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A CLASS OF MEROMORPHIC MULTIVALENT FUNCTIONS WITH POSITIVE COEFFICIENTS

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Abstract. In this paper, we introduce a class $F^n_{\lambda,p}(\alpha,\beta,\gamma)$ of meromorphic multivalent functions in $U^* = \{z : 0 < |z| < 1\}$ by using a differential operator $D^n_{\lambda,p}f(z)$. We obtain coefficient estimates, distortion theorem, radius of convexity, closure theorems and integral transforms for the class $F^n_{\lambda,p}(\alpha,\beta,\gamma)$. Several results involving the Hadamard products of functions belonging to the class $F^n_{\lambda,p}(\alpha,\beta,\gamma)$ are also derived.

1. Introduction

Let Σ^*_p denote the class of functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k} z^{p+k} \, (a_{p+k} \geq 0; p \in N = \{1, 2, ...\}) \tag{1.1}$$

which are analytic and p-valent in the punctured disc $U^* = \{z : z \in C$ and $0 < |z| < 1\} = U \backslash \{0\}$.

For a function $f(z) \in \Sigma^*_p$, we define the following differential operator:

$$\begin{aligned} D^0_{\lambda,p}f(z) &= f(z), \\ D^1_{\lambda,p}f(z) &= (1-\lambda)f(z) + \tfrac{\lambda}{p} z f'(z) + \tfrac{2\lambda}{z^p} \\ &= \frac{1}{z^p} + \sum_{k=0}^{\infty} (\frac{p+\lambda k}{p}) a_{p+k} z^{p+k} \\ &= D_{\lambda,p}f(z)(\lambda \geq 0; p \in N), \\ D^2_{\lambda,p}f(z) &= D_{\lambda,p}(D^1_{\lambda,p}f(z)) \ , \end{aligned}$$

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and

$$(1.2)\quad \begin{aligned} D^n_{\lambda,p}f(z) &= D_{\lambda,p}(D^{n-1}_{\lambda,p}f(z)) \\ &= (1-\lambda)D^{n-1}_{\lambda,p}f(z) + \tfrac{\lambda}{p}z(D^{n-1}_{\lambda,p}f(z))' + \tfrac{2\lambda}{z^p}(\lambda \geq 0; n,p \in N). \end{aligned}$$

It can be easily seen that

$$(1.3)\quad D^n_{\lambda,p}f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty}(\frac{p+\lambda k}{p})^n a_{p+k}z^{p+k}\, (n \in N_0 = N \cup \{0\}; p \in N)\ .$$

Also we can write $D^n_{\lambda,p}f(z)$ as follows:

$$\begin{aligned} D^n_{\lambda,p}f(z) &= \left(f * \left\{ \frac{1}{z^p} + \sum_{k=0}^{\infty}(\frac{p+\lambda k}{p})^n z^{p+k} \right\} \right) \\ &= (f * \Psi^p_n)(z)\,, \end{aligned}$$

where $\Psi^p_n(z) = \frac{1}{z^p} + \sum\limits_{k=0}^{\infty}(\frac{p+\lambda k}{p})^n z^{p+k}$ and $(*)$ denotes convolution.

With the aid of the differential operator $D^n_{\lambda,p}f(z)$ we define the class $F^n_{\lambda,p}(\alpha,\beta,\gamma)$ as follows:

A function $f(z)$ in Σ^*_p is in the class $F^n_{\lambda,p}(\alpha,\beta,\gamma)$ if it satisfies the following inequality:

$$(1.4)\quad \left| \frac{z^{p+1}(D^n_{\lambda,p}f(z))' + p}{(2\gamma-1)z^{p+1}(D^n_{\lambda,p}f(z))' + (2\gamma\alpha - p)} \right| < \beta\ (z \in U^*)$$

$(\alpha(0 \leq \alpha < p); \beta(0 < \beta \leq 1); \gamma(\frac{1}{2} \leq \gamma \leq 1); \lambda \geq 0; p \in N; n \in N_0)$.

