

# CONVERGENCE CRITERION OF INEXACT METHODS FOR OPERATORS WITH HÖLDER CONTINUOUS DERIVATIVES

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**Abstract.** Convergence criterion of the inexact methods is established for operators with hölder continuous first derivatives. An application to a special nonlinear Hammerstein integral equation of the second kind is provided.

## 1. INTRODUCTION

Let  $X$  and  $Y$  be (real or complex) Banach spaces,  $\Omega \subseteq X$  be an open subset and let  $f : \Omega \subseteq X \rightarrow Y$  be a nonlinear operator with the continuous Frechét derivative denoted by  $f'$ . Finding solutions of the nonlinear operator equation

$$(1.1) \quad f(x) = 0$$

in Banach spaces is a very general subject which is widely used in both theoretical and applied areas of mathematics. The most important method to find the approximation of a solution of (1.1) is Newton's method which takes the following form:

$$(1.2) \quad x_{n+1} = x_n - f'(x_n)^{-1}f(x_n), \quad n = 0, 1, 2, \dots$$

One of the important results on Newton's method is the well-known Kantorovich theorem (cf. [10]) which guarantees convergence of Newton's sequence to a solution under very mild conditions. Recent progress on convergence of Newton's method is

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referred to [6, 9, 18, 19] and [20]. Newton's method, as expressed in (1.2), requires the computation of  $f'(x_n)$  and the exact solution of the linear equation:

$$(1.3) \quad f'(x_n)(x_{n+1} - x_n) = -f(x_n),$$

which sometimes make Newton's method inefficient from the point of view of practical calculation. To avoid the drawback of computing the derivative  $f'$ , a number of Newton-like methods have been proposed (cf. [2, 22]). While using linear iterative methods to approximate the solution of (1.3) instead of solving it exactly can also reduce some of the costs of Newton's method which was studied extensively and applied in [5, 1, 15, 21] (such a variant is called the inexact Newton method). Using approximations of the derivative and the solution of (1.3) simultaneously instead of their exact values yields the inexact methods which avoid the both drawbacks mentioned above, see for example [8, 12, 13]). In general, the inexact method has the following general form:

**Algorithm A**[( $B_n, r_n$ );  $x_0$ ]. For  $n = 0$  and a given initial guess  $x_0$  until convergence do

(1) For the residual  $r_n$  and the iteration  $x_n$ , find the step  $s_n$  satisfying

$$(1.4) \quad B_n s_n = -f(x_n) + r_n.$$

(2)  $x_{n+1} = x_n + s_n$ .

(3) set  $n = n + 1$  and turn to step 1.

Here  $\{B_n\}$  is a sequence of invertible operators from  $X$  to  $Y$  while  $\{r_n\}$  is a sequence of elements in  $Y$  (depending on  $\{x_n\}$  in general).

In the special case when  $B_n = f'(x_n)$  for each  $n$ , **Algorithm A**[( $B_n, r_n$ );  $x_0$ ] reduces to the inexact Newton method. As is well known, the convergence behavior of the inexact methods depends on the residuals  $\{r_n\}$ . Several authors (cf. [5, 21]) have analyzed the local convergence behavior in some manner such that the stopping relative residuals  $\{r_n\}$  satisfy  $\|r_n\|/\|f(x_n)\| \leq \eta_n$ . While Morini considered in [12] the relative residual controls:

$$(1.5) \quad \|P_n r_n\| \leq \theta_n \|P_n f(x_n)\| \quad \text{for each } n \in \mathbb{N},$$

and obtained the local linear convergence results of the inexact methods. Such variety of (1.5) leads to an advantage of a relaxation on the forcing terms  $\{\theta_n\}$ .

Motivated by the inexact method for the inverse eigenvalue problem presented in [3, 4], we considered in [17] the residuals  $\{r_n\}$  satisfying

$$(1.6) \quad \|P_n r_n\| \leq \theta_n \|P_n f(x_n)\|^{1+\beta} \quad \text{for each } n \in \mathbb{N}.$$

The local convergence behavior of the inexact methods with the residuals  $\{r_n\}$  satisfying (1.6) was analyzed in [17]. In particular, the results are successfully applied to show the local superlinear convergence of the inexact Newton-like method for the inverse eigenvalue problem.

