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ON GLOBAL SOLUTIONS AND BLOW-UP OF SOLUTIONS FOR A NONLINEARLY DAMPED PETROVSKY SYSTEM

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Abstract. We consider the initial boundary value problem for a Petrovsky system with nonlinear damping

$$u_{tt} + \Delta^2 u + a\,|u_t|^{m-2}\,u_t = b\,|u|^{p-2}\,u,$$

in a bounded domain. We showed that the solution is global in time under some conditions without the relation between m and p. We also prove that the local solution blows-up in finite time if $p > m$ and the initial energy is nonngeative. The decay estimates of the energy function and the estimates of the lifespan of solutions are given. In this way, we can extend the result of ([6]).

1. Introduction

In this paper we are concerned with the initial boundary value problem for the following Petrovsky equation :

$$u_{tt} + \Delta^2 u + a\,|u_t|^{m-2}\,u_t = b\,|u|^{p-2}\,u, \tag{1.1}$$

with initial conditions

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \tag{1.2}$$

and boundary condition

$$u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0, x \in \partial\Omega, t \geq 0, \tag{1.3}$$

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where $\Omega \subset R^N$, $N \geq 1$, is a bounded domain with a smooth boundary $\partial\Omega$ so that Divergence theorem can be applied and ν be the unit normal vector pointing toward the exterior of Ω and let $\frac{\partial}{\partial\nu}$ denotes the normal derivative and a, b are some positive constants and $p, m > 2$.

Guesmia $([2])$ studied the problem

$$u_{tt} + \Delta^2 u + q(x)u + g(u_t) = 0, \tag{1.4}$$

where $q : \Omega \to R^+$ is a bounded function. Under some assumptions, he showed the solution of (1.4) decays exponentially if g behaves like a linear function, whereas the decay is ploynomially otherwise. Later, Guesmia $([3])$ concerned equation (1.4) coupled with a semilinear wave equation and derived similar results. In the related problems, we can cite $([4, 5, 8])$ and the references therein. Recently, Messaoudi ([6]) investigated problem (1.1) and showed the solution blows-up in finite time if $p > m$ in the case that the initial energy is negative. On the other hand, he also proved the solution is global in time if $m \geq p$. However, no decay rate of the global solution is given and no blow-up result is discussed for the initial energy being nonnegative.

In this paper we shall prove the global existence result without the relation between m and p and show that the energy function decays algebraically under some conditions. On the other hand, we also establish the blow-up properties of local solution for problem $(1.1) - (1.3)$ with nonpositive initial energy as well as small positive initial energy. In this way, we can extend the result of $([6])$. The content of this paper is organized as follows. In section 2, we give some lemmas and the local existence theorem 2.3 in ([6]). In section 3, we define an energy function $E(t)$ in (3.3) and show that it is a nonincreasing function of t. We obtain global existence and decay properties of the solutions of $(1.1) - (1.3)$ which are given in Theorem 3.5. Finally, the blow-up properties of $(1.1) - (1.3)$ and the estimates for the blow-up time T^* are also given in the last section.

2. Preliminary Results

In this section, we shall give some lemmas which will be used throughout this work.

Lemma 2.1. (Sobolev-Poincaré inequality)([1]) *If* $1 \leq p \leq \frac{2N}{[N-2m]^+}$ $(1 \leq p < \infty$ *if* $N = 2m)$, *then*

$$\|u\|_p \leq B_1 \left\|(-\Delta)^{\frac{m}{2}} u\right\|_2, \quad \text{for } u \in D((-\Delta)^{\frac{m}{2}})$$

holds with some positive constant B_1, *where we put* $[a]^+ = \max\{0, a\}$, $\frac{1}{[a]^+} = \infty$ *if* $[a]^+ = 0$ *and we denote* $\|\cdot\|_p$ *to be the norm of* $L^p(\Omega)$.

