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ON SOME INEQUALITIES OF HADAMARD'S TYPE AND APPLICATIONS

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Abstract. In this paper, we shall establish some inequalities related to the functions which are studied in $[2, 5, 7, 13]$ and give several applications.

1. Introduction

If $f : [a, b] \to R$ is a convex function, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known as Hadamard inequality $[9]$.

For some interesting results related to Hadamard inequality, see [1-8,10-17].

In $[2]$, Dragomir established the following two theorems which are refinements of the first inequality of (1.1).

Theorem A. *If $f : [a, b] \to R$ is a convex function, and H is defined on $[0, 1]$ by*

$$H(t) = \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx, \tag{1.2}$$

then H is convex, increasing on $[0, 1]$, and for all $t \in [0, 1]$, we have

$$f\left(\frac{a+b}{2}\right) = H(0) \le H(t) \le H(1) = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.3}$$

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Theorem B. *Let f, H be defined as in Theorem A and F be defined on $[0,1]$ by*

$$F(t)=\frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)\,dxdy. \tag{1.4}$$

Then (1) *F is convex on $[0,1]$, symmetric about $\frac{1}{2}$, F is decreasing on $\left[0,\frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2},1\right]$, and for all $t\in[0,1]$,*

$$\sup_{t\in[0,1]} F(t)=F(0)=F(1)=\frac{1}{b-a}\int_a^b f(x)\,dx$$

and

$$\inf_{t\in[0,1]} F(t)=F\left(\frac{1}{2}\right)=\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy;$$

(2) *we have:*

$$f\left(\frac{a+b}{2}\right)\le F\left(\frac{1}{2}\right);\quad H(t)\le F(t),\ \ t\in[0,1]. \tag{1.5}$$

In $[13]$, Yang and Hong established the following theorem which is a refinement of the second inequality of (1.1).

Theorem C. *If $f:[a,b]\to R$ is a convex function, and P is defined on $[0,1]$ by*

$$P(t)=\frac{1}{2(b-a)}\int_a^b\left[f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x\right)\right.$$

$$\left.+f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x\right)\right]dx, \tag{1.6}$$

then P is convex, increasing on $[0,1]$, and for all $t\in[0,1]$, we have

$$\frac{1}{b-a}\int_a^b f(x)\,dx=P(0)\le P(t)\le P(1)=\frac{f(a)+f(b)}{2}. \tag{1.7}$$

In $[7]$, Dragomir, Milosević and Sándor established inequalities related to (1.1).

Theorem D. *Let f and H be defined as in Theorem A. Then:*
(1) *The inequality*

$$f\left(\frac{a+b}{2}\right)\le\frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,dx\le\int_0^1 H(t)\,dt$$

$$\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b f(x)\,dx\right] \tag{1.8}$$

holds;

(2) *If f is differentiable on $[a,b]$, then we have the inequalities*

$$\begin{aligned} 0 &\leq \frac{1}{b-a}\int_a^b f(x)\,dx - H(t) \\ &\leq (1-t)\left[\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right] \end{aligned} \tag{1.9}$$

and

$$0 \leq \frac{f(a)+f(b)}{2} - H(t) \leq \frac{(f'(b)-f'(a))(b-a)}{4} \tag{1.10}$$

for all $t\in[0,1]$.

Theorem E. *Let f, H be defined as in Theorem A and let G be defined on $[0,1]$by*

$$G(t)=\frac{1}{2}\left[f\left(ta+(1-t)\frac{a+b}{2}\right)+f\left(tb+(1-t)\frac{a+b}{2}\right)\right]. \tag{1.11}$$

Then:

(1) *G is convex and increasing on $[0,1]$;*

(2) *We have*

$$\inf_{t\in[0,1]} G(t)=G(0)=f\left(\frac{a+b}{2}\right)$$

and

$$\sup_{t\in[0,1]} G(t)=G(1)=\frac{f(a)+f(b)}{2};$$

(3) *The inequality*

$$H(t)\leq G(t) \tag{1.12}$$

holds for all $t\in[0,1]$;

(4) *The inequality*

$$\begin{aligned} \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,dx &\leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \\ &\leq \int_0^1 G(t)\,dt \leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \end{aligned} \tag{1.13}$$

holds;

(5) *If f is differentiable on $[a,b]$, then we have the inequality*

$$0 \le H(t) - f\left(\frac{a+b}{2}\right) \le G(t) - H(t) \tag{1.14}$$

for all $t \in [0,1]$.

Theorem F. *Let f, H, G be defined as in Theorem D, and let L be defined on $[0,1]$ by*

$$L(t) = \frac{1}{2(b-a)} \int_a^b [f(ta + (1-t)x) + f(tb + (1-t)x)]\, dx.$$

Then:

(1) *L is convex on $[0,1]$.*

(2) *We have the inequality:*

$$G(t) \le L(t) \le \frac{1-t}{b-a} \int_a^b f(x)\, dx + t \cdot \frac{f(a)+f(b)}{2} \le \frac{f(a)+f(b)}{2} \tag{1.15}$$

for all $t \in [0,1]$ and

$$\sup_{t\in[0,1]} L(t) = \frac{f(a)+f(b)}{2}.$$

(3) *One has the inequalities:*

$$H(1-t) \le L(t) \quad \textit{and} \quad \frac{H(t)+H(1-t)}{2} \le L(t)$$

for all $t \in [0,1]$. In this paper, we shall establish some inequalities related to the functions H, F, P, G, L and give several applications.

