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THE GENERALIZED ROPER-SUFFRIDGE EXTENSION OPERATOR ON REINHARDT DOMAIN D_p

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Abstract. We define the generalized Roper-Suffridge extension operator $\Phi_{n,\beta_2,\gamma_2,\cdots,\beta_n,\gamma_n}(f)$ on Reinhardt domain D_p as

$$\Phi_{n,\beta_2,\gamma_2,\cdots,\beta_n,\gamma_n}(f)(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^{\beta_2}\left(f'(z_1)\right)^{\gamma_2} z_2, \cdots, \left(\frac{f(z_1)}{z_1}\right)^{\beta_n}\left(f'(z_1)\right)^{\gamma_n} z_n\right)$$

for $z = (z_1, z_2, \cdots, z_n) \in D_p$, where $D_p = \left\{(z_1, z_2, \cdots, z_n) \in C^n : \sum_{j=1}^{n} |z_j|^{p_j} < 1\right\}$, $p = (p_1, p_2, \cdots, p_n)$, $p_j > 0$, $0 \le \gamma_j \le 1 - \beta_j, 0 \le \beta_j \le 1$, $j = 1, 2, \cdots, n$, and we choose the branch of the power functions such that $\left(\frac{f(z_1)}{z_1}\right)^{\beta_j}|_{z_1=0} = 1$ and $(f'(z_1))^{\gamma_j}|_{z_1=0} = 1, j = 2, \cdots, n$. In the present paper, we show that the operator $\Phi_{n,\beta_2,\gamma_2,\cdots,\beta_n,\gamma_n}(f)$ preserves almost spirallike mapping of type β and order α and spirallike mapping of type β and order α on D_p for some suitable constants β_j, γ_j, p_j. The results improve the corresponding results of earlier authors.

1. Introduction

Let C^n be the vector space of n-complex variables $z = (z_1, z_2, \cdots, z_n)$ with the usual inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$. A domain $\Omega \subset C^n$ is said to be complete circular if $z \in \Omega$ implies $\xi z \in \Omega$ for all $\xi \in C$ with $|\xi| \le 1$. A domain $\Omega \subset C^n$ is said to be complete Reinhardt if $(z_1, z_2, \cdots, z_n) \in \Omega$ implies

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$(\xi_1 z_1, \xi_2 z_2, \cdots, \xi_n z_n) \in \Omega$ for all $\xi_j \in C$ with $|\xi_j| \le 1, j = 1, 2, \cdots, n$. A domain $\Omega \subset C^n$ is said to be starlike if $z \in \Omega$ implies $tz \in \Omega$ for $0 \le t \le 1$. The $Minkowski$ functional of a bounded complete circular domain Ω in C^n is defined by

$$\rho(z) = \inf\Big\{t > 0, \frac{z}{t} \in \Omega\Big\}, z \in C^n.$$

Assume $p = (p_1, p_2, \cdots, p_n)$ with $p_j > 0$ $(j = 1, 2, \cdots, n)$, and let

$$D_p = \Big\{(z_1, z_2, \cdots, z_n) \in C^n : \sum_{j=1}^{n} |z_j|^{p_j} < 1\Big\}. \tag{1.1}$$

Then D_p is a bounded complete Reinhardt domain in C^n, and the $Minkowski$ functional $\rho(z)$ of D_p satisfies

$$\sum_{j=1}^{n} \left|\frac{z_j}{\rho(z)}\right|^{p_j} = 1. \tag{1.2}$$

Suppose that $\Omega \subset C^n$ is a bounded complete circular domain. The class $H(\Omega)$ consists of all holomorphic mappings $f : \Omega \to C^n$. The first Fréchet derivative and the second Fréchet derivative of a mapping $f \in H(\Omega)$ at a point $z \in \Omega$ are denoted by $Df(z)(\cdot)$, $D^2 f(z)(b, \cdot)$ respectively. Their matrix representations are

$$Df(z) = \left(\frac{\partial f_j(z)}{\partial z_k}\right)_{1\le j,k\le n}, \quad D^2 f(z)(b, \cdot) = \left(\sum_{l=1}^{n} \frac{\partial^2 f_j(z)}{\partial z_k \partial z_l} b_l\right)_{1\le j,k\le n},$$

where $f(z) = (f_1(z), \cdots, f_n(z)), b = (b_1, \cdots, b_n) \in C^n$. A mapping $f \in H(\Omega)$ is said to be locally biholomorphic on Ω if f has a local inverse at each point $z \in \Omega$ or, equivalently, if $\det Df(z) \ne 0$ at each point on Ω.

Let $N(\Omega)$ denote the class of all locally biholomorphic mappings $f : \Omega \to C^n$ such that $f(0) = 0, Df(0) = I$, where I is the unit $n \times n$ matrix, and let $S(\Omega)$ be the class of all biholomorphic mappings in $N(\Omega)$. If $f \in S(\Omega)$, and $f(\Omega)$ is a starlike domain in C^n, then we say that f is a biholomorphic starlike mapping on Ω. The class of all biholomorphic starlike mappings on Ω with $f(0) = 0, Df(0) = I$ is denoted by $S^*(\Omega)$.

