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MODULES WHOSE EC-CLOSED SUBMODULES ARE DIRECT SUMMAND

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Abstract. A module M is called ECS if every ec-closed submodule of M is a direct summand. It was shown that the ECS property lies strictly between CS and P-extending properties. We studied modules M such that every homomorphism from an ec-closed submodule of M to M can be lifted to M. Although such modules share some of the properties of ECS-modules, it is shown that they form a substantially bigger class of modules.

0. Introduction

Throughout this paper, all rings are associative with unity and R denotes such a ring. All modules are unital right R-modules. Recall that a module is said to be *extending* or *CS* if every complement (or closed) submodule of M is a direct summand (see [4]). By an *ec-closed* submodule N of a module M, we mean a closed submodule N which contains essentially a cyclic submodule i.e., there exists $x \in N$ such that xR is essential in N (see [8]). Note that every direct summand of an ec-closed submodule of M is ec-closed. Following [8], a module M is said to be *principally extending* (for short *P-extending*) if every cyclic submodule of M is essential in a direct summand.

Let M_1 and M_2 be modules. The module M_2 is M_1-*c-injective* (M_1-*cu-injective*) if every homomorphism $\alpha : K \to M_2$, where K is a closed (closed uniform) submodule of M_1, can be extended to a homomorphism $\beta : M_1 \to M_2$ (see [9] and [10]).

In this paper we are concerned with the study of modules M that every ec-closed submodule is a direct summand. We call such a module as *ECS-module*. Note that clearly CS-modules and (von Neumann) regular rings are ECS-modules.

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In Section 1, we consider connections between the ECS condition and various other conditions. In particular, we give a counterexample which shows that the classes of ECS and P-extending modules are not the same.

Following an idea of [9], in Section 2, we focus on *self-ec-injective* modules, i.e., modules M such that every homomorphism from ec-closed submodules to the module can be extended to the module itself. ECS-modules are an example of modules with this property. We prove general properties of self-ec-injective modules and provide an equivalent condition when M_2 is M_1-ec-injective for modules M_1 and M_2.

Let R be a ring and M a right R-module. If $X \subseteq M$ then $X \leq M$ denotes X is a submodule of M. Moreover $End(M)$ and $M_n(R)$ symbolize the ring of endomorphism of M and the full ring of n-by-n matrices over R, respectively. Other terminology and notation can be found in [1] and [4].

1. Preliminary Results

In this section, we study relationships between the extending condition, ECS and P-extending conditions.

Proposition 1.1. *Let M be a module. Consider the following statements.*

(i) *M is CS*

(ii) *M is ECS*

(iii) *M is P-extending.*

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$. *In general, the converses to these implications do not hold.*

Proof. $(i) \Rightarrow (ii)$. This implication is clear.

$(ii) \Rightarrow (iii)$. Let mR be any cyclic submodule of M. Then the closure of mR in M, L say, is an ec-closed. By hypothesis, L is a direct summand of M. Thus M is P-extending. Let $M_2(R)$ be the ring as in [7, Example 13.8]. Then $M_2(R)$ is a von Neumann regular ring which is not a Baer ring. Hence it is neither left nor right CS, by [2, Example 2.7]. Thus $(ii) \not\Rightarrow (i)$. Finally, let R be the ring as in [3, Example 3.2] i.e., $R = \left[\begin{smallmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z} \end{smallmatrix}\right]$. Then R is right P-extending. However, R_R is not CS, by [11]. Since R_R has finite uniform dimension, R_R has a maximal uniform (and hence an ec-closed) submodule which is not a direct summand of R_R. So R is not right ECS-module. ■

Proposition 1.1 shows that classes of modules with CS, ECS and P-extending are different from each other. In [8], authors assumed that ECS and P-extending conditions are the same and proved several results for P-extending modules. However, the proof of Proposition 1.1 provides a counterexample to these assertions by exhibiting a P-extending module R_R which does not satisfy ECS condition. Actually, most of the results in [8] which are stated for P-extending modules are remaining true only for ECS-modules.

Since the ECS property lies strictly between the CS and P-extending properties, it is natural to seek conditions which ensure that a P-extending module is ECS or that a ECS-module is CS. Such conditions are illustrated in our next result.

Proposition 1.2.

(i) *Let M_R be a nonsingular module. Then M is P-extending if and only if M is ECS.*

(ii) *Let M be a right R-module such that a direct sum of an ec-closed with a direct summand of M is a complement in M. Then M is P-extending if and only if M is ECS.*

(iii) *Let M be a module with finite uniform dimension. Then M is CS if and only if M is ECS.*

Proof.