We note that the class $F^0_{0,1}(\alpha,\beta,\gamma) = \Sigma_d(\alpha,\beta,\gamma)(0 \leq \alpha < 1, 0 < \beta \leq 1$ and $\frac{1}{2} \leq \gamma \leq 1)$ was studied by Cho et al. [4]. Also we note that:

(i) $F^n_{\lambda,p}(\alpha,1,1) = F^n_{\lambda,p}(\alpha) = \left\{ f(z) \in \Sigma^*_p : \mathrm{Re}\left\{ -z^{p+1}(D^n_{\lambda,p}f(z))' \right\} > \alpha, 0 \leq \alpha < p, p \in N, n \in N_0, z \in U^* \right\}$;

(ii) $F^0_{0,p}(\alpha,1,1) = F^*_p(\alpha) = \left\{ f(z) \in \Sigma^*_p : \mathrm{Re}\left\{ -z^{p+1}f'(z) \right\} > \alpha, 0 \leq \alpha < p, p \in N, z \in U^* \right\}$;

(iii) $F^0_{0,p}(\alpha,\beta,1) = F^*_p(\alpha,\beta) = \left\{ f(z) \in \Sigma^*_p : \left| \frac{z^{p+1}f'(z)+p}{z^{p+1}f'(z)+2\alpha-p} \right| < \beta, z \in U^*, 0 \leq \alpha < p, 0 < \beta \leq 1, p \in N, z \in U^* \right\}$.

Meromorphically multivalent functions have been extensively studied (for example) by Mogra ([9] and [10]), Uralegaddi and Ganigi [15], Aouf ([1] and [2]), Aouf et al. [3], Srivastava et al. [14], Owa et al. [11], Joshi and Aouf [5], Joshi and Srivastava [6], Liu [7], Liu and Srivastava [8], Raina and Srivastava [12] and Yang [16].

In the present paper, we obtain coefficient estimates, distortion theorems, radius of convexity, closure theorems and integral transforms for the class $F^n_{\lambda,p}(\alpha,\beta,\gamma)$. Several results involving the Hadamard products of functions belonging to the class $F^n_{\lambda,p}(\alpha,\beta,\gamma)$ are also derived.

2. Coefficients Estimates

Theorem 1. *A function $f(z)$ defined by (1.1) is in the class $F^n_{\lambda,p}(\alpha,\beta,\gamma)$ if and only if*

$$\sum_{k=0}^{\infty}(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)a_{p+k} \le 2\beta\gamma(p-\alpha)\,, \tag{2.1}$$

for $0 \le \alpha < p, 0 < \beta \le 1, \frac{1}{2} \le \gamma \le 1, n \in N_0, p \in N$ and $\lambda \ge 0$.

Proof. Suppose (2.1) holds. Then we have

$$\left|z^{p+1}(D^n_{\lambda,p}f(z))' + p\right| - \beta\left|(2\gamma-1)z^{p+1}(D^n_{\lambda,p}f(z))' + (2\gamma\alpha - p)\right| < 0$$

provided

$$\begin{aligned}&\left|\sum_{k=0}^{\infty}(p+k)(\frac{p+\lambda k}{p})^n\, a_{p+k}z^{2p+k}\right| - \beta\,\Big|2\gamma(p-\alpha)\\ &-\sum_{k=0}^{\infty}(2\gamma-1)(p+k)(\frac{p+\lambda k}{p})^n\;\; a_{p+k}z^{2p+k}\Big| < 0\,.\end{aligned} \tag{2.2}$$

For $|z| = r < 1$ the left hand side of (2.2) is bounded above by

$$\begin{aligned}&\sum_{k=0}^{\infty}(p+k)(\frac{p+\lambda k}{p})^n\, a_{p+k}r^{2p+k} - 2\beta\gamma(p-\alpha)\; +\\ &\sum_{k=0}^{\infty}\beta(2\gamma-1)(p+k)(\frac{p+\lambda k}{p})^n a_{p+k}r^{2p+k}\\ <\; &\sum_{k=0}^{\infty}(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)a_{p+k} - 2\beta\gamma(p-\alpha) \le 0\,.\end{aligned}$$

Hence $f(z) \in F^n_{\lambda,p}(\alpha,\beta,\gamma)$.

Conversely, suppose

$$\left|\frac{z^{p+1}(D^n_{\lambda,p}f(z))' + p}{(2\gamma-1)z^{p+1}(D^n_{\lambda,p}f(z))' - (2\gamma\alpha - p)}\right|$$

$$= \left|\frac{\sum\limits_{k=0}^{\infty}(p+k)(\frac{p+\lambda k}{p})^n a_{p+k}z^{2p+k}}{(2\gamma-1)(p-\sum\limits_{k=0}^{\infty}(p+k)(\frac{p+\lambda k}{p})^n a_{p+k}z^{2p+k}) - (2\gamma\alpha - p)}\right| < \beta\,(z \in U^*).$$