Suppose that $\Omega \subset C^n$ is a bounded complete circular domain, its $Minkowski$ functional $\rho(z)$ is a C^1 function except for a lower dimensional manifold Ω_0 in $\overline{\Omega}$. Let $0 \le \alpha < 1, -\frac{\pi}{2} < \beta < \frac{\pi}{2}$. A mapping $f \in N(\Omega)$ is said to be almost spirallike mapping of type β and order α if

$$\mathrm{Re}\Big[2e^{-i\beta}\Big\langle Df(z)^{-1} f(z), \overline{\frac{\partial\rho(z)}{\partial z}}\Big\rangle\Big] \ge \rho(z)\alpha\cos\beta$$

for $z \in \Omega \setminus \Omega_0$, where $\frac{\partial \rho(z)}{\partial z} = \left(\frac{\partial \rho(z)}{\partial z_1}, \frac{\partial \rho(z)}{\partial z_2}, \cdots, \frac{\partial \rho(z)}{\partial z_n}\right)$. A mapping $f \in N(\Omega)$ is said to be spirallike mapping of type β and order α if

$$\left|4e^{-i\beta}\alpha\left\langle Df(z)^{-1}f(z), \overline{\frac{\partial \rho(z)}{\partial z}}\right\rangle - \rho(z)(\cos\beta - i2\alpha\sin\beta)\right| \le \rho(z)\cos\beta$$

for $z \in \Omega \setminus \Omega_0$ and $0 < \alpha < 1$, and

$$\mathrm{Re}\left[2e^{-i\beta}\left\langle Df(z)^{-1}f(z), \overline{\frac{\partial \rho(z)}{\partial z}}\right\rangle\right] \ge 0$$

for $z \in \Omega \setminus \Omega_0$ and $\alpha = 0$. The class $\widehat{S}_\alpha(\Omega, \beta)$ consists of all normalized spirallike mappings of type β and order α on Ω and the class $A\widehat{S}_\alpha(\Omega, \beta)$ consists of all normalized almost spirallike mappings of type β and order α on Ω for $0 \le \alpha < 1$. Then we have

$$f \in A\widehat{S}_\alpha(U, \beta) \Longleftrightarrow f \in S(U) \text{ and } \mathrm{Re}\left[e^{-i\beta}\frac{f(\xi)}{\xi f'(\xi)}\right] \ge \alpha\cos\beta \text{ for } \xi \in U,$$

and

$$f \in \widehat{S}_0(U, \beta) \Longleftrightarrow f \in S(U) \text{ and } \mathrm{Re}\left[e^{-i\beta}\frac{f(\xi)}{\xi f'(\xi)}\right] \ge 0 \text{ for } \xi \in U,$$

and

$$f \in \widehat{S}_\alpha(U, \beta) \Longleftrightarrow f \in S(U) \text{ and}$$

$$\left|2\alpha(1 - i\tan\beta)\frac{f(\xi)}{\xi f'(\xi)} - 1 + i2\alpha\tan\beta\right| \le 1 \text{ for } \xi \in U$$

for $0 < \alpha < 1$.

Let $S^*_\alpha(\Omega) = \widehat{S}_\alpha(\Omega, 0)$ for $0 < \alpha < 1$ and $S^*_0(\Omega) = S^*(\Omega)$, and let $\widehat{S}_0(\Omega, \beta) = A\widehat{S}_0(\Omega, \beta) = \widehat{S}(\Omega, \beta)$. A mapping $f \in S^*_\alpha(\Omega)$ is called biholomorphic starlike mapping of order α on Ω for $0 \le \alpha < 1$. A mapping $f \in \widehat{S}(\Omega, \beta)$ is called spirallike mapping of type β on Ω. It is evident that $A\widehat{S}_0(\Omega, 0) = \widehat{S}_0(\Omega, 0) = S^*(\Omega)$. From Theorem 1.2.1 in [9], we have $S^*_\alpha(\Omega) \subset S^*(\Omega)$ for $0 \le \alpha < 1$.

In 1995, Roper and Suffridge [19] introduced an extension operator. This operator is defined for normalized locally biholomorphic function f on the unit disc U in C by

$$F(z) = \Phi_n(f)(z) = \left(f(z_1), \sqrt{f'(z_1)}z_0\right), \tag{1.3}$$

where $z = (z_1, z_0)$ belongs to the unit ball B^n in C^n, $z_1 \in U$, $z_0 = (z_2, \cdots, z_n) \in C^{n-1}$, and we choose the branch of the square root such that $\sqrt{f'(0)} = 1$.

