

## ON THE CORES OF SCALAR MEASURE GAMES

Man-Chung Ng, Chi-Ping Mo, and Yeong-Nan Yeh

**Abstract.** A  $CVM(k)$  game is a game of the form  $f \circ \lambda$ , where  $\lambda$  is a  $k$ -dimensional non-atomic measure and  $f$  is a continuously differentiable function on  $R^k$ . For a convex  $CVM(1)$  game, we characterize the “least upper bound” and “greatest lower bound” of the core elements in terms of the distribution function. We also show that the core of a convex  $CVM(1)$  game expands as the underlying measure  $\lambda$  changes in a “convex manner”. These results provide a partial geometric picture for the core and its variations of a convex  $CVM(1)$  game.

### 1. INTRODUCTION

A game  $V$  with a *continuum* of players is a bounded real-valued function defined on  $\Sigma$ , the set of all Borel subsets of  $I = [0, 1]$ , such that  $V(\emptyset) = 0$ . The elements of  $\Sigma$  are interpreted as *coalitions* of players; for each coalition  $S$ ,  $V(S)$  gives the maximum payoff achieved by the efforts of all members in the coalition. With this interpretation, we shall assume that  $V$  is non-negative and that  $V$  is not identically zero throughout this paper. For  $S \in \Sigma$ , we denote by  $S^*$  the indicator function of  $S$ . Let  $\mathbf{B}$  be the Banach space spanned by the set  $\Sigma^* = \{S^* : S \in \Sigma\}$  with the sup norm. Then the space of all bounded additive functions on  $\Sigma$ , denoted by  $\mathbf{BA}$ , is isometrically isomorphic to the norm-dual of  $\mathbf{B}$ . A *payoff* of the game  $V$  is an element  $\mu \in \mathbf{BA}$  with  $\mu(I) = V(I)$ . The *core*  $C_V$  of the game  $V$  consists of all payoffs  $\mu$  such that no coalition can improve upon, i.e.,  $\mu(S) \geq V(S)$  for each  $S \in \Sigma$ . The core is a convex and compact set (in the  $\mathbf{B}$  topology). The assumption that  $V$  is

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non-negative implies that any element in the core is also non-negative. In fact,  $\mu \in C_V$  must be monotonic in the sense that  $\mu(S) \leq \mu(T)$  whenever  $S \subseteq T$ .

Since  $\Sigma$  is a  $\sigma$ -algebra, we can endow a measure  $\lambda$  with it<sup>1</sup>. Then a special class of symmetric games (with respect to the measure  $\lambda$ ) can be defined as follows:

**Definition 1.**  $V$  is called *symmetric* (with respect to  $\lambda$ ) if  $V(S) = V(T)$  for all  $S$  and  $T$  such that  $\lambda(S) = \lambda(T)$ .

It is clear that  $V$  is symmetric if and only if there is a unique real-valued function  $f : [0, \lambda(I)] \rightarrow R$  such that  $V(S) = f \circ \lambda(S)$  for all  $S \in \Sigma$ , and  $f(0) = 0$ . We shall follow Aumann and Shapley [1974] and call such  $V$  a *scalar measure game* (See Aumann and Shapley [1974], p. 14). More generally, we can define:

**Definition 2.** Let  $\mathbf{0}$  be the origin in  $R^k$  and  $\lambda$  be a  $k$ -dimensional vector measure on  $(I, \Sigma)$  such that its range, denoted by  $\text{Range}(\lambda)$ , is of full dimension in  $R^k$ . The class of all  *$k$ -dimensional vector measure games*  $VM(k)$  (with respect to  $\lambda$ ) consists of all games  $V$  of the form:  $V(S) = f \circ \lambda(S)$ , where  $f$  is a non-negative real-valued function defined on  $R^k$  with  $f(\mathbf{0}) = 0$ .