2. Main Results

In order to prove our main results, we need the following lemmas:

Lemma 1. (see [10]). *Let $f : [a,b] \to R$ be a convex function and let $a \le A \le C \le D \le B \le b$ with $A + B = C + D$. Then*

$$f(C) + f(D) \le f(A) + f(B).$$

The assumptions in Lemma 1 can be weakened as in the following lemma:

Lemma 2. *If* $f:[a,b]\to R$ *is a convex function,* $a\le A\le C\le B\le b$ *and* $a\le A\le D\le B\le b$ *with* $A+B=C+D$*, then*

$$f(C)+f(D)\le f(A)+f(B).$$

Lemma 3 (see [3]). *Let* X *be a real linear space and* E *be its convex subset. If* $f:E\to R$ *is convex on* E*, then for all* x,y *in* E *the mapping* $Q:[0,1]\to R$ *given by*

$$Q(t)=\frac{1}{2}\left[f(tx+(1-t)y)+f(tx+(1-t)y)\right],$$

is also convex on $[0,1]$*. In addition, we have the inequality*

$$f\left(\frac{x+y}{2}\right)\le Q(t)\le\frac{f(x)+f(y)}{2}$$

for all x,y *in* E *and* $t\in[0,1]$. Now, we are ready to state and prove our results.

Theorem 1. *Let* $f:[a,b]\to R$ *be a convex function and let* H,P *be defined as above. Then we have the following results:*

(1) *The inequality*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\,dx&\le\frac{2}{b-a}\int_{\left[a,\frac{3a+b}{4}\right]\cup\left[\frac{a+3b}{4},b\right]}f(x)\,dx\\&\le\int_0^1P(t)\,dt\le\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\,dx+\frac{f(a)+f(b)}{2}\right]\end{aligned}\tag{2.1}$$

holds.

(2) *The inequalities*

$$L(t)\le P(t)\le\frac{1-t}{b-a}\int_a^b f(x)\,dx+t\cdot\frac{f(a)+f(b)}{2}\le\frac{f(a)+f(b)}{2}\tag{2.2}$$

and

$$0\le P(t)-G(t)\le\frac{f(a)+f(b)}{2}-P(t)\tag{2.3}$$

hold for all $t\in[0,1]$.

(3) *If* f *is differentiable on* $[a,b]$*, then we have the inequalities*

$$\begin{aligned}0&\le t\left[\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\right]\\&\le P(t)-\frac{1}{b-a}\int_a^b f(x)\,dx,\end{aligned}\tag{2.4}$$

$$0 \leq P(t) - f\left(\frac{a+b}{2}\right) \leq \frac{(f'(b) - f'(a))(b-a)}{4} \tag{2.5}$$

and

$$0 \leq P(t) - H(t) \leq \frac{(f'(b) - f'(a))(b-a)}{4} \tag{2.6}$$

for all $t \in [0,1]$.

Proof. (1) Using simple techniques of integration, we have the following identities

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(x)\,dx \\ &= \frac{2}{b-a}\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}} [f(x) + f(a+b-x)]\,dtdx, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\frac{2}{b-a}\int_{\left[a,\frac{3a+b}{4}\right]\cup\left[\frac{a+3b}{4},b\right]} f(x)\,dx \\ &= \frac{2}{b-a}\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}} \left[f\left(\frac{a+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right)\right] dtdx, \end{aligned} \tag{2.8}$$

$$\begin{aligned} &\int_0^1 P(t)\,dt \\ &= \frac{1}{b-a}\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}} [f(tx + (1-t)a) + f(ta + (1-t)x)]\,dtdx \\ &+ \frac{1}{b-a}\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}} [f(tb + (1-t)(a+b-x)) \\ &+ f(t(a+b-x) + (1-t)b)]\,dtdx \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} &\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + \frac{1}{b-a}\int_a^b f(x)\,dx\right] \\ &= \frac{1}{b-a}\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}} [f(a) + f(x)]\,dtdx \\ &+ \frac{1}{b-a}\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}} [f(a+b-x) + f(b)]\,dtdx. \end{aligned} \tag{2.10}$$

By Lemma 1, the following inequalities hold for all $t \in \left[0, \frac{1}{2}\right]$ and $x \in \left[a, \frac{a+b}{2}\right]$.

$$f(x) + f(a+b-x) \le f\left(\frac{a+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right) \tag{2.11}$$

holds when $A = \frac{a+x}{2}, C = x, D = a+b-x$ and $B = \frac{a+2b-x}{2}$ in Lemma 1.

$$f\left(\frac{a+x}{2}\right) \le \frac{1}{2}\left[f(tx + (1-t)a) + f(ta + (1-t)x)\right] \tag{2.12}$$

holds when $A = tx + (1-t)a, C = D = \frac{a+x}{2}$ and $B = ta + (1-t)x$ in Lemma 1.