However, the semi-local convergence of the inexact methods has been rare explored. In the present paper, we will focus our study on the semi-local convergence of the inexact methods. Assume that the residuals satisfy (1.6) and that  $f'(x_0)^{-1}f'$  satisfies the Hölder condition around  $x_0$ . Then we will use the majorizing function technique to establish a convergence criterion for the inexact method **Algorithm A** $[(B_n, r_n); x_0]$ , which recover the result in [11] and the well-known Kantorovich theorem for Newton's method. We also present an application of the main results to a special nonlinear Hammerstein integral equation of the second kind.

## 2. PRELIMINARIES

Let  $X$  and  $Y$  be (real or complex) Banach spaces. Throughout the whole paper, we use  $\mathbf{B}(x, r)$  and  $\overline{\mathbf{B}}(x, r)$  to stand for the open and closed ball in  $X$  with center  $x$  and radius  $r > 0$  respectively, and assume that  $0 < \beta \leq p \leq 1$ . Let  $a$  and  $b$  be positive constants. We define two real-valued functions  $\varphi_\beta$  and  $\psi_\beta$  respectively by

$$(2.1) \quad \varphi_\beta(t) = \frac{2^{1-\beta}a}{1+\beta}t^{1+\beta} - t + b \quad \text{for each } t \geq 0$$

and

$$(2.2) \quad \psi_p(t) = \frac{a}{1+p}t^{1+p} - t + b \quad \text{for each } t \geq 0.$$

The following lemma describes some properties about the zeros of the function  $\varphi_\beta$ .

**Lemma 2.1.**  $\varphi_\beta$  is decreasing on  $[0, (2^{1-\beta}a)^{-\frac{1}{\beta}}]$  but increasing on  $[(2^{1-\beta}a)^{-\frac{1}{\beta}}, +\infty)$ . Furthermore, if

$$(2.3) \quad 2^{1-\beta}ab^\beta \leq \left(\frac{\beta}{1+\beta}\right)^\beta,$$

then equation  $\varphi_\beta(t) = 0$  has two solutions  $t^*, t^{**}$  in  $(0, +\infty)$  satisfying

$$(2.4) \quad b < t^* \leq \frac{1+\beta}{\beta}b \leq t^{**}.$$

*Proof.* The first assertion is trivial because the derivative of  $\varphi_\beta$  is

$$\varphi'_\beta(t) = 2^{1-\beta}at^\beta - 1$$

and the unique zero in  $[0, +\infty)$  of  $\varphi'_\beta$  is  $(2^{1-\beta}a)^{-\frac{1}{\beta}}$ . Now suppose that (2.3) holds. Then

$$\varphi_\beta((2^{1-\beta}a)^{-\frac{1}{\beta}}) = -\frac{\beta}{1+\beta}(2^{1-\beta}a)^{-\frac{1}{\beta}} + b \leq 0.$$

Since  $\varphi_\beta(0) = b > 0$  and  $\varphi_\beta(+\infty) = +\infty$ , equation  $\varphi_\beta(t) = 0$  has two solutions  $t^* \leq t^{**}$  in  $(0, +\infty)$ . It remains to prove (2.4). Noting that  $\varphi_\beta(b) > 0$  and that (2.3) implies  $b \leq (2^{1-\beta}a)^{-\frac{1}{\beta}}$ , we have

$$(2.5) \quad b < t^* \leq (2^{1-\beta}a)^{-\frac{1}{\beta}} \leq t^{**}.$$

Thus,

$$t^* = \varphi_\beta(t^*) + t^* = \frac{2^{1-\beta}at^{*\beta}}{1+\beta}t^* + b \leq \frac{t^*}{1+\beta} + b.$$

It follows that

$$(2.6) \quad t^* \leq \frac{1+\beta}{\beta}b.$$

Other hand, by (2.3), one has

$$\varphi_\beta\left(\frac{1+\beta}{\beta}b\right) = \frac{b}{\beta} \left( 2^{1-\beta}ab^\beta \left( \frac{1+\beta}{\beta} \right)^\beta - 1 \right) \leq 0.$$