Let  $CVM(k)$  be the class of all games  $V = f \circ \lambda \in VM(k)$  such that  $f$  is continuously differentiable on  $R^k$ , and that  $\lambda$  is a non-atomic vector measure<sup>2</sup> on  $I$ . The *Shapley value* of a game  $V = f \circ \lambda \in CVM(k)$  is a payoff  $\phi_V$  of  $V$  given by the following formula (Aumann and Shapley [1974], p. 23)

$$(1) \quad \phi_V(S) = \int_0^1 f_{\lambda(S)}(t\lambda(I))dt,$$

where  $f_{\lambda(S)}$  is the derivative of  $f$  in the direction  $\lambda(S) \in R^k$ . In the case  $k = 1$ , the Shapley value is in the core if and only if for each  $S \in \Sigma$  such that  $\lambda(S) > 0$ ,  $\frac{v(S)}{\lambda(S)}$  is bounded above by  $\frac{v(I)}{\lambda(I)}$ .

Due to space limitation, the focus of this paper is mostly on the case  $k = 1$ , which carries quite strong geometric intuition. In the second section, we begin with Schmeidler's [1972] result that the core of any convex game consists

<sup>1</sup> Of particular interest is the Lebesgue measure on  $\Sigma$ . We also recall that a measure is a non-negative countably additive function in **BA**.

<sup>2</sup> We recall that an *atom* of a vector measure  $\lambda$  is a coalition  $S \in \Sigma$  such that  $\lambda(S) \neq \mathbf{0}$ , and for each proper subset  $T$  of  $S$ , either  $\lambda(T) = \mathbf{0}$  or  $\lambda(S - T) = \mathbf{0}$ . If  $\lambda$  has no atom, then it is called a non-atomic measure. It is clear that any measure which is absolutely continuous with respect to the  $k$ -dimensional Lebesgue measure on  $I$  must be non-atomic.

entirely of countably additive elements if the game satisfies some type of continuity condition. It is quite obvious that every convex game in  $CVM(1)$  satisfies this condition, so that each core element can be represented by a distribution function. In Section 3, for a convex game in  $CVM(1)$ , we characterize the “least” upper bound, and the “greatest” lower bound for all the distribution functions in the core. This shows a clear geometric picture for the “shape” of the core. Finally, we prove a theorem on expanding core as we change the underlying measure  $\lambda$ .

## 2. GAMES IN $CVM$

To every game  $V$ , we can define an extended real number  $|V|$ , called the norm of  $V$ , by

$$(2) \quad |V| = \sup \left\{ \sum_i a_i V(S_i) : (a_i, S_i) \text{ is a finite sequence in } R_+ \times \Sigma \text{ such that } \sum_i a_i S^* \leq I^* \right\}.$$

For each game  $V$  with finite norm, the *exact envelope* of  $V$  is a game defined by

$$(3) \quad \bar{V}(S) = \min\{\mu(S) : \mu \in \mathbf{BA}, \mu \geq V \text{ and } \mu(I) = |V|\} \text{ for all } S \in \Sigma.$$

**Definition 3.** A game  $V$  is called *balanced* if  $|V| = V(I)$ . It is called *exact* if  $\bar{V} = V$ . It is called *convex* if  $V(S) + V(T) \leq V(S \cup T) + V(S \cap T)$  for all  $S, T \in \Sigma$ .

It is clear that for any convex game  $V = f \circ \lambda \in CVM(1)$ , the Shapley value  $\phi_V(S) = \lambda(S) \frac{V(I)}{\lambda(I)} \geq V(S)$  by convexity of  $f$  for each  $S \in \Sigma$ . Hence, the Shapley value of a convex game in  $CVM(1)$  always lies in its core.

**Definition 4.** A game  $V$  is *continuous* at  $S \in \Sigma$  if  $V(S_n) \rightarrow V(S)$  for any monotone sequence  $\{S_n\}$  in  $\Sigma$  such that  $\bigcup_n S_n = S$ . If  $V$  is continuous at each  $S \in \Sigma$ , then  $V$  is said to be continuous.