$$\begin{aligned} &f\left(\frac{a+2b-x}{2}\right) \\ &\le \frac{1}{2}\left[f(tb + (1-t)(a+b-x)) + f(t(a+b-x) + (1-t)b)\right] \end{aligned} \tag{2.13}$$

holds when $A = tb + (1-t)(a+b-x), C = D = \frac{a+2b-x}{2}$ and $B = t(a+b-x) + (1-t)b$ in Lemma 1.

$$\frac{1}{2}\left[f(tx + (1-t)a) + f(ta + (1-t)x)\right] \le \frac{f(a) + f(x)}{2} \tag{2.14}$$

holds when $A = a, C = tx + (1-t)a, D = ta + (1-t)x$ and $B = x$ in Lemma 1.

$$\begin{aligned} &\frac{1}{2}\left[f(tb + (1-t)(a+b-x)) + f(t(a+b-x) + (1-t)b)\right] \\ &\le \frac{f(a+b-x) + f(b)}{2} \end{aligned} \tag{2.15}$$

holds when $A = a+b-x, C = tb + (1-t)(a+b-x), D = t(a+b-x) + (1-t)b$ and $B = b$ in Lemma 1. Integrating the inequalities $(2.11) - (2.15)$ over t on $\left[0, \frac{1}{2}\right]$, over x on $\left[a, \frac{a+b}{2}\right]$, dividing both sides by $\frac{b-a}{2}$ and using identities $(2.7) - (2.10)$, we derive (2.1).

(2) Using substitution rules for integration, we have the following identities

$$\begin{aligned} P(t) = \frac{1}{b-a}\int_a^{\frac{a+b}{2}} &\left[f(ta + (1-t)x)\right. \\ &\left. + f(tb + (1-t)(a+b-x))\right] dx \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} L(t) = \frac{1}{2}P(t) + \frac{1}{2(b-a)}\int_a^{\frac{a+b}{2}} &\left[f(ta + (1-t)(a+b-x))\right. \\ &\left. + f(tb + (1-t)x)\right] dx \end{aligned} \tag{2.17}$$

for all $t \in [0,1]$.

If we choose $A = ta + (1-t)x, C = ta + (1-t)(a+b-x), D = tb + (1-t)x$ and $B = tb + (1-t)(a+b-x)$ in Lemma 2, then the inequality

$$\begin{aligned} &f(ta + (1-t)(a+b-x)) + f(tb + (1-t)x) \\ &\leq f(ta + (1-t)x) + f(tb + (1-t)(a+b-x)) \end{aligned} \tag{2.18}$$

holds for all $t \in [0,1]$ and $x \in \left[a, \frac{a+b}{2}\right]$. Integrating the inequality (2.18) over x on $\left[a, \frac{a+b}{2}\right]$, dividing both sides by $2(b-a)$ and using $(2.16)-(2.17)$, we derive the first inequality of (2.2). The second and third inequalities of (2.2) can be obtained by the convexity of f and (1.1). This proves (2.2).

Again, using substitution rules for integration, we have the following identity

$$\begin{aligned} P(t) = \frac{1}{b-a} \int_a^{\frac{3a+b}{4}} & [f(ta + (1-t)x) \\ & + f\left(ta + (1-t)\left(\frac{3a+b}{2} - x\right)\right) \\ & + f\left(tb + (1-t)\left(\frac{b-a}{2} + x\right)\right) \\ & + f(tb + (1-t)(a+b-x))] \, dx \end{aligned} \tag{2.19}$$

for all $t \in [0,1]$. By Lemma 1, the following inequalities hold for all $t \in [0,1]$ and $x \in \left[a, \frac{3a+b}{4}\right]$.

$$\begin{aligned} &f(ta + (1-t)x) + f\left(ta + (1-t)\left(\frac{3a+b}{2} - x\right)\right) \\ &\leq f(a) + f\left(ta + (1-t)\frac{a+b}{2}\right) \end{aligned} \tag{2.20}$$

holds when $A = a, C = ta + (1-t)x, D = ta + (1-t)\left(\frac{3a+b}{2} - x\right)$ and $B = ta + (1-t)\frac{a+b}{2}$ in Lemma 1.

$$\begin{aligned} &f\left(tb + (1-t)\left(\frac{b-a}{2} + x\right)\right) + f(tb + (1-t)(a+b-x)) \\ &\leq f\left(tb + (1-t)\frac{a+b}{2}\right) + f(b) \end{aligned} \tag{2.21}$$

holds when $A = tb + (1-t)\frac{a+b}{2}, C = tb + (1-t)\left(\frac{b-a}{2} + x\right), D = tb + (1-t)$ $(a+b-x)$ and $B = b$ in Lemma 1. Integrating the inequalities (2.20) and (2.21)

over x on $\left[a, \frac{3a+b}{4}\right]$, dividing both sides by $(b-a)$ and using identity (2.19), we have

$$P(t) \leq \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+G(t)\right] \tag{2.22}$$

for all $t \in [0,1]$. Using (2.22), we derive the second inequality of (2.3). The first inequality of (2.3) can be obtained from (1.15) and (2.2). This proves (2.3).