Hence,  $\frac{1+\beta}{\beta}b \leq t_{**}$ . Combining this with (2.5) and (2.6), one sees that (2.4) holds and completes the proof.  $\blacksquare$

Let  $\{t_n\}$  denote the sequence generated by the Newton-like method with initial point  $t_0 = 0$ , which is defined by

$$(2.7) \quad t_{n+1} = t_n - \frac{\varphi_\beta(t_n)}{\psi'_p(t_n)} \quad \text{for each } n = 0, 1, \dots$$

The convergence property of the sequence  $\{t_n\}$ , which will play a key role, is described in the following lemma.

**Lemma 2.2.** *Suppose that (2.3) holds. Let  $t^*$  be the smaller nonnegative solution of equation  $\varphi_\beta(t) = 0$ . Suppose also that*

$$(2.8) \quad \left( \frac{1+\beta}{\beta}b \right)^{p-\beta} \leq 1.$$

Then the sequence  $\{t_n\}$  generated by (2.7) satisfies that

$$(2.9) \quad t_{n+1} - t_n \leq \frac{b}{\beta} \quad \text{and} \quad t_n < \frac{1+\beta}{\beta}b \quad \text{for each } n \in \mathbb{N}.$$

Consequently,  $\{t_n\}$  converges increasingly to  $t^*$ .

*Proof.* Suppose that (2.3) holds. Then, by Lemma 2.1, equation  $\varphi_\beta(t) = 0$  has two solutions  $t^* \leq t^{**}$  satisfying

$$(2.10) \quad b < t^* \leq \frac{1+\beta}{\beta}b \leq t^{**}.$$

Below we will verify that for each  $n \in \mathbb{N}$ ,

$$(2.11) \quad t_{n+1} > t_n \quad \text{and} \quad t_n < t^*.$$

Granting this, one sees that sequence  $\{t_n\}$  is increasing and

$$(2.12) \quad t_{n+1} - t_n < t^* - t_1 \leq \frac{1+\beta}{\beta}b - b = \frac{b}{\beta} \quad \text{for each } n = 1, 2, \dots$$

thanks to (2.10). Hence (2.9) follows.

To show (2.11), note that  $t_0 < t_1 = b < t^*$ , which means (2.11) holds for  $n = 0$ . Assume that  $t_1 < t_2 < \dots < t_n < t^*$ . Then, by (2.10) and (2.8), one has

$$t_n^{p-\beta} \leq (t^*)^{p-\beta} \leq \left( \frac{1+\beta}{\beta}b \right)^{p-\beta} \leq 1;$$

hence,

$$(2.13) \quad -\psi'_p(t_n) = 1 - at_n^p \geq 1 - 2^{1-\beta}at_n^\beta = -\varphi'_\beta(t_n) > 0.$$

Moreover, the function  $N$  defined by

$$N(t) := t - \frac{\varphi_\beta(t)}{\varphi'_\beta(t)} \quad \text{for each } t \geq 0$$

is monotonically increasing on  $[0, t^*]$ . It follows that

$$(2.14) \quad t_n < t_n - \frac{\varphi_\beta(t_n)}{\psi'_p(t_n)} = t_{n+1} \leq t_n - \frac{\varphi_\beta(t_n)}{\varphi'_\beta(t_n)} < t^*$$

because of (2.13) and  $\varphi_\beta(t^*) = 0$ . The proof is complete. ■

## 3. CONVERGENCE ANALYSIS

Recall that  $f: \Omega \longrightarrow Y$  is an operator with the continuous Frechét derivative denoted by  $f'$ . Let  $x_0 \in \Omega$  be such that the inverse  $f'(x_0)^{-1}$  exists. In the remainder, we assume that the residuals  $\{r_n\}$  satisfy (1.6) and that  $f'(x_0)^{-1}f'$  satisfies the following  $(L, p)$ -Hölder condition:

$$(3.1) \quad \|f'(x_0)^{-1}(f'(x) - f'(y))\| \leq L\|x - y\|^p \quad \text{for all } x, y \in \mathbf{B}(x_0, r).$$

Moreover, we assume that  $\mathbf{B}(x_0, L^{-\frac{1}{p}}) \subseteq \Omega$  and adopt the convention that  $\frac{1}{0} = +\infty$  throughout the whole paper. Then we have the following lemma, which can be proved by Banach's Lemma with a standard argument, see for example [18].