Lemma 2.2. $([7])$ *Let $\phi(t)$ be a nonincreasing and nonnegative function on $[0, T]$, $T > 1$, such that*

$$\phi(t)^{1+r} \leq \omega_0 \left(\phi(t) - \phi(t+1)\right) \text{ on } [0, T],$$

where ω_0 is a positive constant and r is a nonnegative constant. Then we have (i) if $r > 0$, then

$$\phi(t) \leq \left(\phi(0)^{-r} + \omega_0^{-1} r[t-1]^+\right)^{-\frac{1}{r}} \text{ on } [0, T].$$

(ii) If $r = 0$, then

$$\phi(t) \leq \phi(0) e^{-\omega_1 [t-1]^+} \text{ on } [0, T],$$

where $\omega_1 = ln(\frac{\omega_0}{\omega_0 - 1})$, here $\omega_0 > 1$.

Next, we state the local existence theorem which is proved in ([6]).

Theorem 2.3. **(Local Existence)** *Suppose that $2 < p \leq p^*$, $2 < m \leq m^*$ and that $u_0 \in H_0^2(\Omega)$ and $u_1 \in L^2(\Omega)$, then there exists a unique solution u of $(1.1) - (1.3)$ satisfying*

$$u \in C([0, T]; H_0^2(\Omega))$$

and

$$u_t \in C([0, T]; L^2(\Omega)) \cap L^m(\Omega \times (0, T)).$$

Moreover, at least one of the following statements holds true :

$$(i) T = \infty,$$
$$(ii) \ \|u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2 \to \infty \text{ as } t \to T^-,$$

where

$$p^* = \frac{2(N-2)}{N-4} (\infty, \text{ if } N \leq 4) \text{ and } m^* = \frac{2N}{N-4} (\infty, \text{ if } N \leq 4).$$

3. Global Existence and Energy Decay

In this section, we consider the global existence and energy decay of solutions for problem $(1.1) - (1.3)$.

Let

$$I(t) \equiv I(u(t)) = \|\Delta u(t)\|_2^2 - b\|u(t)\|_p^p, \tag{3.1}$$

and

$$J(t) \equiv J(u(t)) = \frac{1}{2} \|\Delta u(t)\|_2^2 - \frac{b}{p}\|u(t)\|_p^p, \tag{3.2}$$

for $u(t) \in H_0^2(\Omega),\ t \geq 0$.We define the energy function of the solution u of $(1.1)-(1.3)$ by

$$E(t) = \frac{1}{2}\left\|u_t\right\|_2^2 + J(t) \text{ for } t \geq 0. \tag{3.3}$$

Remark: By (3.3), we have

$$\begin{aligned} E\left(t\right) &= \frac{1}{2}\left\|u_t\right\|_2^2 + \frac{1}{2}\left\|\Delta u(t)\right\|_2^2 - \frac{b}{p}\|u\|_p^p \\ &\geq \frac{1}{2}\|\Delta u(t)\|_2^2 - \frac{b}{p}\|u\|_p^p,\ t \geq 0. \end{aligned} \tag{3.4}$$

By Lemma 2.1, we get

$$E(t) \geq G(\|\Delta u(t)\|_2),\ t \geq 0, \tag{3.5}$$

where

$$G(\lambda) = \frac{1}{2}\lambda^2 - \frac{B_1^p b}{p}\lambda^p,$$

here B_1 is the Sobolev's constant given in Lemma 2.1. Note that $G(\lambda)$ has the maximum at $\lambda_1 = \left(\frac{1}{bB_1^p}\right)^{\frac{1}{p-2}}$ and the maximum value is

$$E_1 = G\left(\lambda_1\right) = b^{-\frac{2}{p-2}}\left(\frac{1}{2} - \frac{1}{p}\right)B_1^{\frac{-2p}{p-2}}. \tag{3.6}$$

Lemma 3.1. *$E(t)$ is a nonincreasing function on $[0, T]$ and*

$$E'(t) = -a\int_\Omega |u_t|^m\, dx. \tag{3.7}$$

Proof. By using Divergence theorem and $(1.1)-(1.3)$, we see that (3.7) follows at once.

Lemma 3.2. *Assume that $E\left(0\right) < E_1$. Then*

(i) If $\left\|\Delta u_0\right\|_2 < \lambda_1$, *then* $\left\|\Delta u\left(t\right)\right\|_2 < \lambda_1$ *for* $t \geq 0$.