(3) By integration by parts, we have

$$\begin{aligned} &\frac{1}{b-a}\int_a^{\frac{a+b}{2}}\left[(a-x)f'(x)+(x-a)f'(a+b-x)\right]dx \\ &= \frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right). \end{aligned} \tag{2.23}$$

Using substitution rules for integration, we have the following identity

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\int_a^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)\right]dx. \tag{2.24}$$

Now, using the convexity of f, the inequalities

$$f(ta+(1-t)x)-f(x) \geq t(a-x)f'(x)$$

and

$$f(tb+(1-t)(a+b-x))-f(a+b-x) \geq t(x-a)f'(a+b-x)$$

hold for all $t \in [0,1]$ and $x \in \left[a, \frac{a+b}{2}\right]$. Integrating the above inequalities over x on $\left[a, \frac{a+b}{2}\right]$, dividing both sides by $(b-a)$ and using (2.16), (2.23), (2.24) and (1.1), we derive (2.4).

On the other hand, we have

$$\frac{f(a)-f\left(\frac{a+b}{2}\right)}{2} \leq \frac{1}{2}\left(a-\frac{a+b}{2}\right)f'(a) = \frac{a-b}{4}f'(a)$$

and

$$\frac{f(b)-f\left(\frac{a+b}{2}\right)}{2} \leq \frac{1}{2}\left(b-\frac{a+b}{2}\right)f'(b) = \frac{b-a}{4}f'(b)$$

and taking their sum we obtain:

$$\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right) \leq \frac{(f'(b)-f'(a))(b-a)}{4}. \tag{2.25}$$

Finally, (2.5) and (2.6) follow from (1.1), (1.3), $(1.7)$and (2.25). This completes the proof.

Remark 1. In Theorem 1, the inequality (2.1) gives a new refinement of the inequality (1.1).

Remark 2. In Theorem 1, the inequality (2.2) refines the inequality (1.15).

In the next theorem, we shall point out some inequalities for the functions H, P, G, L, Q considered above:

Theorem 2. *Let $f:[a,b]\to R$ be a convex function and let P, G, L be defined as above. In Lemma 3, let $E=[a,b]$, $x=a, y=b$ and let Q be defined on $[0,1]$ by*

$$Q(t)=\frac{1}{2}\left[f(ta+(1-t)b)+f(tb+(1-t)a)\right].$$

Then we have the following results:

(1) *Q is symmetric about $\frac{1}{2}$, Q is decreasing on $\left[0,\frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2},1\right]$,*

(2.26) $$G(2t)\le Q(t)\quad \left(t\in\left[0,\frac{1}{4}\right]\right),$$

(2.27) $$G(2t)\ge Q(t)\left(t\in\left[\frac{1}{4},\frac{1}{2}\right]\right),$$

(2.28) $$G(2(1-t))\ge Q(t)\quad \left(t\in\left[\frac{1}{2},\frac{3}{4}\right]\right),$$

(2.29) $$G(2(1-t))\le Q(t)\left(t\in\left[\frac{3}{4},1\right]\right)$$

and

(2.30) $$\int_{I_1\cup I_2} f(x)\,dx\le\int_{I_3\cup I_4} f(x)\,dx$$

where $0\le\alpha<\beta\le\frac{1}{4}$,

$$\begin{aligned}
I_1 &= \left[2\beta a+(1-2\beta)\frac{a+b}{2},2\alpha a+(1-2\alpha)\frac{a+b}{2}\right],\\
I_2 &= \left[2\alpha b+(1-2\alpha)\frac{a+b}{2},2\beta b+(1-2\beta)\frac{a+b}{2}\right],\\
I_3 &= [(1-\alpha)a+\alpha b,(1-\beta)a+\beta b]\\
I_4 &= [\beta a+(1-\beta)b,\alpha a+(1-\alpha)b].
\end{aligned}$$

(2) *The inequalities*

(2.31) $$H(t)\le Q(t)\le\frac{f(a)+f(b)}{2}\quad\left(t\in\left[0,\frac{1}{3}\right]\right)$$

and

$$f\left(\frac{a+b}{2}\right) \le Q(t) \le P(t) \quad \left(t \in \left[\frac{1}{3}, 1\right]\right) \tag{2.32}$$

hold.

(3) *The inequality*

$$0 \le L(t) - G(t) \le \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + Q(t)\right] - L(t) \tag{2.33}$$

holds for all $t \in [0,1]$.

Proof. (1) The convexity of Q follows directly from that of f. It is obvious that Q is symmetric about $\frac{1}{2}$.

By Lemma 1, the following inequalities hold for all $0 \le t_1 < t_2 \le \frac{1}{2} \le t_3 < t_4 \le 1$.

$$\begin{aligned} & f(t_2 b + (1-t_2)a) + f(t_2 a + (1-t_2)b) \\ \le\ & f(t_1 b + (1-t_1)a) + f(t_1 a + (1-t_1)b) \end{aligned}$$

holds when $A = t_1 b + (1-t_1)a, C = t_2 b + (1-t_2)a, D = t_2 a + (1-t_2)b$ and $B = t_1 a + (1-t_1)b$ in Lemma 1.

$$\begin{aligned} & f(t_3 a + (1-t_3)b) + f(t_3 b + (1-t_3)a) \\ \le\ & f(t_4 a + (1-t_4)b) + f(t_4 b + (1-t_4)a) \end{aligned}$$

holds when $A = t_4 a + (1-t_4)b, C = t_3 a + (1-t_3)b, D = t_3 b + (1-t_3)a$ and $B = t_4 b + (1-t_4)a$ in Lemma 1. Thus, Q is decreasing on $\left[0, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, 1\right]$.