**Lemma 3.1.** *Let  $r \leq L^{-\frac{1}{p}}$  and let  $x \in \mathbf{B}(x_0, r)$ . Then  $f'(x)$  is invertible and satisfies that*

$$(3.2) \quad \|f'(x)^{-1}f'(x_0)\| \leq (1 - L\|x - x_0\|^p)^{-1}.$$

To estimate the radius of the convergence ball of the inexact methods around solution  $x^*$  of (1.1), Morini introduced the quantity  $\hat{v} := \sup_{n \geq 0} \theta_n \text{cond}(\|P_n B_n\|)$ , and took  $B_n = \hat{B}(x_n)$  in [12], where  $\hat{B}(\cdot)$  is an approximation to  $f'(\cdot)$  around  $x^*$  and satisfies

$$\|\hat{B}(\cdot)^{-1}f'(\cdot) - I\| \leq \tau_1 \quad \text{and} \quad \|\hat{B}(\cdot)^{-1}f'(\cdot)\| \leq \tau_2$$

for some nonnegative constants  $\tau_1$  and  $\tau_2$ . Note that the quantity  $\hat{v}$  is closely related to the sequence  $\{x_n\}$ .

In order to obtain a convergence criterion depending mainly on the initial point  $x_0$ , we use the quantity

$$(3.3) \quad v := \sup_{n \geq 0} \theta_n \|f'(x_0)^{-1}P_n^{-1}\| \|P_n f'(x_0)\|^{1+\beta}.$$

Let  $\omega_1 \geq 1$ ,  $\omega_2 \geq 0$  and assume that the sequence  $\{B_n\}$  satisfies

$$(C1) \quad \|B_n^{-1}f'(x_n)\| \leq \omega_1 \quad \text{for each } n \in \mathbb{N},$$

$$(C2) \quad \|f'(x_0)^{-1}(B_n - f'(x_n))\| \leq \omega_2 \|f'(x_0)^{-1}f'(x_n)\|^\beta \quad \text{for each } n \in \mathbb{N}.$$

Let

$$(3.4) \quad \alpha := \|f'(x_0)^{-1}f(x_0)\|,$$

and write

$$(3.5) \quad a = \omega_1(1 + v^{\frac{1}{1+\beta}})(L + (1 + \beta)(\omega_2 + v)), \quad b = (1 + v^{\frac{1}{1+\beta}})\omega_1\alpha.$$

Recall that  $\{t_n\}$  is the sequence generated by the Newton-like method (2.7). Thus, we obtain the main result of this paper.

**Theorem 3.1.** *Suppose that  $f'(x_0)^{-1}f'$  satisfies the  $(L,p)$ -Hölder condition (3.1) with  $r = \frac{1+\beta}{\beta}b$ . Suppose that*

$$(3.6) \quad \alpha \leq \begin{cases} \frac{\beta}{\omega_1(1+\beta)(1+v^{\frac{1}{1+\beta}})} \min \left\{ 1, 2^{1-\frac{1}{\beta}} a^{-\frac{1}{\beta}} \right\} & 0 < \beta < p, \\ \beta \min \left\{ \frac{1}{v^{\frac{1}{1+\beta}}}, \frac{2^{1-\frac{1}{\beta}} a^{-\frac{1}{\beta}}}{\omega_1(1+\beta)(1+v^{\frac{1}{1+\beta}})} \right\} & \beta = p. \end{cases}$$

*Then the sequence  $\{x_n\}$  generated by the inexact method **Algorithm A**[( $B_n, r_n$ );  $x_0$ ] converges to a solution  $x^*$  of (1.1) satisfying*

$$(3.7) \quad \|x_n - x^*\| \leq t^* - t_n \quad \text{for each } n \in \mathbb{N}.$$