(ii) If $\left\|\Delta u_0\right\|_2 > \lambda_1$, *then there exists* $\lambda_2 > \lambda_1$ *such that* $\left\|\Delta u\left(t\right)\right\|_2 \geq \lambda_2$ *for* $t \geq 0$.

Proof. From the definition of $G\left(\lambda\right)$, we see that $G(\lambda)$ is increasing in $(0, \lambda_1)$, decreasing in (λ_1, ∞) and $G(\lambda) \to -\infty$ as $\lambda \to \infty$. Since $E\left(0\right) < E_1$, so there exist λ_2' and λ_2 such that $\lambda_2' < \lambda_1 < \lambda_2$ and $G(\lambda_2') = G(\lambda_2) = E(0)$. (i) when $\|\Delta u_0\|_2 < \lambda_1$, by (3.5), we have

$$G(\|\Delta u_0\|_2) \leq E(0) = G(\lambda_2').$$

It implies $\|\Delta u_0\|_2 < \lambda_2'$.

We claim that $\|\Delta u(t)\|_2 \leq \lambda_2'$ for $t > 0$. If not, then there exists $t_0 > 0$ such that $\|\Delta u(t_0)\|_2 > \lambda_2'$. Case (a) if $\lambda_2' < \|\Delta u(t_0)\|_2 < \lambda_2$, then $G(\|\Delta u(t_0)\|_2) > E(0) \geq E(t_0)$. It contradicts to (3.5). Case (b) if $\|\Delta u(t_0)\|_2 \geq \lambda_2$, then by contiunity of $\|\Delta u(t)\|_2$, there exists $0 < t_1 < t_0$ such that $\lambda_2' < \|\Delta u(t_1)\|_2 < \lambda_2$, then $G(\|\Delta u(t_1)\|_2) > E(0) \geq E(t_1)$. This is a contracdition. (ii) when $\|\Delta u_0\|_2 > \lambda_1$, as in case (i) we also deduce that $\|\Delta u_0\|_2 > \lambda_1$ implies $\|\Delta u(t)\|_2 \geq \lambda_2$ for $t \geq 0$.

Lemma 3.3. *Let u be the solution of $(1.1) - (1.3)$. Assume the conditions of Theorem 2.3 hold. If $0 < \|\Delta u_0\|_2 < \lambda_1$ and*

$$\alpha = bB_1^p \left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}} < 1, \tag{3.8}$$

then $I(t) > 0$, for $t \in [0, T]$.

Proof. First, we note that $0 < \|\Delta u_0\|_2 < \lambda_1$ implies $I(0) > 0$, then it follows from the continuity of $u(t)$ that

$$I(t) \geq 0, \tag{3.9}$$

for some interval near $t = 0$. Let $t_{\max} > 0$ be a maximal time (possibly $t_{\max} = T$), when (3.9) holds on $[0, t_{\max})$. From (3.2) and (3.1), we have

$$J(t) = \frac{p-2}{2p}\|\Delta u\|_2^2 + \frac{1}{p}I(t). \tag{3.10}$$

By (3.10), (3.3) and (3.7), we deduce

$$\|\Delta u\|_2^2 \leq \frac{2p}{p-2}J(t) \leq \frac{2p}{p-2}E(t) \leq \frac{2p}{p-2}E(0). \tag{3.11}$$

Then, by Lemma 2.1, (3.11) and (3.8), we obtain

$$\begin{aligned} b\|u\|_p^p \leq bB_1^p\|\Delta u\|_2^p \leq bB_1^p \left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}} \|\Delta u\|_2^2 \\ = \alpha\|\Delta u\|_2^2 < \|\Delta u(t)\|_2^2 \text{ on } [0, t_{\max}). \end{aligned} \tag{3.12}$$

Thus

$$I(t) = \|\Delta u(t)\|_2^2 - b\|u\|_p^p > 0 \text{ on } [0, t_{\max}).$$

This implies that we can take $t_{\max} = T$.

Remark. Inequality (3.8) (i.e. $\alpha < 1$) is equivalent to $E(0) < E_1$.