Using the fact that $|\mathrm{Re}(z)| \le |z|$ for all z, we have

$$\mathrm{Re}\left\{\frac{\sum\limits_{k=0}^{\infty}(p+k)(\frac{p+\lambda k}{p})^n a_{p+k}z^{2p+k}}{2\gamma(p-\alpha) - \sum\limits_{k=0}^{\infty}(2\gamma-1)(p+k)(\frac{p+\lambda k}{p})^n a_{p+k}z^{2p+k}}\right\} < \beta\,(z \in U^*)\,. \tag{2.3}$$

Now choose the values of z on the real axis so that $z^{p+1}(D^n_{\lambda,p}f(z))'$ is real. Upon clearing the denominator in (2.3) and letting $z \to 1^-$ through positive values, we obtain

$$\sum_{k=0}^{\infty}(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)a_{p+k} \le 2\beta\gamma(p-\alpha)\,.$$

Hence the result follows.

Corollary 1. *If the function $f(z)$ defined by* (1.1) *is in the class $F^n_{\lambda,p}(\alpha,\beta,\gamma)$, then*

$$a_{p+k} \le \frac{2\beta\gamma(p-\alpha)}{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)}\ (k \ge 0; p \in N; n \in N_0)\,. \tag{2.4}$$

This result is sharp for the function $f(z)$ given by

$$f(z) = \frac{1}{z^p} + \frac{2\beta\gamma(p-\alpha)}{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)}z^{p+k}\ (k \ge 0; p \in N; n \in N_0)\,. \tag{2.5}$$

Putting $n = \lambda = 0$ and $\beta = \gamma = 1$ in Theorem 1, we obtain:

Corollary 2. *A function $f(z)$ defined by* (1.1) *is in the class $F^*_p(\alpha)(0 \le \alpha < p)$ if and only if*

$$\sum_{k=0}^{\infty}(p+k)a_{p+k} \le (p-\alpha)\,. \tag{2.6}$$

We note that this result is obtained also by Mogra [10].

3. Distortion Theorem

Theorem 2. *If the function* $f(z)$ *defined by* (1.1) *is in the class* $F^{n}_{\lambda,p}(\alpha,\beta,\gamma)$*, then for* $0<|z|=r<1$*,*

$$\frac{1}{r^p}-\frac{2\beta\gamma(p-\alpha)}{p(1+2\beta\gamma-\beta)}r^p\le|f(z)|\le\frac{1}{r^p}+\frac{2\beta\gamma(p-\alpha)}{p(1+2\beta\gamma-\beta)}r^p, \tag{3.1}$$

and

$$\frac{p}{r^{p+1}}-\frac{2\beta\gamma(p-\alpha)}{(1+2\beta\gamma-\beta)}r^{p-1}\le|f^{'}(z)|\le\frac{p}{r^{p+1}}+\frac{2\beta\gamma(p-\alpha)}{(1+2\beta\gamma-\beta)}r^{p-1}. \tag{3.2}$$

The bounds in (3.1) and (3.2) are attained for the function $f(z)$ given by

$$f(z)=\frac{1}{z^p}+\frac{2\beta\gamma(p-\alpha)}{p(1+2\beta\gamma-\beta)}z^p\,(p\in N)\,. \tag{3.3}$$

Proof. In view of Theorem 1, we have

$$\begin{aligned}p(1+2\beta\gamma-\beta)\sum_{k=0}^{\infty}a_{p+k}\ &\le\ \sum_{k=0}^{\infty}(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)a_{p+k}\\ &\le\ 2\beta\gamma(p-\alpha)\,,\end{aligned}$$

that is, that

$$\sum_{k=0}^{\infty}a_{p+k}\le\frac{2\beta\gamma(p-\alpha)}{p(1+2\beta\gamma-\beta)}\,. \tag{3.4}$$

Thus, for $0<|z|=r<1$,

$$\begin{aligned}|f(z)|\ &\le\frac{1}{r^p}+\sum_{k=0}^{\infty}a_{p+k}r^{p+k}\\ &\le\frac{1}{r^p}+r^p\sum_{k=0}^{\infty}a_{p+k}\\ &\le\frac{1}{r^p}+\frac{2\beta\gamma(p-\alpha)}{p(1+2\beta\gamma-\beta)}r^p\end{aligned} \tag{3.5}$$

and

$$|f(z)| \geq \frac{1}{r^p} - \sum_{k=0}^{\infty} a_{p+k} r^{p+k}$$

$$\geq \frac{1}{r^p} - r^p \sum_{k=0}^{\infty} a_{p+k} \tag{3.6}$$

$$\geq \frac{1}{r^p} - \frac{2\beta\gamma(p-\alpha)}{p(1+2\beta\gamma-\beta)} r^p ,$$

which, together, yield (3.1). Furthermore, it follows from Theorem 1 that

$$\sum_{k=0}^{\infty} (p+k) a_{p+k} \leq \frac{2\beta\gamma(p-\alpha)}{(1+2\beta\gamma-\beta)} . \tag{3.7}$$