*Proof.* In view of the definition of  $b$ , one sees that

$$(3.8) \quad 2^{1-\beta}ab^\beta \leq \left( \frac{\beta}{1+\beta} \right)^\beta \iff \alpha \leq \frac{\beta 2^{1-\frac{1}{\beta}} a^{-\frac{1}{\beta}}}{\omega_1(1+\beta)(1+v^{\frac{1}{1+\beta}})}$$

and

$$(3.9) \quad \frac{1+\beta}{\beta}b \leq 1 \iff \alpha \leq \frac{\beta}{\omega_1(1+\beta)(1+v^{\frac{1}{1+\beta}})}.$$

Furthermore, it is clear that

$$\alpha \leq \frac{\beta}{\omega_1(1+\beta)(1+v^{\frac{1}{1+\beta}})} \implies \frac{1}{\beta}v^{\frac{1}{1+\beta}}\alpha \leq 1.$$

Hence, (3.6) holds if and only if

$$(3.10) \quad \left( \frac{1+\beta}{\beta}b \right)^{p-\beta} \leq 1, \quad \frac{1}{\beta}v^{\frac{1}{1+\beta}}\alpha \leq 1 \quad \text{and} \quad 2^{1-\beta}ab^\beta \leq \left( \frac{\beta}{1+\beta} \right)^\beta.$$

Now suppose that (3.6) holds. Then (2.3) holds and Lemma 2.2 is applicable to concluding that

$$(3.11) \quad t_{n+1} - t_n \leq \frac{b}{\beta} \quad \text{and} \quad t_n < \frac{1+\beta}{\beta}b \quad \text{for each } n \in \mathbb{N}.$$

To complete the proof, it is sufficient to verify that

$$(3.12) \quad \frac{\omega_1(1+v^{\frac{1}{1+\beta}})}{1-Lt_n^p} \|f'(x_0)^{-1}f(x_n)\| \leq t_{n+1} - t_n$$

and

$$(3.13) \quad \|x_{n+1} - x_n\| \leq t_{n+1} - t_n$$

hold for each  $n \in \mathbb{N}$ . We will proceed by mathematical induction. (3.12) is clear for  $n = 0$  thanks to definitions of  $\alpha$  and  $b$ . Note that, by (1.6), if  $n \in \mathbb{N}$  is such that  $x_n$  is well-defined then

$$(3.14) \quad \begin{aligned} \|f'(x_0)^{-1}r_n\| &\leq \theta_n \|f'(x_0)^{-1}P_n^{-1}\| \|P_n f(x_n)\|^{1+\beta} \\ &\leq v \|f'(x_0)^{-1}f(x_n)\|^{1+\beta}. \end{aligned}$$

It follows from (3.14), (3.10) and definition of  $b$  that

$$(3.15) \quad \begin{aligned} \|x_1 - x_0\| &= \|-B_0^{-1}f(x_0) + B_0^{-1}r_0\| \leq \omega_1(\alpha + v\alpha^{1+\beta}) \\ &\leq (1 + v^{\frac{1}{1+\beta}})\omega_1\alpha = t_1 - t_0, \end{aligned}$$

that is, (3.13) holds for  $n = 0$ . Assume now that (3.12) and (3.13) hold for all  $n \leq m-1$ . We have to prove (3.12) and (3.13) hold for  $n = m$ . For this end, we apply **Algorithm A**  $[(B_n, r_n); x_0]$  to get that

$$\begin{aligned} f(x_m) &= f(x_m) - f(x_{m-1}) - B_{m-1}(x_m - x_{m-1}) + r_{m-1} \\ &= \int_0^1 f'(x_{m-1}^\tau) d\tau (x_m - x_{m-1}) - B_{m-1}(x_m - x_{m-1}) + r_{m-1}, \end{aligned}$$

where  $x_n^\tau = x_n + \tau(x_{n+1} - x_n)$  for each  $0 \leq \tau \leq 1$ . Hence,

$$(3.16) \quad \begin{aligned} &\|f'(x_0)^{-1}f(x_m)\| \\ &\leq \left\| f'(x_0)^{-1} \int_0^1 (f'(x_{m-1}^\tau) - f'(x_{m-1})) d\tau (x_m - x_{m-1}) \right\| \\ &\quad + \|f'(x_0)^{-1}(B_{m-1} - f'(x_{m-1}))(x_m - x_{m-1})\| + \|f'(x_0)^{-1}r_{m-1}\|. \end{aligned}$$