Roper and Suffridge [19] proved that: If $f \in K(U)$, then $F = \Phi_2(f) \in K(B^2)$, where $K(\Omega)$ is the class of all biholomorphic convex mappings on Ω. However, its proof is very complex, Graham and Kohr [1, 2] gave a simple proof of the theorem of Roper and Suffridge. After that, the other properties of Roper-Suffridge operator were studied by Graham, Hamada, Kohr and etc.(see [3, 4, 6, 7, 8]). Moreover, Sheng Gong and Taishun Liu [10, 11], Xiaosong Liu and Taishun Liu [16, 17] generalized Roper-Suffridge operator from the unit ball B^n to Reinhardt domain $D_p = \left\{(z_1, z_2, \cdots, z_n) \in C^n : \sum\limits_{j=1}^{n} |z_j|^{p_j} < 1\right\}$, where $p_1 = 2, p_2 \geq 1, p_3 \geq 1, \cdots, p_n \geq 1$. We also generalized Roper-Suffridge operator to Banach spaces in [12, 13, 22, 23]. All of these papers studied the properties of the generalized Roper-Suffridge operator on a domain D_p with some $p_j = 2$.

Recently, Liu [15] studied the properties of the generalized Roper-Suffridge operator on Reinhardt domain D_p with $0 < p_1 \leq 2$, $p_j \geq 1 (j = 2, 3, \cdots, n)$, which is defined by

$$\begin{aligned} &\Phi_{n,\beta_2,\gamma_2,\cdots,\beta_n,\gamma_n}(f)(z) \\ &= \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^{\beta_2} \left(f'(z_1)\right)^{\gamma_2} z_2, \cdots, \left(\frac{f(z_1)}{z_1}\right)^{\beta_n} \left(f'(z_1)\right)^{\gamma_n} z_n\right) \end{aligned} \tag{1.4}$$

for $z = (z_1, z_2, \cdots, z_n) \in D_p$, where $0 \leq \beta_j \leq 1, 0 \leq \gamma_j \leq 1 - \beta_j$, and we choose the branch of the power functions such that $(\frac{f(z_1)}{z_1})^{\beta_j}|_{z_1=0} = 1$ and $(f'(z_1))^{\gamma_j}|_{z_1=0} = 1, j = 2, \cdots, n$. We studied some properties of the generalized Roper-Suffridge operator on Reinhardt domain D_p with $p_j > 0$ $(j = 1, 2, \cdots, n)$. We proved the following result.

Theorem A. [24]. *Suppose that* $n \geq 2$, $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$, $0 \leq \alpha < 1$, $p_1 > 0$, $p_j > 0$, $\beta_j + \gamma_j \leq 1$, $\beta_j \in [0, 1]$, $\gamma_j \in [0, \frac{1}{ap_j}]$ $(j = 2, \cdots, n)$*, where* $a = a(p_1)$ *is defined by*

$$a = a(p_1) = \begin{cases} 1, & \text{if} \quad 0 < p_1 \leq 2, \\ \dfrac{(\sqrt{2}+1)^{p_1} - 1}{(\sqrt{2}+1)p_1}, & \text{if} \quad p_1 > 2, \end{cases} \tag{1.5}$$

and D_p *is defined by* (1.1). *The operator* $\Phi_{n,\beta_2,\gamma_2,\cdots,\beta_n,\gamma_n}(f)$ *is defined by* (1.4). *Then*

(1) $\Phi_{n,\beta_2,\gamma_2,\cdots,\beta_n,\gamma_n}(\widehat{S}(U, \beta)) \subset \widehat{S}(D_p, \beta)$*, and*

(2) $\Phi_{n,\beta_2,\gamma_2,\cdots,\beta_n,\gamma_n}(S^*_\alpha(U)) \subset S^*_\alpha(D_p)$.

One of the purposes of this paper is to improve Theorem A[14, 24]. Thus we are able to replace the constant $a(p_1)$ in Theorem A by a smaller constant $b(p_1)$. We also shall discuss some properties of the generalized Roper-Suffridge extension operator $\Phi_{n,\beta_2,\gamma_2,\cdots,\beta_n,\gamma_n}(f)$ on the Reinhardt domain D_p with $p_j > 0(j = 1, 2, \cdots, n)$ for almost spirallike mapping of type β and order α.

2. Main Results

In order to state and verify our main results, we need the following lemmas.

Lemma 2.1. *Let* $p_1 > 0$, $p_j > 0, \gamma_j \in [0, \frac{1}{bp_j}](j = 2, \cdots, n)$, *where* $b = b(p_1)$ *is defined by*

$$b = b(p_1) = \begin{cases} 1, & \text{if } 0 < p_1 \le 2, \\ c(p_1), & \text{if } p_1 > 2, \end{cases} \tag{2.1}$$

and

$$c(p_1) = \sup_{\sqrt{2}-1 \le t \le 1} \frac{(t^2 + 2t - 1)(1 - t^{p_1})}{p_1 t^{p_1}(1 - t^2)}. \tag{2.2}$$

Suppose that D_p *is defined by* (1.1), *and* $\rho(z)$ *is the* $Minkowski$ *functional of* D_p. *If the function* $\rho(z)$ *is differentiable at* $z = (z_1, z_2, \cdots, z_n) \in D_p$, *then we have*

$$\frac{\partial\rho(z)}{\partial z_1} z_1 + \frac{1 - 2|z_1| - |z_1|^2}{1 - |z_1|^2} \sum_{j=2}^{n} \gamma_j \frac{\partial\rho(z)}{\partial z_j} z_j \ge 0. \tag{2.3}$$

Proof. By Lemma 2.5 in [24], we only need to prove that Lemma 2.1 holds for $p_1 > 2$.