Hence

$$|f'(z)| \leq \frac{p}{r^{p+1}} + \sum_{k=0}^{\infty} (p+k) a_{p+k} r^{p+k-1}$$

$$\leq \frac{p}{r^{p+1}} + r^{p-1} \sum_{k=0}^{\infty} (p+k) a_{p+k} \tag{3.8}$$

$$\leq \frac{p}{r^{p+1}} + \frac{2\beta\gamma(p-\alpha)}{(1+2\beta\gamma-\beta)} r^{p-1}$$

and

$$|f'(z)| \geq \frac{p}{r^{p+1}} - \sum_{k=0}^{\infty} (p+k) a_{p+k} r^{p+k-1}$$

$$\geq \frac{p}{r^{p+1}} - r^{p-1} \sum_{k=0}^{\infty} (p+k) a_{p+k} \tag{3.9}$$

$$\geq \frac{p}{r^{p+1}} - \frac{2\beta\gamma(p-\alpha)}{(1+2\beta\gamma-\beta)} r^{p-1} ,$$

which, together, yield (3.2). It can easily be seen that the function $f(z)$ defined by (3.3) is extremal for Theorem 2.

4. Radius of Convexity

Theorem 3. *Let the function $f(z)$ defined by (1.1) be in the class $F^n_{\lambda,p}(\alpha, \beta, \gamma)$, then $f(z)$ is meromorphically p-valent convex of order $\delta (0 \leq \delta < p)$ in $0 < |z| <$*

$r(\lambda, p, n, \alpha, \beta, \gamma, \delta)$, *where*

$$r(\lambda, p, n, \alpha, \beta, \gamma) = \inf_k \left[\frac{p(p-\delta)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)(3p+k-\delta)} \right]^{\frac{1}{2p+k}} \quad (k \geq 0; p \in N; n \in N_0)\,. \tag{4.1}$$

The result is sharp.

Proof. It is sufficient to show that

$$\left| \frac{(zf'(z))' + pf'(z)}{f'(z)} \right| \leq p - \delta \quad \text{for } 0 < |z| < r(\lambda, p, n, \alpha, \beta, \gamma, \delta)\,.$$

Note that

$$\left| \frac{(zf'(z))' + pf'(z)}{f'(z)} \right| = \left| \frac{\sum\limits_{k=0}^{\infty}(p+k)(2p+k)a_{p+k}z^{p+k-1}}{-pz^{-(p+1)} + \sum\limits_{k=0}^{\infty}(p+k)a_{p+k}z^{p+k-1}} \right| \leq \frac{\sum\limits_{k=0}^{\infty}(p+k)(2p+k)a_{p+k}r^{2p+k}}{p - \sum\limits_{k=0}^{\infty}(p+k)a_{p+k}r^{2p+k}}\,.$$

Thus $\left| \frac{(zf'(z))' + pf'(z)}{f'(z)} \right| \leq p - \delta$ if

$$\sum_{k=0}^{\infty} \frac{(p+k)(3p+k-\delta)}{p(p-\delta)} a_{p+k} r^{2p+k} \leq 1\,. \tag{4.2}$$

But Theorem 1 ansures that

$$\sum_{k=0}^{\infty} \frac{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\delta)} a_{p+k} \leq 1\,. \tag{4.3}$$

In view of (4.3), it follows that (4.2) will be true if

$$\frac{(p+k)(3p+k-\delta)}{p(p-\delta)} r^{2p+k} \leq \frac{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\delta)} (k \geq; p \in N; n \in N_0) \tag{4.4}$$

or if

$$r \le \left[\frac{p(p-\delta)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)(3p+k-\delta)}\right]^{\frac{1}{2p+k}} \quad (k \ge 0; p \in N; n \in N_0)\,. \tag{4.5}$$

Setting $r(\lambda, p, n, \alpha, \beta, \gamma, \delta)$ in (4.5), the result follows. The result (4.1) is sharp with the extremal function $f(z)$ given by (2.5).