(i) Assume M is right P-extending R-module. Let X be any ec-closed submodule of M. Then xR is essential in X for some $x \in X$. By hypothesis, there exists a direct summand L of M which contains xR as an essential submodule. Since M_R is nonsingular, $X = L$. Thus M is ECS. The converse follows from Proposition 1.1.

(ii) Assume M is P-extending. Let C be an ec-closed submodule of M with cR is essential in C. By hypothesis, there are submodules D, $D^{'}$ of M such that cR is essential in D and $M = D \oplus D^{'}$. It follows that $C \oplus D^{'}$ is essential in M. By hypothesis, $M = C \oplus D'$. Hence M is ECS. The converse follows from Proposition 1.1.

(iii) Assume M is a ECS-module. Let N be any maximal uniform submodule of M. Clearly N is an ec-closed in M. By hypothesis, N is a direct summand of M. Hence M is CS. The converse is clear by Proposition 1.1. ∎

The following definition is needed in our next two results. Let M be a module. Let K, L be two direct summands of M. If $K \cap L$ is also a direct summand of M, then M is said to have *summand intersection property,* SIP (see, for example [12]). Note that the extending version of the following result is appeared in [3].

Theorem 1.3. *Let M be a P-extending module.*

(i) *If M is distributive then every submodule of M is P-extending.*

(*ii*) *If X is a submodule of M such that $e(X) \subseteq X$ for every $e^2 = e \in End(M)$ then X is P-extending.*

(*iii*) *If M has SIP then every direct summand of M is P-extending.*

(*iv*) *If X is a submodule of M such that the intersection of X with any direct summand of M is a direct summand of X, then X is P-extending.*

Proof.

(i) Let X be any submodule of M and xR be a cyclic submodule of X. Then there exists a direct summand D of M such that xR is essential in D. Hence xR is essential in $D \cap X$. Since $X = X \cap (D \oplus D') = (X \cap D) \oplus (X \cap D')$ where D' is a submodule of M, then $X \cap D$ is a direct summand of X. So, X is P-extending.

(ii) Let X be a submodule of M. Let D be any direct summand of M and $e : M \to D$ be the projection with $e(X) \subseteq X$. Then $e(X) = D \cap X$ which is a direct summand of X. By (iv), X is P-extending.

(iii) Let M_1 be a direct summand of M. Then $M = M_1 \oplus M_2$ for some submodule M_2 of M. Let xR be any cyclic submodule of M_1. Then there exists a direct summand D of M such that xR is essential in D. So $M = D \oplus D'$ for some submodule D' of M. Therefore xR is essential in $D \cap M_1$. By SIP, $M = (D \cap M_1) \oplus U$ for some submodule U of M. Since $M_1 = M_1 \cap [(D \cap M_1) \oplus U] = (D \cap M_1) \oplus (M_1 \cap U)$ then M_1 is P-extending.

(iv) Let A be any cyclic submodule of X. Then $A = xR$ for some $x \in X$. Then there exists a direct summand D of M such that A is essential in D. So A is essential in $D \cap X$ and $D \cap X$ is a direct summand of X. Thus X is a P-extending module. ■

By adapting the proof of [5, Lemma 1], we have the following corollary.

Corollary 1.4. *Let R be any ring and M a projective P-extending module which has SIP. Then there exists an index set I such that M is a direct sum $\oplus_{i \in I} M_i$ of submodules M_i $(i \in I)$ of M such that each submodule M_i is ec-closed in M.*

Proof. By Kaplansky's Theorem (see [6, p.120]), the module M is a direct sum of countably generated submodules. By Theorem 1.3 (iii), we may suppose that M is countably generated. There exists a countably set of elements $m_1, m_2, ...$ in M such that $M = \sum_i m_i R$. By hypothesis, there exists submodules M_1, N_1 of M such that $M = M_1 \oplus N_1$ and $m_1 R$ is essential in M_1. Suppose that n_i is the projection of m_i in N_1 for all $i \geq 2$. By Theorem 1.3 (iii) again, there exists a direct summand M_2 of N_1 which contains n_2R as an essential submodule. Continuing in this manner we obtain a direct sum $M_1 \oplus M_2 \oplus M_3 \oplus ...$ of submodules in the module M such that $m_1R + m_2R + ... + m_kR \subseteq M_1 \oplus M_2 \oplus ... \oplus M_k$, for all positive integers k. It follows that $M = \oplus_i M_i$. Moreover, by construction, each submodule M_i is ec-closed in M. ■

2. EC-INJECTIVITY

Motivated by lifting homomorphisms in [9] and [10] for closed uniform submodules and complement submodules respectively, we study lifting property for ec-closed submodules. Let M_1 and M_2 be modules. The module M_2 is M_1-*ec-injective* if every homomorphism $\varphi : K \to M_2$, where K is an ec-closed submodule of M_1, can be extended to a homomorphism $\theta : M_1 \to M_2$ (see [8]). Clearly, if M_2 is M_1-c-injective (or M_1-injective), then M_2 is M_1-ec-injective. A module M is called *self-ec-injective* when it is M-ec-injective. Recall that extending modules can be characterized by the lifting of homomorphisms from certain submodules to the module itself, as was shown in [10]. We begin by mentioning analogous fact about ECS-modules.