Lemma 3.4. *Let u satisfies the assumptions of Lemma 3.3. Then there exists $0 < \eta < 1$ such that*

$$b\|u(t)\|_p^p \leq (1-\eta)\,\|\Delta u(t)\|_2^2 \text{ on } [0,T],$$

where $\eta = 1-\alpha$.

Proof. From (3.12), we get

$$b\|u(t)\|_p^p \leq \alpha\,\|\Delta u(t)\|_2^2\,,\ t \in [0,T].$$

Let $\eta = 1-\alpha$, then we have the result.

Remark. From Lemma 3.4, we can deduce that

$$\|\Delta u(t)\|_2^2 \leq \frac{1}{\eta} I(t), t \in [0,T]. \tag{3.13}$$

Theorem 3.5. ([Global Existence and Energy decay]). *Suppose that $0 < \|\Delta u_0\|_2 < \lambda_1$ and $0 < E(0) < E_1$, then the problem $(1.1)-(1.3)$ admits a global solution u if $u_0 \in H_0^2(\Omega)$ and $u_1 \in L^2(\Omega)$. Furthermore, we have the following decay estimates:*

$$E(t) \leq \left(E(0)^{-\frac{m-2}{2}} + \frac{(m-2)\tau}{2}[t-1]^+\right)^{-\frac{2}{m-2}} \text{ on } [0,\infty),$$

where τ is given in (3.27).

Proof. First, we want to show that the solution of $(1.1)-(1.3)$ is global in time in the sense of Theorem 2.3. From (3.3) and (3.11), we see that

$$\|u_t\|_2^2 + \|\Delta u\|_2^2 \leq (2 + \frac{2p}{p-2})E(0).$$

Then, by Theorem 2.3, we have the global existence result. By integrating (3.7) over $[t, t+1]$, $t > 0$, we have

$$E(t) - E(t+1) \equiv D(t)^m, \tag{3.14}$$

where

$$D(t)^m = a\int_t^{t+1} \|u_t\|_m^m \,dt. \tag{3.15}$$

By virture of (3.15) and Hölder inequality, we observe that

$$\int_t^{t+1}\int_\Omega |u_t|^2\,dxdt \le c(\Omega)D(t)^2, \tag{3.16}$$

where $c(\Omega) = (\text{vol}(\Omega))^{\frac{m-2}{m}}$. Hence, from (3.16), there exist $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|u_t(t_i)\|_2^2 \le 4c(\Omega)D(t)^2, \quad i=1,2. \tag{3.17}$$

Next, multiplying (1.1) by u and integrating it over $\Omega \times [t_1, t_2]$, we get

$$\int_{t_1}^{t_2} I(t)dt = -\int_{t_1}^{t_2}\int_\Omega u_{tt}udxdt - a\int_{t_1}^{t_2}\int_\Omega |u_t|^{m-2}u_t udxdt. \tag{3.18}$$

Then, by using (1.1) and integrating by parts on the first term of the right hand side of (3.18), we obtain

$$\begin{aligned}&\int_{t_1}^{t_2} I(t)dt \\ &\le \sum_{i=1}^{2}\|u_t(t_i)\|_2\|u(t_i)\|_2 + \int_{t_1}^{t_2}\|u_t\|_2^2\,dt - a\int_{t_1}^{t_2}\int_\Omega |u_t|^{m-2}u_t udxdt.\end{aligned} \tag{3.19}$$

By Hölder inequality and Poincaré inequality, it follows that

$$\begin{aligned}&\left|a\int_{t_1}^{t_2}\int_\Omega |u_t|^{m-2}u_t udxdt\right| \le a\int_{t_1}^{t_2}\|u\|_m\|u_t\|_m^{m-1}\,dt \\ &\le aB_1\int_{t_1}^{t_2}\|\Delta u\|_2\|u_t\|_m^{m-1}\,dt \\ &\le aB_1\left(\frac{2p}{p-2}\right)^{\frac{1}{2}}\sup_{t_1\le s\le t_2}E(s)^{\frac{1}{2}}\int_{t_1}^{t_2}\|u_t\|_m^{m-1}\,dt \\ &\le aB_1\left(\frac{2p}{p-2}\right)^{\frac{1}{2}}D(t)^{m-1}\sup_{t_1\le s\le t_2}E(s)^{\frac{1}{2}}.\end{aligned} \tag{3.20}$$