5. Closure Theorems

Theorem 4. *Let*

$$f_{p-1}(z) = \frac{1}{z^p} \tag{5.1}$$

and

$$f_{p+k}(z) = \frac{1}{z^p} + \frac{2\beta\gamma(p-\alpha)}{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)} z^{p+k} (k \ge 0; p \in N; n \in N_0). \tag{5.2}$$

Then $f(z)$ is in the class $F^n_{\lambda,p}(\alpha,\beta,\gamma)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=-1}^{\infty} \mu_{p+k} f_{p+k}(z)\,, \tag{5.3}$$

where

$$\mu_{p+k} \ge 0 \text{ and } \sum_{k=-1}^{\infty} \mu_{p+k} = 1\,.$$

Proof. Let $f(z) = \sum\limits_{k=-1}^{\infty} \mu_{p+k} f_{p+k}(z)$, where $\mu_{p+k} \ge 0$ and $\sum\limits_{k=-1}^{\infty} \mu_{p+k} = 1$. Then

$$\begin{aligned} f(z) &= \sum_{k=-1}^{\infty} \mu_{p+k} f_{p+k}(z) \\ &= \frac{1}{z^p} + \sum_{k=0}^{\infty} \mu_{p+k} \frac{2\beta\gamma(p-\alpha)}{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)} z^{p+k}\,. \end{aligned}$$

Then

$$\sum_{k=0}^{\infty} \mu_{p+k} \frac{2\beta\gamma(p-\alpha)}{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)} \cdot \frac{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)}$$
$$= \sum_{k=0}^{\infty} \mu_{p+k} = 1 - \mu_{p-1} \le 1\,,$$

which shows, that $f(z) \in F^n_{\lambda,p}(\alpha,\beta,\gamma)$.

Conversely, suppose $f(z) \in F^n_{\lambda,p}(\alpha,\beta,\gamma)$. Then

$$a_{p+k} \leq \frac{2\beta\gamma(p-\alpha)}{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)}(k \geq 0; p \in N; n \in N_0)\,.$$

Setting

$$\mu_{p+k} = \frac{2\beta\gamma(p-\alpha)}{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)}a_{p+k}\,(k \geq 0; p \in N; n \in N_0)$$

and

$$\mu_{p-1} = 1 - \sum_{k=0}^{\infty} \mu_{p+k}\ ,$$

it follows that $f(z) = \sum\limits_{k=0}^{\infty} \mu_{p+k} f_{p+k}(z)$. This completes the proof of Theorem 4.

Theorem 5. *The class* $F^n_{\lambda,p}(\alpha,\beta,\gamma)$ *is closed under convex linear combinations.*

Proof. Let each of the functions

(5.4) $$f_j(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k,j} z^{p+k}\ (a_{p+k,j} \geq 0; j = 1, 2)$$

be in the class $F^n_{\lambda,p}(\alpha,\beta,\gamma)$. It is sufficient to show that the function $h(z)$ defined by

(5.5) $$h(z) = (1-t)f_1(z) + tf_2(z)\,(0 \leq t \leq 1)$$

is also in the class $F^n_{\lambda,p}(\alpha,\beta,\gamma)$. Since

(5.6) $$h(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty}[(1-t)a_{p+k,1} + ta_{p+k,2}]z^{p+k}(0 \leq t \leq 1)\,,$$

with the aid of Theorem 1, we have

$$\begin{aligned}
&\sum_{k=0}^{\infty}(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)[(1-t)a_{p+k,1} + ta_{p+k,2}]\\
=\ &(1-t)\sum_{k=0}^{\infty}(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)a_{p+k,1} +\\
&+t\sum_{k=0}^{\infty}(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)a_{p+k,2}\\
\leq\ &(1-t)2\beta\gamma(p-\alpha) + t2\beta\gamma(p-\alpha) = 2\beta\gamma(p-\alpha),
\end{aligned}$$

which shows that $h(z) \in F^n_{\lambda,p}(\alpha,\beta,\gamma)$. Hence we have Theorem 5.

6. Integral Transforms

In this section we consider integral transforms of function in the class $F^n_{\lambda,p}(\alpha,\beta,\gamma)$.