Lemma 2.1. *Let M be a module. Then M is ECS if and only if for each ec-closed submodule K of M there exists a complement L of K in M such that every homomorphism $\varphi : K \oplus L \to M$ can be lifted to a homomorphism $\theta : M \to M$.*

Proof. This is a direct consequence of [10, Lemma 2] ■

Lemma 2.2. *Let M be a module and let K be an ec-closed submodule of M. If K is M-ec-injective, then K is a direct summand of M.*

Proof. By hypothesis, there exists a homomorphism $\theta : M \to K$ that extends the identity $i : K \to K$. It is easy to see that $M = K \oplus Ker\theta$, so that K is a direct summand of M. ■

Proposition 2.3. *The following are equivalent for a module M.*

(i) *M is ECS.*

(ii) *Every module is M-ec-injective.*

(iii) *Every ec-closed submodule of M is M-ec-injective.*

Proof. It is clear that (i) implies (ii) and, obviously (ii) implies that (iii). The implication $(iii) \Rightarrow (i)$ follows by Lemma 2.2. ■

In particular, by proposition 2.3, every ECS-module is self-ec-injective. However, our next example shows that not every self-ec-injective module is ECS. Note that this example also shows that the condition in [8] which is supposed to be equivalent to [8, Definition 2.2] is not valid. The cited assumption states that, if $M = M_1 \oplus M_2$, then M_2 is M_1-ec-injective if and only if for every ec-(closed) submodule N of M such that $N \cap M_2 = 0$, there exists $N^{'} \leq M$ such that $N \leq N^{'}$, and $M = N^{'} \oplus M_2$.

Example 2.4. Let p be any prime integer and let R denote the local ring $\mathbb{Z}_p$. Let M denote the $\mathbb{Z}$-module $\mathbb{Q} \oplus (\mathbb{Z}/\mathbb{Z}p)$. Then M is self-ec-injective but not ECS. Moreover M does not have the condition mentioned above.

Proof. Recall that $M_{\mathbb{Z}}$ is not extending, by [10, Example 10]. Since $M_{\mathbb{Z}}$ has finite uniform dimension, $M_{\mathbb{Z}}$ is not ECS-module from Proposition 1.2 and self-ec-injective, by [10]. For the last part, let $M_1 = \mathbb{Q} \oplus 0$ and $M_2 = 0 \oplus \mathbb{Z}/\mathbb{Z}p$. Since M_1, M_2 are uniform modules, M_2 is M_1-ec-injective. Let $N = R(1, 1+\mathbb{Z}p)$. Note that N is an ec-closed submodule of $M = M_1 \oplus M_2$. By [10, Example 10], N is not a direct summand of M and $N \cap M_2 = 0$. Assume there exists $N^{'} \leq M$ such that $N \leq N^{'}$ and $M = N^{'} \oplus M_2$. Since N is a maximal uniform in M, N' has uniform dimension 2, which yields a contradiction. Thus there is no such submodule $N^{'}$. ■

In conjunction with Example 2.4, we provide a condition in our next Theorem which is equivalent to M_2 is M_1-ec-injective. First note that we use π_i to denote the projections from $M = M_1 \oplus M_2$ to M_i for $i = 1, 2$. Compare the following result with [4, Lemma 7.5] and [9, Lemma 2.3].