**Proposition 1.** (Shapley [1971], Schmeidler [1972]) *A game  $V$  has a non-empty core if and only if it is balanced. Every convex game is exact, and every exact game is balanced.*

**Proposition 2.** (Schmeidler [1972]) *Let  $V$  be an exact game. Then every element in the core of  $V$  is countably additive if and only if  $V$  is continuous at  $I$ .*

**Theorem 1.** *Every exact (hence convex) game in  $CVM(k)$  has a non-empty core with countably additive elements only.*

*Proof.* We first prove that every game in  $CVM(k)$  is continuous at  $I$  for all  $k$ . Let  $V = f \circ \lambda \in CVM(k)$ , and  $\{S_n\}$  be any monotone sequence in  $\Sigma$  such that  $\bigcup_n S_n = I$ . Countable additivity of  $\lambda$  implies that  $\lambda(I) = \lim_n \lambda(S_n)$ . Continuity (continuous differentiability is not required here) of  $f$  implies that  $f \circ \lambda(I) = \lim_n f \circ \lambda(S_n)$ . Hence, the assertion of Theorem 1 follows immediately from Propositions 1 and 2. ■

It is easy to show that a game  $V = f \circ \lambda \in CVM(1)$  is convex if and only if the derivative  $f'$  is non-decreasing on  $\text{Range}(\lambda)$ .<sup>3</sup> The implication of Theorem 1 is that there is a measure on  $\Sigma$  such that every element of an exact game in  $CVM(1)$  is absolutely continuous with respect to it. We can, then, via the Radon-Nikodym theorem, represent every element in the core by a real-valued function on  $I$ . In fact, the following theorem shows that for a game  $V = f \circ \lambda$ ,  $\lambda$  is exactly the measure we are seeking.

**Theorem 2.** *Let  $V = f \circ \lambda \in CVM(1)$  be an exact game. Then for each  $\mu \in C_V$ , there is a non-negative Borel measurable function  $g_\mu$  (unique up to the measure  $\lambda$ ) such that*

$$\mu(S) = \int_S g_\mu d\lambda.$$

*Proof.* Because of Theorem 1 and the Radon-Nikodym theorem, we are left with proving that  $\mu$  is absolutely continuous with respect to  $\lambda$ . We first recall that any element in the core is non-negative since  $\mu \geq V \geq 0$ . Now, let  $S \in \Sigma$  with  $\lambda(S) = 0$ . Then

$$\mu(I) - \mu(S) = \mu(I - S) \geq V(I - S) = V(I) = \mu(I),$$

which implies  $\mu(S) \leq 0$ . It follows that  $\mu(S) = 0$ , and hence  $\mu$  is absolutely continuous with respect to  $\lambda$ . ■

According to Theorem 2, for an exact game in  $CVM(1)$ , we may identify each  $\mu \in C_V$  with its *density function*  $g_\mu$ . Two measures  $\mu_1$  and  $\mu_2$  in the core will always be regarded as equivalent if  $\mu_1 = \mu_2$  except on a measurable set  $E$ , with  $\lambda(E) = 0$ .

**Definition 5.** A measurable auto-map  $\pi$  on  $I$  is *measure-preserving* for  $\lambda$  if it is one-one, onto, and  $\lambda(S) = \lambda(\pi(S))$  for each  $S \in \Sigma$ . (Note we have used the notation  $\pi(S) = \{\pi(x) : x \in S\}$ .)

<sup>3</sup> Curiously, for  $k > 1$ , there is no corresponding statement in either directions.

It is clear that the set of all measure-preserving maps for  $\lambda$  forms a group consisting of idempotent elements only. An easy example of a measure-preserving map for the Lebesgue measure on  $I$  is the function  $\pi(x) = 1 - x$ . We note that  $\pi^{-1} = \pi$ .

**Theorem 3.** *Let  $\pi$  be any measure-preserving map for  $\lambda$ . Then for any game  $V \in CVM(1)$ ,  $\mu \in C_V$  implies  $\mu \circ \pi \in C_V$ . In fact,  $g_{\mu \circ \pi} = g_\mu \circ \pi$ .*

*Proof.* Let  $\mu \in C_V$  and  $\pi$  be a measure-preserving map for  $\lambda$ . Then for each  $S \in \Sigma$ , we have

$$(\mu \circ \pi)(S) = \mu(\pi(S)) \geq V(\pi(S)) = V(S).$$

Hence,  $\mu \circ \pi \in C_V$ . The second statement follows from:

$$\int_S g_\mu \circ \pi d\lambda = \int_{\pi(S)} g_\mu d(\lambda \circ \pi^{-1}) = \int_{\pi(S)} g_\mu d\lambda,$$

where  $\pi^{-1}(S) = \{x \in I : \pi(x) \in S\}$ . ■

Theorem 3 has quite concrete geometric interpretation. Furthermore, for any convex game in  $CVM(1)$ , we can classify the “shape” of the core in terms of *distribution functions*. This is the main purpose of the next section.

