 0$, one has

$$W(\cdot)\varphi \in C([0,+\infty), \dot{H}^s(\mathbb{R})) \bigcap C((0,+\infty), L^2(\mathbb{R})).$$

Let u be the solution of (1.1) associated with φ. It follows from the estimate $u \in \dot{X}^{\frac{1}{2},s}_{\alpha,T_0}$ together with Theorem 3.2 that $\partial_x(u^2) \in \dot{X}^{-\frac{1}{2}+\delta,s}_{\alpha,T_0}$. Using Proposition 2.4, one deduces

$$t \mapsto \int_0^t W(t-t')\partial_x(u^2(t-t'))dt' \in C((0,T_0], \dot{H}^{s+\delta}),$$

and so

$$u \in C([0,T_0], \dot{H}^s) \bigcap C((0,T_0], \dot{H}^{s+\delta}). \tag{4.4}$$

That $u \in C([0, T_0], \dot{H}^s) \cap C((0, T_0], L^2(\mathbb{R}))$ is deduced from (4.4) and the uniqueness result by induction. Finally, the mapping that associated to initial-data the corresponding solution is analytic as a function between appropriate Banach spaces directly from Proposition 2.4 and the implicit function theorem (see [5], [7] and references therein).

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Ruying Xue and Sufen Hu
Department of Mathematics,
Zhejiang University,
Hangzhou 310027, Zhejiang,
P. R. China
E-mail: ryxue@zju.edu.cn