Suppose that the function $\rho(z)$ is differentiable at $z = (z_1, z_2, \cdots, z_n) \in D_p$. From Lemma 1.1 in [21], we have

$$\frac{\partial\rho(z)}{\partial z_j} = \frac{p_j \overline{z_j} |\frac{z_j}{\rho(z)}|^{p_j - 2}}{2\rho(z) \sum\limits_{k=1}^{n} p_k |\frac{z_k}{\rho(z)}|^{p_k}}, \ j = 1, 2, \cdots, n. \tag{2.4}$$

This implies that

$$\frac{\partial\rho(z)}{\partial z_j} z_j \ge 0, \ j = 1, 2, \cdots, n, \tag{2.5}$$

and

$$\rho(z)=2\sum_{j=1}^{n}\frac{\partial\rho(z)}{\partial z_j}z_j. \tag{2.6}$$

Now we split inequality (2.3) into two cases to prove.

Case 1. If $0\le|z_1|\le\sqrt{2}-1$, then we have $\frac{2|z_1|}{1-|z_1|^2}\le 1$. Noting that $\gamma_j\ge 0(j=2,\cdots,n)$, from (2.5), we obtain

$$\begin{aligned}&\frac{\partial\rho(z)}{\partial z_1}z_1+\frac{1-2|z_1|-|z_1|^2}{1-|z_1|^2}\sum_{j=2}^{n}\gamma_j\frac{\partial\rho(z)}{\partial z_j}z_j\\ =&\frac{\partial\rho(z)}{\partial z_1}z_1+\Big(1-\frac{2|z_1|}{1-|z_1|^2}\Big)\sum_{j=2}^{n}\gamma_j\frac{\partial\rho(z)}{\partial z_j}z_j\ge 0.\end{aligned}$$

Case 2. If $\sqrt{2}-1\le|z_1|<1$, we let $w_j=\frac{z_j}{\rho(z)}, j=1,2,\cdots,n, A=\sum\limits_{k=1}^{n}p_k|w_k|^{p_k}$. By the definition of $\rho(z)$, we have $\rho(z)\le 1$ for $z\in D_p$. Hence we have $\sqrt{2}-1\le|z_1|\le|w_1|<1$ and

$$1-2|w_1|-|w_1|^2\le 0. \tag{2.7}$$

Since $\gamma_j p_j\le\frac{1}{c(p_1)}$, $\gamma_j\ge 0$ $(j=2,\cdots,n)$ and $\sum\limits_{j=1}^{n}|w_j|^{p_j}=1$, from (2.2), (2.4), (2.5),(2.7), we obtain

$$\begin{aligned}&\frac{\partial\rho(z)}{\partial z_1}z_1+\frac{1-2|z_1|-|z_1|^2}{1-|z_1|^2}\sum_{j=2}^{n}\gamma_j\frac{\partial\rho(z)}{\partial z_j}z_j\\ =&\frac{\partial\rho(z)}{\partial z_1}z_1+\Big(1-\frac{2|z_1|}{1-|z_1|^2}\Big)\sum_{j=2}^{n}\gamma_j\frac{\partial\rho(z)}{\partial z_j}z_j\\ \ge&\frac{\partial\rho(z)}{\partial z_1}z_1+\Big(1-\frac{2|w_1|}{1-|w_1|^2}\Big)\sum_{j=2}^{n}\gamma_j\frac{\partial\rho(z)}{\partial z_j}z_j\\ =&\frac{\rho(z)}{2A}\Big\{p_1|w_1|^{p_1}+\frac{1-2|w_1|-|w_1|^2}{1-|w_1|^2}\sum_{j=2}^{n}\gamma_jp_j|w_j|^{p_j}\Big\}\\ \ge&\frac{\rho(z)}{2A}\Big\{p_1|w_1|^{p_1}+\frac{1-2|w_1|-|w_1|^2}{1-|w_1|^2}\frac{1}{c(p_1)}\sum_{j=2}^{n}|w_j|^{p_j}\Big\}\\ =&\frac{\rho(z)p_1|w_1|^{p_1}}{2Ac(p_1)}\Big\{c(p_1)-\frac{(|w_1|^2+2|w_1|-1)(1-|w_1|^{p_1})}{p_1|w_1|^{p_1}(1-|w_1|^2)}\Big\}\\ \ge&\ 0.\end{aligned}$$

This completes the proof.

Lemma 2.2. ([1, 24]).

(1) *Suppose that* $g \in H(U)$ *satisfies* $g(U) \subset U$. *Then*

$$|g'(\xi)| \le \frac{1-|g(\xi)|^2}{1-|\xi|^2}$$

for each $\xi \in U$.