And by using (3.17), Poincaré inequality and (3.11), we also have

$$\|u_t(t_i)\|_2\|u(t_i)\|_2 \le c_1 D(t)\sup_{t_1\le s\le t_2}E(s)^{\frac{1}{2}}, \tag{3.21}$$

where $c_1 = 2B_1\sqrt{c(\Omega)}\left(\frac{2p}{p-2}\right)^{\frac{1}{2}}$. Then by $(3.20)-(3.21)$ and (3.16), we obtain from (3.19)

$$\begin{aligned}&\int_{t_1}^{t_2} I(t)dt\\ &\leq 2c_1D(t)\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}} + aB_1\left(\frac{2p}{p-2}\right)^{\frac{1}{2}}D(t)^{m-1}\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}}\\ &+c(\Omega)D(t)^2.\end{aligned} \tag{3.22}$$

On the other hand, from (3.3) and using (3.10) and (3.13), we deduce

$$E(t) \leq \frac{1}{2}\left\|u_t\right\|_2^2 + c_2I(t), \tag{3.23}$$

where $c_2 = \left(\frac{p-2}{2p\eta}+\frac{1}{p}\right)$. By integrating (3.23) over (t_1, t_2), we obtain

$$\int_{t_1}^{t_2} E(t)dt \leq \frac{1}{2}\int_{t_1}^{t_2}\left\|u_t\right\|_2^2 dt + c_2\int_{t_1}^{t_2} I(t)dt.$$

Hence, by (3.16) and (3.22), we have

$$\begin{aligned}\textstyle\int_{t_1}^{t_2} E(t)dt \leq\ & \frac{1}{2}c(\Omega)D(t)^2 + c_2[2c_1D(t)\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}}\\ &+aB_1\left(\frac{2p}{p-2}\right)^{\frac{1}{2}}D(t)^{m-1}\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}} + c(\Omega)D(t)^2].\end{aligned} \tag{3.24}$$

Moreover, multiplying (1.1) by u_t and then integrating it over $[t, t_2]\times\Omega$, we get

$$E(t) = E(t_2) + a\int_t^{t_2}\left\|u_t\right\|_m^m ds.$$

Since $t_2 - t_1 \geq \frac{1}{2}$, it follows that

$$E(t_2) \leq 2\int_{t_1}^{t_2} E(t)dt.$$

Then, thanks to (3.14), we arrive at

$$\begin{aligned}E(t) &\leq 2\int_{t_1}^{t_2} E(t)dt + a\int_t^{t_2}\left\|u_t\right\|_m^m ds\\ &= 2\int_{t_1}^{t_2} E(t)dt + D(t)^m.\end{aligned} \tag{3.25}$$

Thus, by using (3.24) and Lemma 3.2, we see that

$$\begin{aligned} E(t) \leq & \, (c(\Omega) + 4c_1c_2)\, D(t)^2 + D(t)^m \\ & +2c_2\left[2c_1D(t) + aB_1\left(\frac{2p}{p-2}\right)^{\frac{1}{2}} D(t)^{m-1}\right] E(t)^{\frac{1}{2}},\ t \geq 0. \end{aligned}$$

Hence, by Young's inequality, we deduce

$$E(t) \leq c_3\left[D(t)^2 + D(t)^m + D(t)^{2(m-1)}\right], \tag{3.26}$$

where c_3 is some positive constant. Therefore, we have the following decay estimate. From (3.26) and (3.14), we get

$$\begin{aligned} E(t) & \leq c_3\left[1 + D(t)^{m-2} + D(t)^{2m-4}\right] D(t)^2 \\ & \leq c_3\left[1 + E(0)^{\frac{m-2}{m}} + E(0)^{\frac{2m-4}{m}}\right] D(t)^2. \end{aligned}$$

This implies that

$$E(t)^{\frac{m}{2}} \leq (c_4(E(0)))^{\frac{m}{2}}\, D(t)^m,$$

where $c_4(E(0)) = c_3\left[1 + E(0)^{\frac{m-2}{m}} + E(0)^{\frac{2m-4}{m}}\right]$. Note that $\lim_{E(0)\to 0} c_4(E(0)) = c_3 > 0$. Hence, by applying Lemma 2.2 yields

$$E(t) \leq \left(E(0)^{-\frac{m-2}{2}} + \frac{(m-2)\tau}{2}[t-1]^+\right)^{-\frac{2}{m-2}} \text{ on } [0, \infty), \tag{3.27}$$

where $\tau = (c_4(E(0)))^{-\frac{m}{2}}$.