Theorem 6. *If $f(z)$ is in the class $F^n_{\lambda,p}(\alpha,\beta,\gamma)$, then the integral transforms*

$$F_{c+p-1}(z) = c\int_0^1 u^{c+p-1} f(uz)du, \quad 0 < c < \infty \tag{6.1}$$

are in the class $F^*_p(\theta), 0 \le \theta < p$, where

$$\theta = \theta(p,\alpha,\beta,\gamma,c) = \frac{p(2p+c)(1+2\beta\gamma-\beta) - 2\beta\gamma(p-\alpha)c}{(2p+c)(1+2\beta\gamma-\beta)}\,. \tag{6.2}$$

The result is best possible for the function $f(z)$ given by (3.3).

Proof. Suppose $f(z) = \frac{1}{z^p} + \sum\limits_{k=0}^{\infty} a_{p+k} z^{p+k} \in F^n_{\lambda,p}(\alpha,\beta,\gamma)$. Then we have

$$\begin{aligned} F_{c+p-1}(z) &= c\int_0^1 u^{c+p-1} f(uz)du \\ &= \frac{1}{z^p} + \sum_{k=0}^{\infty} \frac{c}{2p+k+c} a_{p+k} z^{p+k}\,. \end{aligned}$$

In view of Corollary 2, it is sufficient to show that

$$\sum_{k=0}^{\infty} \frac{k+p}{p-\theta} \cdot \frac{c}{2p+k+c} a_{p+k} \le 1\,. \tag{6.3}$$

Since $f(z) \in F^n_{\lambda,p}(\alpha,\beta,\gamma)$, we have

$$\sum_{k=0}^{\infty} \frac{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)} a_{p+k} \le 1\,. \tag{6.4}$$

Thus (6.3) will be satisfied if

$$\frac{(k+p)c}{(p-\theta)(2p+k+c)} \le \frac{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)} \text{ for each } k\,,$$

or

$$\theta \leq \frac{p(2p+k+c)(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)-(k+p)2\beta\gamma(p-\alpha)c}{(2p+k+c)(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)}\,. \tag{6.5}$$

Since the right hand side of (6.5) is an increasing function of k, putting $k=0$ in (6.5), we get

$$\theta \leq \frac{p(2p+c)(1+2\beta\gamma-\beta)-2\beta\gamma(p-\alpha)c}{(2p+c)(1+2\beta\gamma-\beta)}\,.$$

Hence the theorem.

7. Convolution Properties

For the functions $f_j(z)(j=1,2)$ defined by (5.4) belonging to the class Σ_p^*, we denote by $(f_1 * f_2)(z)$ the Hadamard product (or the convolution) of the functions $f_1(z)$ and $f_2(z)$, that is

$$(f_1 * f_2)(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k,1}a_{p+k,2}z^{p+k}\,. \tag{7.1}$$

Theorem 7. *Let the functions $f_j(z)(j=1,2)$ defined by (5.4) be in the class $F^n_{\lambda,p}(\alpha,\beta,\gamma)$, then $(f_1 * f_2)(z) \in F^n_{\lambda,p}(\phi,\beta,\gamma)$, where*

$$\varphi = p - \frac{2\beta\gamma(p-\alpha)^2}{p(1+2\beta\gamma-\beta)}\,. \tag{7.2}$$

The result is sharp for the functions $f_j(z)(j=1,2)$ given by

$$f_j(z) = \frac{1}{z^p} + \frac{2\beta\gamma(p-\alpha)}{p(1+2\beta\gamma-\beta)}z^p\,(j=1,2;p\in N)\,. \tag{7.3}$$

Proof. Employing the technique used ealier by Schild and Silverman [13], in order to prove Theorem 7, we need to find the largest φ such that

$$\sum_{k=0}^{\infty} \frac{(k+p)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\phi)}a_{p+k,1}a_{p+k,2} \leq 1 \tag{7.4}$$

for $f_j(z) \in F^n_{\lambda,p}(\phi,\beta,\gamma)(j=1,2)$. Since $f_j(z) \in F^n_{\lambda,p}(\alpha,\beta,\gamma)(j=1,2)$, we readily see that

$$\sum_{k=0}^{\infty} \frac{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)}a_{p+k,j} \leq 1\ (j=1,2)\,. \tag{7.5}$$

Therefore, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{k=0}^{\infty} \frac{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)} \sqrt{a_{p+k,1}a_{p+k,2}} \leq 1\,. \tag{7.6}$$