### 3. CONVEX GAMES IN $CVM(1)$

**Definition 6.** For any measure  $\mu$  on  $\Sigma$ , one can define the *distribution function* of  $\mu$  on  $I$  by  $F_\mu(x) = \mu([0, x])$  for  $x \in I$ .

It can be easily seen that  $F_\mu$  has the following properties:

- (P1)  $F_\mu$  is a nondecreasing function;
- (P2)  $F_\mu$  is right continuous and has a left-hand limit everywhere;
- (P3)  $F_\mu(0) = 0$  and  $F_\mu(1) = \mu(I)$ .

It turns out the converse is also true. Given a distribution function  $F$  on  $I$  with properties (P1), (P2),  $F(0) = 0$ , and  $F(1)$  finite, one can define<sup>4</sup> a unique measure  $\gamma_F$  on  $\Sigma$  such that  $\gamma_F((x, y]) = F(y) - F(x)$  for any  $0 \leq x < y \leq 1$ .

Since  $\lambda$  is a non-atomic measure on  $I$ ,  $F_\lambda$  is a continuous function on  $I$ . Assuming that  $F_\lambda$  is strictly increasing, we can define the following auto map on  $I$ :

$$(4) \quad \hat{\pi}(x) = \{y \in I : F_\lambda(x) + F_\lambda(y) = \lambda(I)\}, \quad x \in I.$$

<sup>4</sup> This is a standard fact in probability theory. See, e.g., Shiriyayev [1984].

We now show that  $\hat{\pi}$  is a measure-preserving map for  $\lambda$ . Since  $F_\lambda(x) = \lambda(I) - F_\lambda(\hat{\pi}(x))$ , we have  $\lambda([0, x]) = \lambda([\hat{\pi}(x), 1])$ . It can be readily seen that  $\lambda(S) = \lambda \circ \hat{\pi}(S)$  for all open intervals  $S \in \Sigma$ . The collection  $\Sigma$  is the smallest  $\sigma$ -algebra generated by all the open intervals in  $I$ . By Carathéodory's extension theorem,  $\lambda(S) = \lambda \circ \hat{\pi}(S)$  for all  $S \in \Sigma$ . Since  $F_\lambda$  is strictly increasing,  $\hat{\pi}$  is one-one. Now,  $\hat{\pi}(0) = 1$ ,  $\hat{\pi}(1) = 0$  and so by the Intermediate Value Theorem,  $\hat{\pi}$  is onto as well. It follows that  $\hat{\pi}$  is a measure-preserving map for  $\lambda$ .

For each monotonic game  $V \in CVM(1)$ , we can define  $F_V(x) = V([0, x])$  for each  $x \in I$ . Clearly,  $F_V$  satisfies properties (P1) to (P3), and there is a unique measure  $\gamma_{F_V}$  associated with it. Note that  $\gamma_{F_V}$  is different from  $V$  itself unless  $V$  is countably additive.

**Theorem 4.** *Suppose that  $V = f \circ \lambda$  is a convex game in  $CVM(1)$  and that  $F_\lambda$  is strictly increasing. Then for any  $\mu \in C_V$ , we have*

$$(5) \quad F_V(x) \leq F_\mu(x) \leq V(I) - F_V(\hat{\pi}(x)) \quad \text{for each } x \in I.$$

*Furthermore, the unique measures associated with  $F_V(x)$  and  $V(I) - F_V(\hat{\pi}(x))$ , respectively, are elements of the core.*

*Proof.* Since a convex game must be monotonic, we know that  $F_V$  is well-defined. We first prove  $\gamma_{F_V} \in C_V$ . It is clear that for each  $S$ ,  $\gamma_{F_V}(S) = \int_S f' d\lambda$ . Thus,  $\gamma_{F_V} \in C_V$  if and only if

$$(6) \quad \int_S f' d\lambda \geq f(\lambda(S)) \quad \text{for all } S \in \Sigma.$$