4. Blow-up Poperty

In this section, we shall show that the solution of problem (1.1) blows up in finite time if $p > m$ and $E(0) < E_1$.

Theorem 4.1. **(Nonexistence of Global Solutions)** *Suppose that $p > m$. If one of the following is satisfied*

(i) $E(0) < 0$

(ii) $0 \leq E(0) < E_1$ *and* $\|\Delta u_0\|_2 > \lambda_1$,

then the local solution of the problem $(1.1) - (1.3)$ *blows up at a finite time* T. *The lifespan T is estimated by* $0 < T \leq \frac{L(0)^{1-\theta}}{c_{11}(\theta-1)}$, *here $L(t)$ and c_{11} are given in* (4.13) *and* (4.22) *respectively. θ is a constant given in* (4.17).

Proof. (I) For $0 \leq E(0) < E_1$, we set

$$H(t) = E_2 - E(t), \ t \geq 0, \tag{4.1}$$

where $E_2 = \frac{E(0)+E_1}{2}$. By (3.7), we see that

$$H'(t) = a \int_\Omega |u_t|^m \, dx \geq 0. \tag{4.2}$$

Thus, we have

$$H(t) \geq H(0) = E_2 - E(0) > 0, \ t \geq 0. \tag{4.3}$$

Let

$$A(t) = \int_\Omega u u_t dx. \tag{4.4}$$

By differentiating (4.4) and using (1.1), we obtain

$$A'(t) = \|u_t\|_2^2 - \|\Delta u\|_2^2 - a \int_\Omega |u_t|^{m-2} u_t u dx + b \|u\|_p^p. \tag{4.5}$$

Hence, by (3.3), we deduce

$$\begin{aligned} A'(t) &= a_1 \|u_t\|_2^2 + a_2 \|\Delta u(t)\|_2^2 - a \int_\Omega |u_t|^{m-2} u_t u dx \\ &\quad + pH(t) - pE_2. \end{aligned} \tag{4.6}$$

where $a_1 = 1 + \frac{p}{2}$ and $a_2 = \frac{p}{2} - 1$. We observe that $a_i > 0, i = 1, 2$. Moreover

$$\begin{aligned} & a_2 \|\Delta u(t)\|_2^2 - pE_2 \\ = \ & a_2 \frac{\lambda_2^2 - \lambda_1^2}{\lambda_2^2} \|\Delta u(t)\|_2^2 + a_2 \lambda_1^2 \frac{\|\Delta u(t)\|_2^2}{\lambda_2^2} - pE_2 \\ \geq \ & c_1 \|\Delta u(t)\|_2^2 + c_2, \end{aligned}$$

where λ_2 is given in Lemma 3.1, $c_1 = a_2 \frac{\lambda_2^2 - \lambda_1^2}{\lambda_2^2}$ and $c_2 = a_2 \lambda_1^2 - pE_2$. By Lemma 3.1 (ii), we have $c_1 > 0$ and by (3.6), we see that

$$\begin{aligned} c_2 &= \frac{(p-2)\lambda_1^2}{2} - \frac{p(E_1 + E(0))}{2} \\ &= \frac{p(E_1 - E(0))}{2} > 0. \end{aligned} \tag{4.7}$$

Thus, by (4.6) and (4.7), we arrive at

$$A'(t) \geq a_1 \|u_t\|_2^2 + c_1 \|\Delta u(t)\|_2^2 - a \int_\Omega |u_t|^{m-2} u_t u dx + pH(t). \tag{4.8}$$