Below we will show that

$$(3.17) \quad \begin{aligned} &\left\| f'(x_0)^{-1} \int_0^1 (f'(x_{m-1}^\tau) - f'(x_{m-1})) (x_m - x_{m-1}) d\tau \right\| \\ &\leq \frac{L}{1+p} (t_m - t_{m-1})^{1+\beta}, \end{aligned}$$

$$(3.18) \quad \|f'(x_0)^{-1}(B_{m-1} - f'(x_{m-1}))(x_m - x_{m-1})\| \leq \omega_2(t_m - t_{m-1})^{1+\beta}$$

and

$$(3.19) \quad \|f'(x_0)^{-1}r_{m-1}\| \leq v(t_m - t_{m-1})^{1+\beta}.$$



Since (3.13) holds for all  $n \leq m-1$ , one has that

$$\begin{aligned}
 (3.20) \quad \|x_{m-1}^\tau - x_0\| &\leq \sum_{i=0}^{m-2} \|x_{i+1} - x_i\| + \tau \|x_m - x_{m-1}\| \\
 &\leq \sum_{i=0}^{m-2} (t_{i+1} - t_i) + \tau(t_m - t_{m-1}).
 \end{aligned}$$

Hence, by (3.11),

$$(3.21) \quad \|x_{m-1}^\tau - x_0\| \leq \tau t_m + (1 - \tau)t_{m-1} < \frac{1 + \beta}{\beta} b.$$

In particular,

$$(3.22) \quad \|x_m - x_0\| \leq t_m < \frac{1 + \beta}{\beta} b \quad \text{and} \quad \|x_{m-1} - x_0\| \leq t_{m-1} < \frac{1 + \beta}{\beta} b.$$

Then, (3.1) is applicable and

$$\begin{aligned}
 (3.23) \quad &\left\| f'(x_0)^{-1} \int_0^1 (f'(x_{m-1}^\tau) - f'(x_{m-1})) (x_m - x_{m-1}) d\tau \right\| \\
 &\leq \int_0^1 L \|x_{m-1}^\tau - x_{m-1}\|^p \|x_m - x_{m-1}\| d\tau \\
 &= \frac{L}{1 + p} \|x_m - x_{m-1}\|^{1+p} \\
 &\leq \frac{L}{1 + p} (t_m - t_{m-1})^{1+p},
 \end{aligned}$$

where the last inequality holds because of (3.13) (with  $n = m-1$ ). By (3.10),

$$\left(\frac{b}{\beta}\right)^{p-\beta} \leq \left(\frac{1 + \beta}{\beta} b\right)^{p-\beta} \leq 1.$$

This together with (3.11) gives that,

$$(3.24) \quad (t_m - t_{m-1})^{1+p} \leq (t_m - t_{m-1})^{1+\beta} \left(\frac{b}{\beta}\right)^{p-\beta} \leq (t_m - t_{m-1})^{1+\beta};$$

and (3.17) follows. To show (3.18) and (3.19), we note by (3.12) (with  $n = m-1$ ) that

$$\|f'(x_0)^{-1} f(x_{m-1})\| \leq t_m - t_{m-1}.$$

Thus, we have that

$$\begin{aligned} & \|f'(x_0)^{-1}(B_{m-1} - f'(x_{m-1}))(x_m - x_{m-1})\| \\ & \leq \omega_2 \|f'(x_0)^{-1}f(x_{m-1})\|^\beta \|x_m - x_{m-1}\| \\ & \leq \omega_2 (t_m - t_{m-1})^{1+\beta} \end{aligned}$$

thanks to **(C2)** and (3.13) (with  $n = m - 1$ ), and

$$(3.25) \quad \|f'(x_0)^{-1}r_{m-1}\| \leq v \|f'(x_0)^{-1}f(x_{m-1})\|^{1+\beta} \leq v(t_m - t_{m-1})^{1+\beta}$$

thanks to (3.14). Therefore, (3.18) and (3.19) are proved. Combining (3.16)-(3.19), one has that

$$(3.26) \quad \|f'(x_0)^{-1}f(x_m)\| \leq \left( \frac{L}{1+p} + \omega_2 + v \right) (t_m - t_{m-1})^{1+\beta}.$$