Next, we shall discuss the following two cases.

Case 1. $t \in \left[0, \frac{1}{4}\right]$.

If we choose $A = tb + (1-t)a, C = 2ta + (1-2t)\frac{a+b}{2}, D = 2tb + (1-2t)\frac{a+b}{2}$ and $B = ta + (1-t)b$ in Lemma 1, then the inequality

$$\begin{aligned} & f\left(2ta + (1-2t)\frac{a+b}{2}\right) + f\left(2tb + (1-2t)\frac{a+b}{2}\right) \\ \le\ & f(tb + (1-t)a) + f(ta + (1-t)b) \end{aligned}$$

holds for all $t \in \left[0, \frac{1}{4}\right]$, which is equivalent to the inequality (2.26).

Case 2. $t \in \left[\frac{1}{4}, \frac{1}{2}\right]$.

If we choose $A = 2ta + (1-2t)\frac{a+b}{2}, C = tb + (1-t)a, D = ta + (1-t)b$ and $B = 2tb + (1-2t)\frac{a+b}{2}$ in Lemma 1, then the inequality

$$\begin{aligned} & f(tb + (1-t)a) + f(ta + (1-t)b) \\ & \le f\left(2ta + (1-2t)\frac{a+b}{2}\right) + f\left(2tb + (1-2t)\frac{a+b}{2}\right) \end{aligned}$$

holds for all $t \in \left[\frac{1}{4}, \frac{1}{2}\right]$, which is equivalent to the inequality (2.27).

Using the symmetricity of Q, (2.28) and (2.29) follow from (2.27) and (2.26), respectively.

Next, let $0 \le \alpha < \beta \le \frac{1}{4}$. Using substitution rules for integration, we have the following identities

$$\begin{aligned} & \int_\alpha^\beta \left[f\left(2ta + (1-2t)\frac{a+b}{2}\right) + f\left(2tb + (1-2t)\frac{a+b}{2}\right)\right] dt \\ & = \frac{1}{b-a}\int_{I_1 \cup I_2} f(x)\, dx \end{aligned} \tag{2.34}$$

and

$$\int_\alpha^\beta [f(tb + (1-t)a) + f(ta + (1-t)b)]\, dt = \frac{1}{b-a}\int_{I_3 \cup I_4} f(x)\, dx. \tag{2.35}$$

Integrating the inequality (2.26) over t on $[\alpha, \beta]$, multipling both sides by $2(b-a)$ and using $(2.34) - (2.35)$, we derive (2.30).

(2) We shall discuss the following two cases.

Case 1. $t \in \left[0, \frac{1}{3}\right]$.

Using substitution rules for integration, we have the following identity

$$\begin{aligned} H(t) = \frac{1}{b-a}\int_a^{\frac{a+b}{2}} & \left[f\left(tx + (1-t)\frac{a+b}{2}\right)\right. \\ & \left. + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right)\right] dx. \end{aligned} \tag{2.36}$$

If we choose $A = (1-t)a + tb, C = tx + (1-t)\frac{a+b}{2}, D = t(a+b-x) + (1-t)\frac{a+b}{2}$ and $B = ta + (1-t)b$ in Lemma 1, then the inequality

$$\begin{aligned} & f\left(tx + (1-t)\frac{a+b}{2}\right) + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right) \\ & \le f((1-t)a + tb) + f(ta + (1-t)b) \end{aligned} \tag{2.37}$$

holds for all $t \in \left[0, \frac{1}{3}\right]$ and $x \in \left[a, \frac{a+b}{2}\right]$. Integrating the inequality (2.37) over x on $\left[a, \frac{a+b}{2}\right]$, dividing both sides by $(b-a)$ and using (2.36), we derive the frist inequality of (2.31). From $\sup\limits_{t \in \left[0, \frac{1}{3}\right]} Q(t) = \frac{f(a)+f(b)}{2}$, the second inequality of (2.31) can be obtained. This proves (2.31).

Case 2. $t \in \left[\frac{1}{3}, 1\right]$.

If we choose $A = ta + (1-t)x, C = ta + (1-t)b, D = (1-t)a + tb$ and $B = tb + (1-t)(a+b-x)$ in Lemma 2, then the inequality

$$\begin{aligned} & f(ta + (1-t)b) + f(tb + (1-t)a) \\ \le\; & f(ta + (1-t)x) + f(tb + (1-t)(a+b-x)) \end{aligned} \tag{2.38}$$

holds for all $t \in \left[\frac{1}{3}, 1\right]$ and $x \in \left[a, \frac{a+b}{2}\right]$. Integrating the inequality (2.38) over x on $\left[a, \frac{a+b}{2}\right]$, dividing both sides by $(b-a)$ and using (2.16), we derive the second inequality of (2.32). From $\inf\limits_{t \in \left[\frac{1}{3}, 1\right]} Q(t) = f\left(\frac{a+b}{2}\right)$, the first inequality of (2.32) can be obtained. This proves (2.32).