On the other hand, by Hölder inequality, we have

$$a\left|\int_\Omega |u_t|^{m-2}\,u_t u dx\right| \le a\,\|u\|_m\,\|u_t\|_m^{m-1} \le c_3\,\|u\|_p^{1-\frac{p}{m}}\,\|u\|_p^{\frac{p}{m}}\,\|u_t\|_m^{m-1}, \tag{4.9}$$

where $c_3 = a(\mathrm{vol}(\Omega))^{\frac{p-m}{mp}}$. Note that, from (4.1) and (3.4), we get

$$H\,(t) \le E_1 - \frac{1}{2}\,\|\Delta u\|_2^2 + \frac{b}{p}\|u\|_p^p \le E_1 - \frac{1}{2}\lambda_1^2 + \frac{b}{p}\|u\|_p^p,$$

where the last inequality is derived by Lemma 3.1(ii). Thus, by (3.6) and (4.3), we see that

$$0 < H\,(0) \le H\,(t) \le \frac{b}{p}\,\|u\|_p^p \text{ for } t \ge 0. \tag{4.10}$$

Then, using (4.10), we have from (4.9)

$$a\left|\int_\Omega |u_t|^{m-2}\,u_t u dx\right| \le c_4\,\|u\|_p^{\frac{p}{m}}\,H(t)^{\frac{1}{p}-\frac{1}{m}}\,\|u_t\|_m^{m-1}.$$

Hence by Young's inequality and (4.2), we obtain

$$a\left|\int_\Omega |u_t|^{m-2}\,u_t u dx\right| \le c_5\left(\varepsilon^m\,\|u\|_p^p + \varepsilon^{-m'}H'(t)\right)H(t)^{-\alpha_1}, \tag{4.11}$$

where $\alpha_1 = \frac{1}{m} - \frac{1}{p} > 0$, $\varepsilon > 0$, $m' = \frac{m}{m-1}$, $c_4 = c_3(\frac{p}{b})^{\frac{1}{p}-\frac{1}{m}}$ and $c_5 = c_4\max(1, \frac{1}{a})$. Letting $0 < \alpha < \alpha_1$ and by (4.10), we see that

$$a\left|\int_\Omega |u_t|^{m-2}\,u_t u dx\right| \le c_5\left(\varepsilon^m H\,(0)^{-\alpha_1}\,\|u\|_p^p + \varepsilon^{-m'}H\,(0)^{\alpha-\alpha_1}\,H(t)^{-\alpha}H'(t)\right). \tag{4.12}$$

Now, we define

$$L\,(t) = H\,(t)^{1-\alpha} + \delta_1 A(t),\ t \ge 0, \tag{4.13}$$

where δ_1 is a positive constant to be specified later. By differentiating (4.13), and then by (4.12) and (4.8), we see that

$$L'\,(t) \ge \left(1 - \alpha - \delta_1 c_5 \varepsilon^{-m'}H\,(0)^{\alpha-\alpha_1}\right)H(t)^{-\alpha}H'(t) + \delta_1\left[a_1\,\|u_t\|_2^2 + c_1\,\|\Delta u(t)\|_2^2 + pH(t)\right] - \delta_1 c_5 \varepsilon^m H\,(0)^{-\alpha_1}\,\|u\|_p^p. \tag{4.14}$$

Letting $a_3 = \min\{a_1, c_1, \frac{p}{2}\}$ and decomposing $\delta_1 pH(t)$ in (4.14) by

$$\delta_1 pH(t) = 2a_3\delta_1 H(t) + (p - 2a_3)\delta_1 H(t).$$

Thus, by (4.1) and (3.3), we obtain

$$\begin{aligned} L'(t) \geq & \left(1 - \alpha - \delta_1 c_5 \varepsilon^{-m'} H(0)^{\alpha-\alpha_1}\right) H(t)^{-\alpha} H'(t) \\ & +\delta_1 \left[\frac{2a_3 b}{p} - c_5 \varepsilon^m H(0)^{-\alpha_1}\right] \|u\|_p^p + \delta_1 (a_1 - a_3) \|u_t\|_2^2 \\ & \delta_1 (c_1 - a_3) \|\Delta u(t)\|_2^2 + (p - 2a_3)\delta_1 H(t). \end{aligned} \tag{4.15}$$