(2) *Suppose that* $p \in H(U)$ *satisfies* $\mathrm{Re}p(\xi) > 0$ *for* $\xi \in U$ *with* $p(0) = 1$. *Then*

$$\mathrm{Re}[p(\xi) + \xi p'(\xi)] \ge \frac{1-2|\xi|-|\xi|^2}{1-|\xi|^2}\mathrm{Re}p(\xi)$$

for all $\xi \in U$.

Theorem 2.1. *Suppose that* $n \ge 2$, $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$, $0 \le \alpha < 1$, $p_1 > 0$, $p_j > 0$, $\beta_j + \gamma_j \le 1$, $\beta_j \in [0, 1]$, $\gamma_j \in [0, \frac{1}{bp_j}]$ $(j = 2, \cdots, n)$*, where* $b = b(p_1)$ *is defined by* (2.1)*, and* D_p *is defined by* (1.1)*. The operator* $\Phi_{n,\beta_2,\gamma_2,\cdots,\beta_n,\gamma_n}(f)$ *is defined by* (1.4)*. Then* $\Phi_{n,\beta_2,\gamma_2,\cdots,\beta_n,\gamma_n}(f) \in A\widehat{S}_\alpha(D_p, \beta)$ *if and only if* $f \in A\widehat{S}_\alpha(U, \beta)$.

Proof. Firstly, we prove that $\Phi_{n,\beta_2,\gamma_2,\cdots,\beta_n,\gamma_n}(f) \in A\widehat{S}_\alpha(D_p, \beta)$ when $f \in A\widehat{S}_\alpha(U, \beta)$.

Let $F(z) = \Phi_{n,\beta_2,\gamma_2,\cdots,\beta_n,\gamma_n}(f)(z)$. Suppose that $\rho(z)$ is the $Minkowski$ functional of D_p. From (1.2), we may obtain that $\rho(z)$ is a C^1 function except for a lower dimensional manifold Ω_0 in $\overline{D_p}$. By computing the Fr$\acute{e}$chet derivative of $F(z)$ directly, we have

$$DF(z) = \begin{pmatrix} f'(z_1) & 0 & \cdots & 0 \\ a_2 & \left(\frac{f(z_1)}{z_1}\right)^{\beta_2}(f'(z_1))^{\gamma_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ a_n & 0 & \cdots & \left(\frac{f(z_1)}{z_1}\right)^{\beta_n}(f'(z_1))^{\gamma_n} \end{pmatrix},$$

where

$$a_j = \left[\beta_j\left(\frac{f'(z_1)}{f(z_1)} - \frac{1}{z_1}\right) + \gamma_j\frac{f''(z_1)}{f'(z_1)}\right]\left(\frac{f(z_1)}{z_1}\right)^{\beta_j}(f'(z_1))^{\gamma_j} z_j, \quad j = 2, \cdots, n.$$

Noting $f \in A\widehat{S}_\alpha(U, \beta)$ with $f(0) = 0, f'(0) = 1$, we have $F(0) = 0, DF(0) = I$ and $\det DF(z) \neq 0$, where I is the unit $n \times n$ matrix. Hence we obtain $F \in N(\Omega)$ and

$$DF(z)^{-1}=\begin{pmatrix} \frac{1}{f'(z_1)} & 0 & \cdots & 0 \\ b_2 & \left(\frac{f(z_1)}{z_1}\right)^{-\beta_2}(f'(z_1))\Big)^{-\gamma_2} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ b_n & 0 & \cdots & \left(\frac{f(z_1)}{z_1}\right)^{-\beta_n}(f'(z_1))\Big)^{-\gamma_n} \end{pmatrix},$$

where

$$b_j=\left[\beta_j\left(\frac{1}{z_1f'(z_1)}-\frac{1}{f(z_1)}\right)-\gamma_j\frac{f''(z_1)}{(f'(z_1))^2}\right]z_j, \quad j=2,\cdots,n.$$

Therefore we get

$$DF(z)^{-1}F(z)=\begin{pmatrix} \frac{f(z_1)}{f'(z_1)} \\ \left(1-\beta_2+\beta_2\frac{f(z_1)}{z_1f'(z_1)}-\gamma_2\frac{f''(z_1)f(z_1)}{(f'(z_1))^2}\right)z_2 \\ \vdots \\ \left(1-\beta_n+\beta_n\frac{f(z_1)}{z_1f'(z_1)}-\gamma_n\frac{f''(z_1)f(z_1)}{(f'(z_1))^2}\right)z_n \end{pmatrix}.$$

It follows that

$$\begin{aligned} &\left\langle DF(z)^{-1}F(z),\overline{\frac{\partial\rho(z)}{\partial z}}\right\rangle \\ &=\frac{f(z_1)}{z_1f'(z_1)}\frac{\partial\rho(z)}{\partial z_1}z_1+\sum_{j=2}^{n}\left[1-\beta_j+\beta_j\frac{f(z_1)}{z_1f'(z_1)}-\gamma_j\frac{f''(z_1)f(z_1)}{(f'(z_1))^2}\right]\frac{\partial\rho(z)}{\partial z_j}z_j. \end{aligned} \tag{2.8}$$