(3) Using substitution rules for integration, we have the following identity

$$\begin{aligned} & \int_a^{\frac{3a+b}{4}} \left[f(ta + (1-t)x) + f\left(ta + (1-t)\left(\frac{3a+b}{2} - x\right)\right) \right. \\ & + f\left(ta + (1-t)\left(\frac{b-a}{2} + x\right)\right) + f(ta + (1-t)(a+b-x)) \\ & + f(tb + (1-t)x) + f\left(tb + (1-t)\left(\frac{3a+b}{2} - x\right)\right) \\ & \left. + f\left(tb + (1-t)\left(\frac{b-a}{2} + x\right)\right) + f(tb + (1-t)(a+b-x)) \right] dx \\ & = 2(b-a)L(t) \end{aligned} \tag{2.39}$$

for all $t \in [0,1]$.

By Lemma 1, the following inequalities hold for all $t \in [0,1]$ and $x \in \left[a, \frac{3a+b}{4}\right]$.

$$\begin{aligned} & f(ta + (1-t)x) + f\left(ta + (1-t)\left(\frac{3a+b}{2} - x\right)\right) \\ \le\; & f(a) + f\left(ta + (1-t)\frac{a+b}{2}\right) \end{aligned} \tag{2.40}$$

holds when $A = a, C = ta + (1-t)x, D = ta + (1-t)\left(\frac{3a+b}{2} - x\right)$ and $B =$

$ta+(1-t)\frac{a+b}{2}$ in Lemma 1.

$$\begin{aligned} &f\left(ta+(1-t)\left(\frac{b-a}{2}+x\right)\right)+f(ta+(1-t)(a+b-x)) \\ &\le f\left(ta+(1-t)\frac{a+b}{2}\right)+f(ta+(1-t)b) \end{aligned} \tag{2.41}$$

holds when $A=ta+(1-t)\frac{a+b}{2}, C=ta+(1-t)\left(\frac{b-a}{2}+x\right), D=ta+(1-t)$ $(a+b-x)$ and $B=ta+(1-t)b$ in Lemma 1.

$$\begin{aligned} &f(tb+(1-t)x)+f\left(tb+(1-t)\left(\frac{3a+b}{2}-x\right)\right) \\ &\le f(tb+(1-t)a)+f\left(tb+(1-t)\frac{a+b}{2}\right) \end{aligned} \tag{2.42}$$

holds when $A=tb+(1-t)a, C=tb+(1-t)x, D=tb+(1-t)\left(\frac{3a+b}{2}-x\right)$ and $B=tb+(1-t)\frac{a+b}{2}$ in Lemma 1.

$$\begin{aligned} &f\left(tb+(1-t)\left(\frac{b-a}{2}+x\right)\right)+f(tb+(1-t)(a+b-x)) \\ &\le f\left(tb+(1-t)\frac{a+b}{2}\right)+f(b) \end{aligned} \tag{2.43}$$

holds when $A=tb+(1-t)\frac{a+b}{2}, C=tb+(1-t)\left(\frac{b-a}{2}+x\right), D=tb+(1-t)$ $(a+b-x)$ and $B=b$ in Lemma 1. Integrating the inequalities $(2.40)-(2.43)$ over x on $\left[a,\frac{3a+b}{4}\right]$, dividing both sides by $(b-a)$ and using identity (2.39), we have

$$2L(t)\le G(t)+\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+Q(t)\right] \tag{2.44}$$

for all $t\in[0,1]$. Using (1.15) and (2.44), we derive (2.33). This completes the proof.

The following corollary holds:

Corollary. *Let f,H,P,G,L,Q be defined as above. Then we have:*
(1) *For all $t\in\left[0,\frac{1}{4}\right]$ one has the inequality*

$$f\left(\frac{a+b}{2}\right)\le H(t)\le H(2t)\le G(2t)\le Q(t)\le\frac{f(a)+f(b)}{2}.$$

(2) *For all $t\in\left[\frac{1}{4},\frac{1}{3}\right]$ one has the inequality*

$$f\left(\frac{a+b}{2}\right)\le H(t)\le Q(t)\le G(2t)\le L(2t)\le P(2t)$$

$$\leq \frac{1-2t}{b-a}\int_a^b f(x)\,dx + 2t\cdot\frac{f(a)+f(b)}{2} \leq \frac{f(a)+f(b)}{2}.$$

(3) *For all* $t\in\left[\frac{1}{3},\frac{1}{2}\right]$ *one has the inequalities*

$$f\left(\frac{a+b}{2}\right) \leq Q(t) \leq G(2t) \leq L(2t) \leq P(2t)$$

$$\leq \frac{1-2t}{b-a}\int_a^b f(x)\,dx + 2t\cdot\frac{f(a)+f(b)}{2} \leq \frac{f(a)+f(b)}{2}$$

and

$$f\left(\frac{a+b}{2}\right) \leq Q(t) \leq P(t) \leq P(2t)$$

$$\leq \frac{1-2t}{b-a}\int_a^b f(x)\,dx + 2t\cdot\frac{f(a)+f(b)}{2} \leq \frac{f(a)+f(b)}{2}.$$