Thus, in view of definition of  $a$ ,

$$(3.27) \quad \frac{\omega_1(1+v^{\frac{1}{1+\beta}})}{1-Lt_m^p} \|f'(x_0)^{-1}f(x_m)\| \leq \frac{a(t_m - t_{m-1})^{1+\beta}}{(1+\beta)(1-Lt_m^p)}.$$

Moreover, by (3.10),

$$(3.28) \quad at_m^p < a \left( \frac{1+\beta}{\beta} b \right)^p \leq a \left( \frac{1+\beta}{\beta} b \right)^\beta \leq 2^{\beta-1} \leq 1.$$

It follows from (3.27) and the fact that  $a \geq L$  that

$$(3.29) \quad \frac{\omega_1(1+v^{\frac{1}{1+\beta}})}{1-Lt_m^p} \|f'(x_0)^{-1}f(x_m)\| \leq \frac{a(t_m - t_{m-1})^{1+\beta}}{(1+\beta)(1-at_m^p)}.$$

Since

$$(3.30) \quad \frac{\omega_1(1+v^{\frac{1}{1+\beta}})}{1-Lt_1^p} \|f'(x_0)^{-1}f(x_1)\| \leq \frac{2^{1-\beta}at_1^{1+\beta}}{(1+\beta)(1-at_1^p)} = -\frac{\varphi_\beta(t_1)}{\psi'_p(t_1)} = t_2 - t_1,$$

(3.12) is seen to hold in the case when  $m = 1$ . For the case when  $m > 1$ , as

$$\frac{t^{1+\beta}}{1+\beta} + t \leq 2^{1-\beta} \frac{(1+t)^{1+\beta} - 1}{1+\beta} \quad \text{for each } t \geq 0,$$

we have that

$$\begin{aligned} (3.31) \quad & \frac{1}{1+\beta} \left( \frac{t_m - t_{m-1}}{t_{m-1}} \right)^{1+\beta} + \frac{t_m - t_{m-1}}{t_{m-1}} \\ & \leq \frac{2^{1-\beta}}{1+\beta} \left( \left( 1 + \frac{t_m - t_{m-1}}{t_{m-1}} \right)^{1+\beta} - 1 \right). \end{aligned}$$

Note by (3.10) and (3.11) that

$$(3.32) \quad -at_{m-1}^p = -at_{m-1}^\beta t_{m-1}^{p-\beta} \geq -at_{m-1}^\beta \left( \frac{1+\beta}{\beta} b \right)^{p-\beta} \geq -at_{m-1}^\beta.$$

It follows from (3.31) and (3.32) that

$$(3.33) \quad \begin{aligned} & \frac{a}{1+\beta} (t_m - t_{m-1})^{1+\beta} \\ &= at_{m-1}^{1+\beta} \left( \frac{1}{1+\beta} \left( \frac{t_m - t_{m-1}}{t_{m-1}} \right)^{1+\beta} + \frac{t_m - t_{m-1}}{t_{m-1}} \right) - at_{m-1}^\beta (t_m - t_{m-1}) \\ &\leq \frac{2^{1-\beta} at_m^{1+\beta}}{1+\beta} - \frac{2^{1-\beta} at_{m-1}^{1+\beta}}{1+\beta} - at_{m-1}^p (t_m - t_{m-1}) \\ &= \varphi_\beta(t_m) - \varphi_\beta(t_{m-1}) - \psi'_p(t_{m-1})(t_m - t_{m-1}). \end{aligned}$$

This together with (3.29) implies that

$$(3.34) \quad \begin{aligned} & \frac{\omega_1(1+v^{\frac{1}{1+\beta}})}{1-Lt_m^p} \|f'(x_0)^{-1}f(x_m)\| \\ &\leq -\frac{\varphi_\beta(t_m) - \varphi_\beta(t_{m-1}) - \psi'_p(t_{m-1})(t_m - t_{m-1})}{\psi'_p(t_m)} \\ &= t_{m+1} - t_m, \end{aligned}$$

and (3.12) holds for  $n = m$ . To verify (3.13) holds for  $n = m$ , we note by (3.10) and definition of  $a$  that

$$(3.35) \quad \left( \frac{1+\beta}{\beta} b \right)^p \leq \left( \frac{1+\beta}{\beta} b \right)^\beta \leq 2^{\beta-1} a^{-1} \leq L^{-1}.$$