Theorem 2.5. *Let M_1 and M_2 be modules and let $M = M_1 \oplus M_2$. Then M_2 is M_1-ec-injective if and only if for every ec-closed submodule N of M such that $N \cap M_2 = 0$ and $\pi_1(N)$ is ec-closed in M_1, there exists a submodule $N^{'}$ of M such that $N \leq N^{'}$ and $M = N^{'} \oplus M_2$.*

Proof. Assume that M_2 is M_1-ec-injective and let N be an ec-closed submodule of M such that $N \cap M_2 = 0$ and $\pi_1(N)$ is ec-closed in M_1. As $N \cap M_2 = 0$, the restriction of π_1 to N is an isomorphism between N and $\pi_1(N)$. Let $\alpha : \pi_1(N) \to M_2$ be the homomorphism defined by $\alpha(x) = \pi_2(\pi_1|_N)^{-1}(x)$, where $x \in \pi_1(N)$. The map α can be extended to a homomorphism $\theta : M_1 \to M_2$, since M_2 is M_1-ec-injective and $\pi_1(N)$ is ec-closed in M_1. Define $N^{'} = \{x + \theta(x) : x \in M_1\}$. Clearly, $N^{'}$ is a submodule of M and $M = N^{'} \oplus M_2$. For any $x \in N$, $\theta\pi_1(x) = \alpha\pi_1(x) = \pi_2(x)$ and hence $x = \pi_1(x) + \theta\pi_1(x) \in N'$. Thus $N \leq N^{'}$.

Conversely, suppose that, for every ec-closed submodule N of M such that

$N \cap M_2 = 0$ and $\pi_1(N)$ is ec-closed in M_1, there exists a submodule $N^{'}$ of M such that $N \leq N'$ and $M = N^{'} \oplus M_2$. Let K be an ec-closed submodule of M_1 and let $\alpha : K \to M_2$ be a homomorphism. Let $N = \{x - \alpha(x) : x \in K\}$. It is clear that N is a submodule of M and $N \cap M_2 = 0$. Since $\pi_1(N) = K$, $\pi_1(N)$ is ec-closed in M_1. By hypothesis, there exists a submodule $N^{'}$ of M such that $N \leq N^{'}$ and $M = N^{'} \oplus M_2$. Let $\pi : M \to M_2$ be the projection with kernel N' and let $\theta : M_1 \to M_2$ be the restriction of π to M_1. Now, for any $x \in K$, $\theta(x) = \pi(x) = \pi(x - \alpha(x) + \alpha(x)) = \alpha(x)$. Hence θ extends α. So, M_2 is M_1-ec-injective. ∎

Lemma 2.6. *Let M_1 and M_2 be modules. If M_2 is M_1-ec-injective, then, for every ec-closed submodule N of M_1, M_2 is N-ec-injective and (M_1/N)-ec-injective.*

Proof. Let N be an ec-closed in M_1. Since every ec-closed submodule of N is also an ec-closed submodule of M_1, M_2 is N-ec-injective. Now, let K/N be an ec-closed submodule of M_1/N and let $\alpha : K/N \to M_2$ be a homomorphism. It is easy to see that K is an ec-closed submodule of M_1 (see, [4]). Let $\pi : M_1 \to M_1/N$ and $\pi^{'} : K \to K/N$ be the canonical epimorphisms. Since M_2 is M_1-ec-injective, there exists a homomorphism $\theta : M_1 \to M_2$ that extends $\alpha\pi^{'}$. Now $N \leq Ker\theta$ gives that there exists a homomorphism $\gamma : M_1/N \to M_2$ such that $\gamma\pi = \theta$. For any $a \in K$, $\gamma(a + N) = \gamma\pi(a) = \theta(a) = \alpha\pi^{'}(a) = \alpha(a + N)$. Hence M_2 is (M_1/N)-ec-injective. ∎

Lemma 2.7. *Let M be any self-ec-injective module. Then a direct summand of M is also self-ec-injective.*

Proof. Let L be any direct summand of M. Hence $M = L \oplus L^{'}$ for some submodule $L^{'}$ of M. Let X be an ec-closed in L and $\varphi : X \to L$ be any homomorphism. Since X is an ec-closed in M, then there exists a homomorphism $\theta : M \to M$ such that $\theta|_X = \varphi$. Let $\pi : M \to L$ be the projection. Define $\alpha : L \to L$ by $\alpha(l) = \pi(\theta(l))$, for any $l \in L$. It is clear that $\alpha_{|X} = \varphi$. Hence L is self-ec-injective. ∎

The converse of Lemma 2.7 is not true, in general. Let us consider for example, the $\mathbb{Z}$-modules $M_1 = \mathbb{Z}$ and $M_2 = \mathbb{Z}/\mathbb{Z}p$ for a prime integer p. Then M_1 and M_2 are uniform modules, so that they are self-ec-injective. However, since $M_{\mathbb{Z}}$ has finite uniform dimension, M is not self-ec-injective, because it is not self-c-injective by [9, Corollary 3.5]. However we have the following observation.