(4) *For all* $t\in\left[\frac{1}{2},\frac{2}{3}\right]$ *one has the inequality*

$$f\left(\frac{a+b}{2}\right) \leq Q(t) \leq G(2(1-t)) \leq L(21-t) \leq P(2(1-t))$$

$$\leq \frac{2t-1}{b-a}\int_a^b f(x)\,dx + 2(1-t)\cdot\frac{f(a)+f(b)}{2} \leq \frac{f(a)+f(b)}{2}.$$

(5) *For all* $t\in\left[\frac{2}{3},\frac{3}{4}\right]$ *one has the inequality*

$$f\left(\frac{a+b}{2}\right) \leq Q(t) \leq G(2(1-t)) \leq G(t) \leq L(t) \leq P(t)$$

$$\leq \frac{1-t}{b-a}\int_a^b f(x)\,dx + t\cdot\frac{f(a)+f(b)}{2} \leq \frac{f(a)+f(b)}{2}.$$

(6) *For all* $t\in\left[\frac{3}{4},1\right]$ *one has the inequality*

$$f\left(\frac{a+b}{2}\right) \leq H(2(1-t)) \leq G(2(1-t)) \leq Q(t) \leq P(t)$$

$$\leq \frac{1-t}{b-a}\int_a^b f(x)\,dx + t\cdot\frac{f(a)+f(b)}{2} \leq \frac{f(a)+f(b)}{2}.$$

The proof follows by Theorems A, C, E, F, 1 and 2 and we shall omit the details. Now, we shall give a simple proof of the Dragomir's result (see [5]):

Theorem 3. *Let f, F, H, L be defined as above. Then we have the inequality*

$$0 \leq F(t) - H(t) \leq L(1-t) - F(t) \tag{2.45}$$

for all $t \in [0,1]$.

Proof. Using substitution rules for integration, we have the following identity

$$\begin{aligned} F(t) = \frac{1}{(b-a)^2} \int_a^b \int_a^{\frac{3a+b}{4}} & \left[f(tx + (1-t)y) \right. \\ & + f\left(tx + (1-t)\left(\frac{3a+b}{2} - y\right)\right) \\ & + f\left(tx + (1-t)\left(\frac{b-a}{2} + y\right)\right) \\ & \left. + f(tx + (1-t)(a+b-y)) \right] dydx \end{aligned} \tag{2.46}$$

for all $t \in [0,1]$.

By Lemma 1, the following inequalities hold for all $t \in [0,1]$, $x \in [a,b]$ and $y \in \left[a, \frac{3a+b}{4}\right]$.

$$\begin{aligned} & f(tx + (1-t)y) + f\left(tx + (1-t)\left(\frac{3a+b}{2} - y\right)\right) \\ & \leq f(tx + (1-t)a) + f\left(tx + (1-t)\frac{a+b}{2}\right) \end{aligned} \tag{2.47}$$

holds when $A = tx + (1-t)a, C = tx + (1-t)y, D = tx + (1-t)\left(\frac{3a+b}{2} - y\right)$ and $B = tx + (1-t)\frac{a+b}{2}$ in Lemma 1.

$$\begin{aligned} & f\left(tx + (1-t)\left(\frac{b-a}{2} + y\right)\right) + f(tx + (1-t)(a+b-y)) \\ & \leq f\left(tx + (1-t)\frac{a+b}{2}\right) + f(tx + (1-t)b) \end{aligned} \tag{2.48}$$

holds when $A = tx + (1-t)\frac{a+b}{2}, C = tx + (1-t)\left(\frac{b-a}{2} + y\right), D = tx + (1-t)$ $(a+b-y)$ and $B = tx + (1-t)b$ in Lemma 1. Integrating the inequalities $(2.47)-(2.48)$ over x on $[a,b]$, over y on $\left[a, \frac{3a+b}{4}\right]$, dividing both sides by $(b-a)^2$ and using identity (2.46), we have

$$F(t) \leq \frac{1}{2}(L(1-t) + H(t)) \tag{2.49}$$

for all $t \in [0,1]$. Using (1.5) and (2.49), we derive (2.45). This completes the proof.