We observe that the collection of all subsets in  $\Sigma$  satisfying inequality (6) is closed under finite disjoint union. Now suppose that  $S = (x, y]$  for some  $0 \leq x \leq y \leq 1$ . Then

$$\begin{aligned} \int_x^y f' d\lambda &= f \circ F_\lambda(y) - f \circ F_\lambda(x) \\ &\geq f(F_\lambda(y) - F_\lambda(x)) \quad (\text{by convexity and } f(0) = 0) \\ &= f \circ \lambda((x, y]), \end{aligned}$$

which is exactly inequality (6). Since  $\lambda$  is a non-atomic measure, and  $f'$  is bounded on  $[0, \lambda(I)]$ , inequality (6) holds for all closed intervals contained in  $I$  as well. Hence, inequality (6) is true for any finite disjoint union of closed intervals in  $I$ . Suppose that for some  $S \in \Sigma$ , inequality (6) is false. Let

$$\epsilon = f \circ \lambda(S) - \int_S f' d\lambda > 0.$$

Using the facts that  $\lambda$  is a non-atomic measure on  $\Sigma$ ,  $f'$  is bounded on  $I$ , and  $f$  is continuous, we can find a set  $T$  which is a finite disjoint union of closed intervals in  $I$  such that

$$\max \left\{ |f \circ \lambda(S) - f \circ \lambda(T)|, \left| \int_S f' d\lambda - \int_T f' d\lambda \right| \right\} < \frac{\epsilon}{2}.$$

This is a contradiction since  $f \circ \lambda(T) = \int_T f' d\lambda$ . Hence we have proved that  $\gamma_{F_V} \in C_V$ .

By Theorem 3,  $\gamma_{F_V} \circ \hat{\pi} \in C_V$ . It is easy to check that its distribution function is given by  $V(I) - F_V \circ \hat{\pi}$ .

For the first part of the theorem, let  $\mu$  be an element in  $C_V$ . By definition,  $F_\mu(x) \geq F_V(x)$  for each  $x \in I$ . From Theorem 3,  $\mu \circ \hat{\pi} \in C_V$ , and hence

$$F_\mu(x) = V(I) - F_{\mu \circ \hat{\pi}}(\hat{\pi}(x)) \leq V(I) - F_V(\hat{\pi}(x)) \quad \text{for each } x \in I. \quad \blacksquare$$

**Definition 7.** An element in the core of a game  $V$  is called a *vertex* if it cannot be written as the convex combination of two distinct elements in the core.

According to the above definition, if  $V = f \circ \lambda \in CVM(1)$ , then both  $\gamma_{F_V}$  and  $\gamma_{F_V} \circ \hat{\pi}$  are vertices. The convex hull of these two elements forms part of  $C_V$ . There are also other vertices. In the example below we show a method of constructing some vertices.

**Example.** Consider the convex game  $V = f \circ \lambda$ , where  $f(x) = x^2$  and  $\lambda$  is the Lebesgue measure. Then  $\hat{\pi}(x) = 1 - x$  in Theorem 4. All distribution functions in the core are bounded by  $y = x^2$  and  $y = 1 - f \circ \hat{\pi}(x) = 1 - (1 - x)^2$ . Now for each  $0 < a \in I$ , let us consider the measure-preserving map  $\hat{\pi}_a$  for  $\lambda$  defined by

$$\hat{\pi}_a(x) = \begin{cases} a - x, & \text{if } x \in [0, a]; \\ x, & \text{if } x \in (a, 1]. \end{cases}$$

Through  $\hat{\pi}_a$ , we obtain the following density function for a distribution function which we call  $F_a$ :

$$g_{F_a}(x) = \begin{cases} 2(a - x), & \text{if } x \in [0, a]; \\ 2x, & \text{if } x \in (a, 1]. \end{cases}$$

We claim that  $F_a$  is a vertex of the core. Suppose that there are two elements  $F_1$  and  $F_2$  in the core such that  $F_a = \alpha F_1 + (1 - \alpha)F_2$  for some  $\alpha \in (0, 1)$ . Since  $F_1$  and  $F_2$  are bounded below by  $F_V$ , we must have  $F_1(x) = F_2(x) = F_V(x)$  for all  $x \geq a$  (since all distribution functions are right continuous). Since