This implies that we need only to show that

$$\frac{a_{p+k,1}a_{p+k,2}}{(p-\varphi)} \leq \frac{\sqrt{a_{p+k,1}a_{p+k,2}}}{(p-\alpha)} \ (k \geq 0; p \in N) \tag{7.7}$$

or, equivalently, that

$$\sqrt{a_{p+k,1}a_{p+k,2}} \leq \frac{(p-\varphi)}{(p-\alpha)} \ (k \geq 0; p \in N)\,. \tag{7.8}$$

Hence, by the inequality (7.6), it is sufficient to prove that

$$\frac{2\beta\gamma(p-\alpha)}{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)} \leq \frac{(p-\varphi)}{(p-\alpha)} \ (k \geq 0)\,. \tag{7.9}$$

It follows from (7.9) that

$$\phi \leq p - \frac{2\beta\gamma(p-\alpha)^2}{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)} \ (k \geq 0)\,. \tag{7.10}$$

Now, defining the function $\Psi(k)$ by

$$\Psi(k) = p - \frac{2\beta\gamma(p-\alpha)^2}{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)} \ (k \geq 0)\,. \tag{7.11}$$

We see that $\Psi(k)$ is an increasing function of k. Therefore, we conclude that

$$\phi \leq \Psi(0) = p - \frac{2\beta\gamma(p-\alpha)^2}{p(1+2\beta\gamma-\beta)}\,, \tag{7.12}$$

which evidently completes the proof of Theorem 7.

Using arguments similar to those in the proof of Theorem 7, we obtain the following result.

Theorem 8. *Let the function $f_1(z)$ defined by (5.4) be in the class $F^n_{\lambda,p}(\alpha_1,\beta,\gamma)$, and the function $f_2(z)$ defined by (5.4) be in the class $F^n_{\lambda,p}(\alpha_2,\beta,\gamma)$. Then $(f_1 * f_2)(z) \in F^n_{\lambda,p}(\eta,\beta,\gamma)$, where*

$$\eta = p - \frac{2\beta\gamma(p-\alpha_1)(p-\alpha_2)}{p(1+2\beta\gamma-\beta)}\,. \tag{7.13}$$

The result is sharp for the functions $f_j(z)(j = 1, 2)$ *given by*

(7.14) $$f_1(z) = \frac{1}{z^p} + \frac{2\beta\gamma(p-\alpha_1)}{p(1+2\beta\gamma-\beta)} z^p \; (p \in N)$$

and

(7.15) $$f_2(z) = \frac{1}{z^p} + \frac{2\beta\gamma(p-\alpha_2)}{p(1+2\beta\gamma-\beta)} z^p \; (p \in N)\,.$$

Theorem 9. *If* $f_1(z) = \frac{1}{z^p} + \sum\limits_{k=0}^{\infty} a_{p+k,1} z^{p+k} \in F_{\lambda,p}^n(\alpha,\beta,\gamma)$*, and* $f_2(z) = \frac{1}{z^p} + \sum\limits_{k=0}^{\infty} a_{p+k,2} z^{p+k}$ *with* $|a_{p+k,2}| \le 1; k = 0, 1, 2, ...; p \in N$*, then* $(f_1 * f_2)(z) \in F_{\lambda,p}^n(\alpha,\beta,\gamma)$.

Proof. Since

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{(p+k)(\frac{p+\lambda k}{p})^n (1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)} \, |a_{p+k,1} a_{p+k,2}| \\ = &\sum_{k=0}^{\infty} \frac{(p+k)(\frac{p+\lambda k}{p})^n (1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)} a_{p+k,1} \, |a_{p+k,2}| \\ \le &\sum_{k=0}^{\infty} \frac{(p+k)(\frac{p+\lambda k}{p})^n (1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)} \, a_{p+k,1} \; \le 1\,, \end{aligned}$$

by Theorem 1, it follows that $(f_1 * f_2)(z) \in F_{\lambda,p}^n(\alpha,\beta,\gamma)$.

Corollary 3. *If* $f_1(z) = \frac{1}{z^p} + \sum\limits_{k=0}^{\infty} a_{p+k,1} z^{p+k} \in F_{\lambda,p}^n(\alpha,\beta,\gamma)$*, and* $f_2(z) = \frac{1}{z^p} + \sum\limits_{k=0}^{\infty} a_{p+k,2} z^{p+k}$ $(0 \le a_{p+k,2} \le 1;\; k = 0, 1, 2, ...; p \in N)$*, then* $(f_1 * f_2)(z) \in F_{\lambda,p}^n(\alpha,\beta,\gamma)$.