Since $f\in A\widehat{S}_\alpha(U,\beta)$, we let

$$p(z_1)=\frac{e^{-i\beta}\frac{f(z_1)}{z_1f'(z_1)}+i\sin\beta-\alpha\cos\beta}{(1-\alpha)\cos\beta},$$

then $p(z_1)$ is analytic on U such that $\mathrm{Re}\,p(z_1)>0$ for $z_1\in U$ with $p(0)=1$ and

$$\begin{aligned} &e^{-i\beta}\left[1-\frac{f(z_1)f''(z_1)}{(f'(z_1))^2}\right] \\ &=(1-\alpha)\cos\beta\cdot[p(z_1)+z_1p'(z_1)]+\alpha\cos\beta-i\sin\beta. \end{aligned} \tag{2.9}$$

From (2.5), (2.8), (2.9), and Lemma 2.2(2), noting the fact that $0 \le \beta_j \le 1, 0 \le \gamma_j \le 1-\beta_j, j = 2, \cdots, n$, we obtain

$$\begin{aligned}
&\mathrm{Re}\left[2e^{-i\beta}\left\langle DF(z)^{-1}F(z), \overline{\frac{\partial\rho(z)}{\partial z}}\right\rangle\right] \\
&= \left[2\sum_{j=1}^{n}\frac{\partial\rho(z)}{\partial z_j}z_j\right]\alpha\cos\beta + 2(1-\alpha)\cos\beta\sum_{j=2}^{n}(1-\beta_j-\gamma_j)\frac{\partial\rho(z)}{\partial z_j}z_j \\
&\quad +2(1-\alpha)\cos\beta\mathrm{Re}[p(z_1)]\sum_{j=2}^{n}\beta_j\frac{\partial\rho(z)}{\partial z_j}z_j + 2(1-\alpha)\cos\beta\frac{\partial\rho(z)}{\partial z_1}z_1\mathrm{Re}p(z_1) \\
&\quad +2(1-\alpha)\cos\beta\mathrm{Re}[p(z_1)+z_1p'(z_1)]\sum_{j=2}^{n}\gamma_j\frac{\partial\rho(z)}{\partial z_j}z_j \\
&\ge \rho(z)\alpha\cos\beta + 2(1-\alpha)\cos\beta\mathrm{Re}p(z_1) \\
&\quad \cdot\left\{\frac{\partial\rho(z)}{\partial z_1}z_1 + \frac{1-2|z_1|-|z_1|^2}{1-|z_1|^2}\sum_{j=2}^{n}\gamma_j\frac{\partial\rho(z)}{\partial z_j}z_j\right\} \\
&\ge \rho(z)\alpha\cos\beta, \ z \in D_p\setminus\Omega_0,
\end{aligned}$$

where Lemma 2.1 is used in the last inequality. Hence $\Phi_{n,\beta_2,\gamma_2,\cdots,\beta_n,\gamma_n}(f) \in A\widehat{S}_\alpha(D_p,\beta)$.

Conversely, if $F(z) = \Phi_{n,\beta_2,\gamma_2,\cdots,\beta_n,\gamma_n}(f) \in A\widehat{S}_\alpha(D_p,\beta)$, then we prove that $f \in A\widehat{S}_\alpha(U,\beta)$.

In fact, let $\widetilde{z} = (z_1, 0, \cdots, 0) \in D_p$ with $z_1 \ne 0$, from (2.4) and (2.8), we obtain

$$\mathrm{Re}\left[e^{-i\beta}\frac{f(z_1)}{z_1f'(z_1)}\right] = \frac{2}{\rho(z)}\mathrm{Re}\left[e^{-i\beta}\left\langle DF(z)^{-1}F(z), \overline{\frac{\partial\rho(z)}{\partial z}}\right\rangle\right]_{z=\widetilde{z}} \ge \alpha\cos\beta$$

for $0 < |z_1| < 1$. This completes the proof. ■

Theorem 2.2. *Suppose that* $n \ge 2$, $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$, $0 \le \alpha < 1$, $p_1 > 0$, $p_j > 0$, $\beta_j + \gamma_j \le 1$, $\beta_j \in [0,1]$, $\gamma_j \in [0, \frac{1}{bp_j}]$ $(j = 2, \cdots, n)$*, where* $b = b(p_1)$ *is defined by* (2.1)*, and* D_p *is defined by* (1.1)*. Let the operator* $\Phi_{n,\beta_2,\gamma_2,\cdots,\beta_n,\gamma_n}(f)$ *be defined by* (1.4)*. Then* $\Phi_{n,\beta_2,\gamma_2,\cdots,\beta_n,\gamma_n}(f) \in \widehat{S}_\alpha(D_p,\beta)$ *if and only if* $f \in \widehat{S}_\alpha(U,\beta)$*.*

Proof. We first prove that $\Phi_{n,\beta_2,\gamma_2,\cdots,\beta_n,\gamma_n}(f) \in \widehat{S}_\alpha(D_p,\beta)$ when $f \in \widehat{S}_\alpha(U,\beta)$.