$F_1 \in C_V$ , the distribution defined by  $\overline{F}_1 = F_1 \circ \hat{\pi}_a \in C_V$ . Then for each  $x \leq a$ , we have  $\overline{F}_1(x) = V(a) - F_1(\hat{\pi}_a(x))$  and hence

$$F_1(x) = V(a) - \overline{F}_1(\hat{\pi}_a(x)) \leq V(a) - F_V(\hat{\pi}_a(x)) = F_a(x).$$

Similarly,  $F_2(x)$  is bounded above by  $F_a$  on the range  $[0, a]$ . It follows that  $F_1 = F_2 = F_a$ , a contradiction. Hence,  $F_a$  is a vertex of the core (c.f. Figure 1). ■

**Remark.** As regards the preceding example, let us consider a finite union of disjoint intervals  $E_i \subseteq I$ ,  $i = 1, 2, \dots, n$ . On each  $E_i$ , one can define a measure-preserving map  $\pi_i$  by renaming the players of  $E_i$  in the “opposite” direction. Then define a measure-preserving map  $\pi$  on  $I$  by letting  $\pi = \pi_i$  on each  $E_i$ , and  $\pi$  be the identity map elsewhere. It is clear that the distribution function with density function  $f \circ \pi$  is a vertex. ■

For each  $V = f \circ \lambda$ , and each  $0 < y \in I$ , we can define a subgame  $V_y$  on  $[0, y]$  by  $V_y(S) = V(S)$  for all  $S \subseteq [0, y]$  and  $S \in \Sigma$ . Then, we can easily prove that any element in  $C_{V_y}$  can be extended to some element in  $C_{V_x}$  whenever  $y \leq x$ . The ideas of the proof are clear from Example 1 and the proof of Theorem 4. We now state another theorem in which we change the underlying measure rather than the underlying subspace  $[0, y]$ .

FIG. 1.

**Theorem 5.** (Core Expansion) *Let  $V_1 = f \circ \lambda$ , and  $V_2 = f \circ \lambda_2$  be two convex games in  $CVM(1)$ . Assume that  $F_{\lambda_1}$  and  $F_{\lambda_2}$  are both strictly increasing on  $I$ , and the function  $h = F_{\lambda_1}^{-1} \circ F_{\lambda_2}$  is convex with  $\lambda_1(I) \leq \lambda_2(I)$ . Then, there exists a one-one map  $\Phi : C_{V_1} \rightarrow C_{V_2}$ .*

*Proof.* Let  $b$  be the unique point in  $I$  satisfying  $h(b) = 1$ . Take a distribution  $F_1 \in C_{V_1}$ . Define the following distribution function:

$$F_2(x) = \begin{cases} F_1(h(x)) & \text{if } x \leq b; \\ F_V(x) & \text{if } x > b. \end{cases}$$

We claim that  $F_2$  is a distribution in  $C_{V_2}$ . As in the proof of Theorem 4, observing that the collection of all Borel measurable sets satisfying

$$\gamma_{F_2}(S) \geq V_2(S) \quad \text{for each } S \in \Sigma$$

is closed under finite disjoint union, it suffices to prove that for each  $0 < x \leq y \leq 1$ ,

$$(7) \quad F_2(y) - F_2(x) \geq f(F_{\lambda_2}(y) - F_{\lambda_2}(x)).$$

Inequality (7) is true from Theorem 4 if both  $x, y > b$ . Now consider the case when both  $x, y \leq b$ . In this case,

$$\begin{aligned} F_2(y) - F_2(x) &= F_1(h(y)) - F_1(h(x)) \geq f \circ F_{\lambda_1}(h(y) - h(x)) \\ &\geq f \circ F_{\lambda_2}(y - x), \end{aligned}$$

where the last inequality follows from the convexity of  $h$ , and the fact that  $F_{\lambda_1}$ ,  $F_{\lambda_2}$  and  $f$  are increasing. To complete the proof, note that when  $x \leq b < y$ ,  $[x, y]$  can be written as the disjoint union of two intervals  $[x, b]$ , and  $(b, y]$ . The map  $\Phi$ , which maps  $F_1 \in C_{V_1}$  to  $F_2 \in C_{V_2}$  is clearly one-one. ■

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