Theorem 10. *Let the functions* $f_j(z)(j = 1, 2)$ *defined by (5.4) be in the class* $F_{\lambda,p}^n(\alpha,\beta,\gamma)$ *and*

(7.16) $$p(1+2\beta\gamma-\beta) - 4\beta\gamma(p-\alpha) \ge 0\,,$$

then the function $h(z)$ defined by

(7.17) $$h(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} (a_{p+k,1}^2 + a_{p+k,2}^2) z^{p+k}$$

belongs to the class $F^n_{\lambda,p}(\alpha,\beta,\gamma)$.

Proof. Since $f_1(z) \in F^n_{\lambda,p}(\alpha,\beta,\gamma)$, we have

$$\sum_{k=0}^{\infty} \frac{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)} a_{p+k,1} \le 1$$

and so

$$\sum_{k=0}^{\infty} \left[\frac{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)}\right]^2 a^2_{p+k,1} \le 1\,.$$

Similarly, since $f_2(z) \in F^n_{\lambda,p}(\alpha,\beta,\gamma)$, we have

$$\sum_{k=0}^{\infty} \left[\frac{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)}\right]^2 a^2_{p+k,2} \le 1\,.$$

Hence

$$\sum_{k=0}^{\infty} \frac{1}{2}\left[\frac{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)}\right]^2 (a^2_{p+k,1}+a^2_{p+k,2}) \le 1\,.$$

In view of Theorem 1, it is sufficient to show that

$$\sum_{k=0}^{\infty} \left[\frac{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)}\right] (a^2_{p+k,1}+a^2_{p+k,2}) \le 1\,. \tag{7.18}$$

Thus the inequality (7.18) will be satisfied if, for $k = 0, 1, 2, ...$

$$\frac{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)} \le \frac{1}{2}\left[\frac{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)}\right]^2, \tag{7.19}$$

or if

$$(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta) - 4\beta\gamma(p-\alpha) \ge 0\,, \tag{7.20}$$

for $k = 0, 1, 2, \dots$. The left hand side of (7.20) is an increasing function of k, hence (7.20) is satisfied for all k if

$$p(1+2\beta\gamma-\beta) - 4\beta\gamma(p-\alpha) \ge 0\,,$$

which is true by our assumetion. Hence the theorem.

Theorem 11. *Let the functions* $f_j(z)(j = 1,2)$ *defined by (5.4) be in the class* $F^n_{\lambda,p}(\alpha,\beta,\gamma)$*. Then the function* $h(z)$ *defined by (7.17) belongs to the class* $F^n_{\lambda,p}(\Psi,\beta,\gamma)$*, where*

$$\Psi = p - \frac{4\beta\gamma(p-\alpha)^2}{p(1+2\beta\gamma-\beta)} \ . \tag{7.21}$$

This result is sharp for the functions $f_j(z)(j = 1,2)$ defined by (7.3).

Proof. Noting that

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{\left[(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)\right]^2}{[2\beta\gamma(p-\alpha)]^2} a^2_{p+k,j} \\ &\le \left[\sum_{k=0}^{\infty} \frac{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)} a_{p+k,j}\right]^2 \le 1(j=1,2), \end{aligned} \tag{7.22}$$

for $f_j(z) \in F^n_{\lambda,p}(\alpha,\beta,\gamma)(j = 1,2)$, we have

$$\sum_{k=0}^{\infty} \frac{\left[(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)\right]^2}{2[2\beta\gamma(p-\alpha)]^2}(a^2_{p+k,1}+a^2_{p+k,2}) \le 1 \, . \tag{7.23}$$

Therefore, we have to find the largest Ψ such that

$$\frac{1}{(p-\Psi)} \le \frac{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)}{4\beta\gamma(p-\alpha)^2} \, (k \ge 0) \, , \tag{7.24}$$

that is, that

$$\Psi \le p - \frac{4\beta\gamma(p-\alpha)^2}{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)} \, (k \ge 0) \, . \tag{7.25}$$

Now, definig a function $H(k)$ by

$$H(k) = p - \frac{4\beta\gamma(p-\alpha)^2}{(p+k)(\frac{p+\lambda k}{p})^n(1+2\beta\gamma-\beta)} \, (k \ge 0) \ . \tag{7.26}$$

We observe that $H(k)$ is an increasing function of k. We thus conclude that

$$\Psi \leq H(0) = p - \frac{4\beta\gamma(p-\alpha)^2}{p(1+2\beta\gamma-\beta)} \tag{7.27}$$

which completes the proof of Theorem 11.

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