Theorem 2.8. *Let $M = M_1 \oplus M_2$ be $\mathbb{Z}$-module where M_1 is torsion and M_2 is infinite cyclic. If M is self-ec-injective then $M_1 = pM_1$ for each prime p.*

Proof. Let $M_2 = \mathbb{Z}m_2$ for some $0 \neq m_2 \in M_2$. Suppose $M_1 \neq pM_1$ for some prime p. Let $m_1 \in M_1,\ m_1 \notin pM_1$. Let $K = \mathbb{Z}(m_1, pm_2)$. Suppose K is essential in L for some $L \leq M$. Then for any $n \in \mathbb{Z}$, $n(m_1, pm_2) = (nm_1, npm_2) = (0, 0)$ implies that $n = 0$. Therefore K is infinite cyclic, and hence K is a uniform $\mathbb{Z}$-module. Let $x \in L$ and $a = (m_1, pm_2)$. Then $K + \mathbb{Z}x = \mathbb{Z}a + \mathbb{Z}x$ is finitely generated, so that $K + \mathbb{Z}x \leq L$, and is a direct sum of cyclic modules. But $K + \mathbb{Z}x$ is uniform, hence $K + \mathbb{Z}x$ is cyclic. Then $\mathbb{Z}a \subseteq K + \mathbb{Z}x = \mathbb{Z}y$ for some $y \in M$. Suppose $y = (m_1^{'}, km_2)$ for some $m_1^{'} \in M_1$ and $k \in \mathbb{Z}$. Then $a = sy$ for some $s \in \mathbb{Z}$. Hence $(m_1, pm_2) = s(m_1^{'}, km_2)$, which gives $m_1 = sm_1^{'}$, $pm_2 = skm_2$. Since M_2 is infinite cyclic, $s = \pm 1$ or $k = \pm 1$. If $k = \pm 1$ then $s = \mp p$, so that $m_1 = \pm pm_1^{'} \in pM_1$, a contradiction. Thus $s = \pm 1$. Therefore $y \in \mathbb{Z}a$ and hence $x \in \mathbb{Z}y \subseteq \mathbb{Z}a$, i.e., $L \subseteq \mathbb{Z}a = K$. Hence $K = L$, so K is a complement in M. Since K is cyclic, K is ec-closed. Now define a homomorphism $\varphi : K \to M$ by $\varphi(m_1, pm_2) = (0, m_2)$. Suppose that φ can be lifted to $\theta : M \to M$. Then $\theta(m_1, 0) = (u, 0)$ for some $u \in M_1$ and $\theta(0, m_2) = (v, tm_2)$ for some $v \in M_1$, $t \in \mathbb{Z}$. Hence $(0, m_2) = \varphi(m_1, pm_2) = \theta(m_1, pm_2) = \theta(m_1, 0) + p\theta(0, m_2) = (u, 0) + p(v, tm_2)$. Then we obtain, $0 = u + pv$, $m_2 = ptm_2$, so that $1 = pt$, a contradiction. Therefore φ cannot be lifted. It follows that $M_1 = pM_1$ for each prime p. ■

We finish this section by showing that there are self-ec-injective modules which are not self-c-injective. For our next result, first recall that the module M_2 is *essentially* M_1*-injective* if every homomorphism $\alpha : A \to M_2$, where A is a submodule of M_1 and $Ker\alpha$ is essential in A, can be extended to a homomorphism, $\beta : M_1 \to M_2$ (see, [9]).

Proposition 2.9. *Let M_1 be an extending module and let M_2 be a uniform module such that M_2 is essentially M_1-injective. Then the following statements are equivalent.*

(i) *$M_1 \oplus M_2$ is self-c-injective.*

(ii) *$M_1 \oplus M_2$ is self-ec-injective.*

(iii) *$M_1 \oplus M_2$ is self-cu-injective.*

Proof. $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are clear. $(iii) \Rightarrow (i)$. By [9, Proposition 2.9]. ■

Corollary 2.10. *Let R be a Prüfer domain which is not a field. Then any non-finitely generated free R-module is self-ec-injective, but not self-c-injective.*

Proof. Let M be a free R-module with infinite basis $\{m_i : i \in I\}$. Let U be an ec-closed submodule of M. If $U = 0$ then nothing to prove. So, assume that

$U \neq 0$. Hence there exists $0 \neq x \in U$ such that xR is essential in U. There exists a finite subset F of I such that $x \in \oplus_{i \in F} m_i R$. Since U/xR is a torsion module, it follows that $U \subseteq \oplus_{i \in F} m_i R$. By [4, Corollary 12.10], U is a direct summand of $\oplus_{i \in F} m_i R$ and hence also of M. Thus M is self-ec-injective. By [9, Theorem 3.1], M is not self-c-injective. ■

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