3. Applications

1. Let $p \geq 1$ and $0 < a < b$, then the inequalities

$$\begin{aligned}
&\frac{(ta+(1-t)\,b)^{p+1}+b^{p+1}-((1-t)\,a+tb)^{p+1}-a^{p+1}}{2\,(1-t)\,(p+1)\,(b-a)} \\
&\leq \frac{\left(ta+(1-t)\,\frac{a+b}{2}\right)^{p+1}+b^{p+1}-\left(tb+(1-t)\,\frac{a+b}{2}\right)^{p+1}-a^{p+1}}{(1-t)\,(p+1)\,(b-a)} \\
&\leq \frac{(1-t)\left(b^{p+1}-a^{p+1}\right)}{(p+1)\,(b-a)}+\frac{t\,(a^p+b^p)}{2} \\
&\leq \frac{a^p+b^p}{2},
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
0 &\leq \frac{\left(ta+(1-t)\,\frac{a+b}{2}\right)^{p+1}+b^{p+1}-\left(tb+(1-t)\,\frac{a+b}{2}\right)^{p+1}-a^{p+1}}{(1-t)\,(p+1)\,(b-a)} \\
&\quad-\frac{\left(ta+(1-t)\,\frac{a+b}{2}\right)^{p}+\left(tb+(1-t)\,\frac{a+b}{2}\right)^{p}}{2} \\
&\leq \frac{a^p+b^p}{2} \\
&\quad-\frac{\left(ta+(1-t)\,\frac{a+b}{2}\right)^{p+1}+b^{p+1}-\left(tb+(1-t)\,\frac{a+b}{2}\right)^{p+1}-a^{p+1}}{(1-t)\,(p+1)\,(b-a)},
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
0 &\leq t\left[\frac{b^{p+1}-a^{p+1}}{(p+1)\,(b-a)}-\left(\frac{a+b}{2}\right)^p\right] \\
&\leq \frac{\left(ta+(1-t)\,\frac{a+b}{2}\right)^{p+1}+b^{p+1}-\left(tb+(1-t)\,\frac{a+b}{2}\right)^{p+1}-a^{p+1}}{(1-t)\,(p+1)\,(b-a)} \\
&\quad-\frac{b^{p+1}-a^{p+1}}{(p+1)\,(b-a)}
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
0 &\leq \frac{(ta+(1-t)\,b)^{p+1}+b^{p+1}-((1-t)\,a+tb)^{p+1}-a^{p+1}}{2\,(1-t)\,(p+1)\,(b-a)} \\
&\quad-\frac{\left(ta+(1-t)\,\frac{a+b}{2}\right)^{p}+\left(tb+(1-t)\,\frac{a+b}{2}\right)^{p}}{2} \\
&\leq \frac{a^p+b^p+(ta+(1-t)\,b)^p+(tb+(1-t)\,a)^p}{4} \\
&\quad-\frac{(ta+(1-t)\,b)^{p+1}+b^{p+1}-((1-t)\,a+tb)^{p+1}-a^{p+1}}{2\,(1-t)\,(p+1)\,(b-a)}
\end{aligned} \tag{3.4}$$

hold for all $t \in [0, 1)$.

The proof of $(3.1)-(3.4)$ follows from the inequalities $(2.2)-(2.4)$ and (2.33) applied to $f(x) = x^p$ $(x > 0,\ p \geq 1)$.

2. Let $0 < a < b$, then the inequalities

$$
\begin{aligned}
&\frac{1}{2(1-t)(b-a)} \ln\left(\frac{b(ta+(1-t)b)}{a((1-t)a+tb)}\right) \\
&\leq \frac{1}{(1-t)(b-a)} \ln\left(\frac{b\left(ta+(1-t)\frac{a+b}{2}\right)}{a\left(tb+(1-t)\frac{a+b}{2}\right)}\right) \\
&\leq \frac{(1-t)(\ln b - \ln a)}{(b-a)} + \frac{t(a+b)}{2ab} \\
&\leq \frac{a+b}{2ab},
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
0 \leq{}& \frac{1}{(1-t)(b-a)} \ln\left(\frac{b\left(ta+(1-t)\frac{a+b}{2}\right)}{a\left(tb+(1-t)\frac{a+b}{2}\right)}\right) \\
&- \frac{a+b}{2\left(ta+(1-t)\frac{a+b}{2}\right)\left(tb+(1-t)\frac{a+b}{2}\right)} \\
\leq{}& \frac{a+b}{2ab} - \frac{1}{(1-t)(b-a)} \ln\left(\frac{b\left(ta+(1-t)\frac{a+b}{2}\right)}{a\left(tb+(1-t)\frac{a+b}{2}\right)}\right),
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
0 &\leq t\left[\frac{(\ln b - \ln a)}{(b-a)} - \frac{2}{a+b}\right] \\
&\leq \frac{\ln\left(\frac{b\left(ta+(1-t)\frac{a+b}{2}\right)}{a\left(tb+(1-t)\frac{a+b}{2}\right)}\right)}{(1-t)(b-a)} - \frac{(\ln b - \ln a)}{(b-a)}
\end{aligned} \tag{3.7}
$$

and

$$
\begin{aligned}
0 \leq{}& \frac{1}{2(1-t)(b-a)} \ln\left(\frac{b(ta+(1-t)b)}{a((1-t)a+tb)}\right) \\
&- \frac{a+b}{2\left(ta+(1-t)\frac{a+b}{2}\right)\left(tb+(1-t)\frac{a+b}{2}\right)} \\
\leq{}& \frac{a+b}{4ab} + \frac{a+b}{4(ta+(1-t)b)(tb+(1-t)a)} \\
&- \frac{1}{2(1-t)(b-a)} \ln\left(\frac{b(ta+(1-t)b)}{a((1-t)a+tb)}\right)
\end{aligned} \tag{3.8}
$$

hold for all $t \in [0, 1)$.

The proof of $(3.5)-(3.8)$ follows from the inequalities $(2.2)-(2.4)$ and (2.33) applied to $f(x)=\frac{1}{x}$ $(x>0)$.

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