Now, we choose $\varepsilon > 0$ small such that $\frac{2a_3 b}{p} - c_5 \varepsilon^m H(0)^{-\alpha_1} \geq \frac{a_3 b}{2p}$, and $0 < \delta_1 < \frac{(1-\alpha)}{c_5} \varepsilon^{m'} H(0)^{\alpha_1 - \alpha}$. Then (4.15) becomes

$$L'(t) \geq c_6 \delta_1 \left(\|u\|_p^p + \|u_t\|_2^2 + H(t) + \|\Delta u\|_2^2\right), \tag{4.16}$$

here $c_6 = \min\left\{\frac{a_3 b}{2p},\ a_1 - a_3,\ c_1 - a_3,\ p - 2a_3\right\}$. Thus $L(t)$ is a nondecreasing function on $t \geq 0$. Letting δ_1 be small enough in (4.13), then we have $L(0) > 0$. Hence

$$L(t) > 0, \text{ for } t \geq 0.$$

Now set

$$\theta = \frac{1}{1-\alpha}. \tag{4.17}$$

Since $\alpha < \alpha_1 < 1$, it is evident that $1 < \theta < \frac{1}{1-\alpha_1}$. By Young's inequality and Hölder inequality, it follows that

$$L(t)^\theta \leq 2^{\theta-1} \left[H(t) + \left(\delta_1 \int_\Omega u_t u dx\right)^\theta\right]. \tag{4.18}$$

On the other hand, for $p > 2$ and using Hölder inequality, we have

$$\begin{aligned} \int_\Omega u_t u dx \leq & \|u_t\|_2 \|u\|_2 \\ \leq & c_7 \|u_t\|_2 \|u\|_p, \end{aligned}$$

here $c_7 = (\mathrm{vol}(\Omega))^{\frac{(p-2)}{2p}}$. And by Young's inequality, we obtain

$$\begin{aligned} \left(\int_\Omega u_t u dx\right)^\theta \leq & c_8 \|u_t\|_2^\theta \|u\|_p^\theta \\ \leq & c_9 \left(\|u\|_p^{\theta\beta_1} + \|u_t\|_2^{\theta\beta_2}\right), \end{aligned} \tag{4.19}$$

where $\frac{1}{\beta_1}+\frac{1}{\beta_2}=1$, $c_8=c_7^{\theta}$, and $c_9=c_9(c_8,\ \beta_1,\ \beta_2)>0$. In particular, we take $\theta_1\beta_2=2$, i.e. $\beta_2=2\,(1-\alpha)$.

Therefore, for α small enough, the numbers β_1 and β_2 are close to 2. Now choose $\alpha\in\left(0,\ (\alpha_1,\ \frac{1}{2}-\frac{1}{p})\right)$. Note that from (4.10), we see that

$$\frac{b}{pH(0)}\,\|u\|_p^p\geq 1.$$

Thus we obtain

$$\|u\|_p^{\theta\beta_1}\leq\left(\frac{b}{pH(0)}\right)^{1-\frac{\theta\beta_1}{p}}\|u\|_p^p\,, \tag{4.20}$$

because

$$\theta\beta_1=\frac{2}{1-2\alpha}<p.$$

Consequently by $(4.18)-(4.20)$, we have

$$L\,(t)^{\theta}\leq c_{10}\left[H\,(t)+\|u\|_p^p+\|u_t\|_2^2\right], \tag{4.21}$$

here c_{10} is some positive constant. From (4.16) and (4.21), we get

$$L'\,(t)\geq c_{11}L\,(t)^{\theta}\,,\ \ t\geq 0, \tag{4.22}$$

here $c_{11}=\frac{c_6\delta_1}{c_{10}}$. An integration of (4.22) over $(0,t)$ then yields

$$L\,(t)\geq\left(L\,(0)^{1-\theta}-c_{11}\,(\theta-1)\,t\right)^{-\frac{1}{\theta-1}}. \tag{4.23}$$

Since $L\,(0)>0$, (4.23) shows that L becomes infinite in a finite time $T\leq T^*=\frac{L(0)^{1-\theta}}{c_{11}(\theta-1)}$. (II) For $E(0)<0$, we set

$$H(t)=-E(t),$$

instead of (4.2). Then, applying the same argunments as in part (I), we have the result.

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