Let $F(z) = \Phi_{n,\beta_2,\gamma_2,\cdots,\beta_n,\gamma_n}(f)(z)$. Suppose that $\rho(z)$ is the $Minkowski$ functional of D_p. From (1.2), we may obtain that $\rho(z)$ is a C^1 function except for a lower dimensional manifold Ω_0 in $\overline{D_p}$. From the proof of Theorem 2.1, we have $F \in N(\Omega)$.

Case 1. When $\alpha = 0$, noting that $\widehat{S}_0(U,\beta) = A\widehat{S}_0(U,\beta) = \widehat{S}(U,\beta)$, from Theorem 2.1, we obtain that $F \in A\widehat{S}_0(D_p,\beta) = \widehat{S}_0(D_p,\beta)$.

Case 2. When $0 < \alpha < 1$, we set $q(z_1) = 2\alpha(1 - i\tan\beta)\frac{f(z_1)}{z_1 f'(z_1)} - 1 + i2\alpha\tan\beta$, then we have $q \in H(U)$ and $|q(z_1)| < 1$ for $z_1 \in U$ and

$$1 - \frac{f''(z_1)f(z_1)}{(f'(z_1))^2} = \frac{1 - i2\alpha\tan\beta + q(z_1) + z_1 q'(z_1)}{2\alpha(1 - i\tan\beta)}. \tag{2.10}$$

From (2.5), (2.6), (2.10), and Lemma 2.2(1), noting the fact that $|2\alpha - 1| < 1, 0 \leq \beta_j \leq 1, 0 \leq \gamma_j + \beta_j \leq 1, \gamma_j \geq 0, j = 2, \cdots, n$, we obtain

$$\begin{aligned}
&\left|4\alpha(1 - i\tan\beta)\left\langle DF(z)^{-1}F(z), \overline{\frac{\partial\rho(z)}{\partial z}}\right\rangle - \rho(z)(1 - i2\alpha\tan\beta)\right| \\
&= \left|2q(z_1)\frac{\partial\rho(z)}{\partial z_1}z_1 + 2(2\alpha - 1)\sum_{j=2}^{n}(1 - \beta_j - \gamma_j)\frac{\partial\rho(z)}{\partial z_j}z_j \right.\\
&\quad \left. +2q(z_1)\sum_{j=2}^{n}(\beta_j + \gamma_j)\frac{\partial\rho(z)}{\partial z_j}z_j + 2z_1 q'(z_1)\sum_{j=2}^{n}\gamma_j\frac{\partial\rho(z)}{\partial z_j}z_j\right| \\
&\leq 2|q(z_1)|\frac{\partial\rho(z)}{\partial z_1}z_1 + 2\sum_{j=2}^{n}(1 - \beta_j - \gamma_j)\frac{\partial\rho(z)}{\partial z_j}z_j \\
&\quad +2|q(z_1)|\sum_{j=2}^{n}(\beta_j + \gamma_j)\frac{\partial\rho(z)}{\partial z_j}z_j + 2\frac{|z_1|(1 - |q(z_1)|^2)}{1 - |z_1|^2}\sum_{j=2}^{n}\gamma_j\frac{\partial\rho(z)}{\partial z_j}z_j \\
&\leq 2|q(z_1)|\frac{\partial\rho(z)}{\partial z_1}z_1 + 2\sum_{j=2}^{n}(1 - \beta_j - \gamma_j)\frac{\partial\rho(z)}{\partial z_j}z_j \\
&\quad +2|q(z_1)|\sum_{j=2}^{n}(\beta_j + \gamma_j)\frac{\partial\rho(z)}{\partial z_j}z_j + 2\frac{2|z_1|(1 - |q(z_1)|)}{1 - |z_1|^2}\sum_{j=2}^{n}\gamma_j\frac{\partial\rho(z)}{\partial z_j}z_j \\
&= 2\sum_{j=1}^{n}\frac{\partial\rho(z)}{\partial z_j}z_j - 2(1 - |q(z_1)|)\left\{\frac{\partial\rho(z)}{\partial z_1}z_1 + \sum_{j=2}^{n}(\beta_j + \gamma_j)\frac{\partial\rho(z)}{\partial z_j}z_j \right.\\
&\quad \left. -\frac{2|z_1|}{1 - |z_1|^2}\sum_{j=2}^{n}\gamma_j\frac{\partial\rho(z)}{\partial z_j}z_j\right\} \\
&\leq \rho(z) - 2(1 - |q(z_1)|)\left\{\frac{\partial\rho(z)}{\partial z_1}z_1 + \frac{1 - 2|z_1| - |z_1|^2}{1 - |z_1|^2}\sum_{j=2}^{n}\gamma_j\frac{\partial\rho(z)}{\partial z_j}z_j\right\} \\
&\leq \rho(z),\ z \in D_p \setminus \Omega_0,
\end{aligned}$$

where we have used Lemma 2.1 in the last inequality. This implies that $F \in \widehat{S}_\alpha(D_p,\beta)$ for $0 < \alpha < 1$.

Conversely, if $F(z) = \Phi_{n,\beta_2,\gamma_2,\cdots,\beta_n,\gamma_n}(f)(z) \in \widehat{S}_\alpha(D_p, \beta)$, then we prove that $f \in \widehat{S}_\alpha(U, \beta)$.

In fact, letting $\widetilde{z} = (z_1, 0, \cdots, 0) \in D_p$ with $z_1 \neq 0$, from (2.4) and (2.8), we obtain

$$\mathrm{Re}\left[e^{-i\beta}\frac{f(z_1)}{z_1 f'(z_1)}\right] = \frac{2}{\rho(z)}\mathrm{Re}\left[e^{-i\beta}\left\langle DF(z)^{-1}F(z), \overline{\frac{\partial\rho(z)}{\partial z}}\right\rangle\right]_{z=\widetilde{z}} \geq 0$$

for $0 < |z_1| < 1$ and $\alpha = 0$, and

$$\begin{aligned}&\left|2\alpha(1 - i\tan\beta)\frac{f(z_1)}{z_1 f'(z_1)} - 1 + i2\alpha\tan\beta\right| \\ &= \left|\frac{4\alpha(1 - i\tan\beta)}{\rho(z)}\left\langle DF(z)^{-1}F(z), \overline{\frac{\partial\rho(z)}{\partial z}}\right\rangle\Big|_{z=\widetilde{z}} - (1 - i2\alpha\tan\beta)\right| \leq 1\end{aligned}$$

for $0 < |z_1| < 1$ and $0 < \alpha < 1$. This completes the proof.

Remark 2.1. Setting $0 < p_1 \leq 2$ and $p_j \geq 1 (j = 2, \cdots, n)$ in Theorem 2.2, we get Corollary 2.1 and Theorem 3.1 in [15]. From Theorem 11 in [20], we have $\widehat{S}(D_p, \beta) \subset S(D_p)$ for $p_j > 1 (j = 1, 2, \cdots, n)$. Setting $p_1 = 2, p_j \geq 1 (j = 2, \cdots, n)$ in Theorem 2.2, we obtain Theorem 2.1 and Theorem 2.2 in [16] for $0 < \alpha < 1$ and Theorem 2.1 in [17]. The proof of Theorem 2.2 is different from Theorem 2.1 in [16, 17].

Remark 2.2. Suppose that $a = a(p_1)$ is defined by

$$a = a(p_1) = \begin{cases} 1, & \text{if} \quad 0 < p_1 \leq 2, \\ \dfrac{(\sqrt{2}+1)^{p_1} - 1}{(\sqrt{2}+1)p_1}, & \text{if} \quad p_1 > 2, \end{cases} \tag{2.11}$$

Let $p > 2$ and $g(t) = \frac{1-t^p}{1-t^2}$ for $t > 1$. Direct computation yields

$$g'(t) = \frac{-pt^{p-1}(1-t^2) + (1-t^p)2t}{(1-t^2)^2} = \frac{t[2 - pt^{p-2} + (p-2)t^p]}{(1-t^2)^2}.$$

Let $h(t) = 2 - pt^{p-2} + (p-2)t^p$ for $t > 1$. Then we have

$$h'(t) = -p(p-2)t^{p-3}(1-t^2) > 0$$

for $t > 1$. Hence we obtain $h(t) > h(1) = 0$ for $t > 1$. This implies that $g'(t) > 0$

for $t > 1$. Hence we have

$$\begin{aligned} c(p_1) &= \sup_{\sqrt{2}-1\le t\le 1} \frac{(t^2+2t-1)(1-t^{p_1})}{p_1 t^{p_1}(1-t^2)} = \sup_{1\le t\le\sqrt{2}+1} \frac{(1+2t-t^2)(t^{p_1}-1)}{p_1(t^2-1)} \\ &\le \frac{(\sqrt{2}+1)^{p_1}-1}{p_1((\sqrt{2}+1)^2-1)} \sup_{1\le t\le\sqrt{2}+1} (1+2t-t^2) \\ &= \frac{(\sqrt{2}+1)^{p_1}-1}{p_1(2\sqrt{2}+2)} \cdot \sup_{1\le t\le\sqrt{2}+1} (2-(1-t)^2) \\ &= \frac{(\sqrt{2}+1)^{p_1}-1}{p_1(2\sqrt{2}+2)} \cdot 2 = a(p_1) \end{aligned}$$

for $p_1 > 2$. In particular, from (2.11) and (2.1), direct computation yields

$$c(4) = \max_{\sqrt{2}-1\le t\le 1} \frac{(t^2+2t-1)(1+t^2)}{4t^4} = \frac{(t^2+2t-1)(1+t^2)}{4t^4}\Big|_{t=0.59607} \approx 1.469328.$$

Hence we have

$$\frac{1}{a(4)} = 1-\frac{\sqrt{2}}{2} \approx 0.292893, \quad \frac{1}{c(4)} \approx 0.680583.$$

Setting $\beta = 0$ or $\alpha = 0$ in Theorem 2.2, we get a result which improves Theorem 2.3 in [14, 24] for $p_1 > 2$.

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