Hence,

$$(3.36) \quad r = \frac{1+\beta}{\beta} b \leq L^{-\frac{1}{p}}.$$

Thus, by (3.22), Lemma 3.1 is applicable to concluding that  $f'(x_m)^{-1}$  exists and

$$(3.37) \quad \|f'(x_m)^{-1}f'(x_0)\| \leq \frac{1}{1-L\|x_m - x_0\|^p} \leq \frac{1}{1-Lt_m^p}.$$

Combining this with **(C1)** and (3.14), we get that

$$(3.38) \quad \begin{aligned} \|x_{m+1} - x_m\| &\leq \|B_m^{-1}f'(x_m)\| \|f'(x_m)^{-1}f'(x_0)\| \\ &\leq \frac{\omega_1}{1-Lt_m^p} (\|f'(x_0)^{-1}f(x_m)\| + \|f'(x_0)^{-1}r_m\|) \\ &\leq \frac{\omega_1}{1-Lt_m^p} (\|f'(x_0)^{-1}f(x_m)\| + v\|f'(x_0)^{-1}f(x_m)\|^{1+\beta}). \end{aligned}$$

Since  $\omega_1 \geq 1$  and  $b = (1 + v^{\frac{1}{1+\beta}})\omega_1\alpha$ , it follows from (3.12) (with  $n = m$ ), (3.11) and (3.10) that

$$\begin{aligned} v^{\frac{1}{1+\beta}} \|f'(x_0)^{-1} f(x_m)\| &\leq \frac{v^{\frac{1}{1+\beta}} (1 - Lt_m^p)}{\omega_1 (1 + v^{\frac{1}{1+\beta}})} (t_{m+1} - t_m) \\ &\leq \frac{v^{\frac{1}{1+\beta}}}{\omega_1 (1 + v^{\frac{1}{1+\beta}})} \left(\frac{b}{\beta}\right) = \frac{1}{\beta} v^{\frac{1}{1+\beta}} \alpha \leq 1, \end{aligned}$$

which implies that  $v \|f'(x_0)^{-1} f(x_m)\|^{1+\beta} \leq v^{\frac{1}{1+\beta}} \|f'(x_0)^{-1} f(x_m)\|$ . This in turn together with (3.38) implies that

$$(3.39) \quad \|x_{m+1} - x_m\| \leq \frac{\omega_1 (1 + v^{\frac{1}{1+\beta}})}{1 - Lt_m^p} \|f'(x_0)^{-1} f(x_m)\| \leq t_{m+1} - t_m,$$

where we have used (3.12) just proved for  $n = m$ . Therefore, (3.13) holds for  $n = m$  and the proof is complete.  $\blacksquare$

In particular, in the case when  $B_n = f'(x_n)$  for each  $n \in \mathbb{N}$ , one has that conditions (C1) and (C2) are satisfied with  $\omega_1 = 1$  and  $\omega_2 = 0$ . Hence,  $a = (1 + v^{\frac{1}{1+\beta}})(L + (1 + \beta)v)$  and  $b = (1 + v^{\frac{1}{1+\beta}})\alpha$ . Consequently, the following corollary for the inexact Newton method results directly from Theorem 3.1.

**Corollary 3.1.** *Suppose that  $f'(x_0)^{-1} f'$  satisfies the  $(L, p)$ -Hölder condition (3.1) with  $r = \frac{1+\beta}{\beta}b$ . Suppose that*

$$\alpha \leq \begin{cases} \frac{\beta}{(1 + \beta)(1 + v^{\frac{1}{1+\beta}})} \min \left\{ 1, 2^{1-\frac{1}{\beta}} a^{-\frac{1}{\beta}} \right\} & 0 < \beta < p, \\ \beta \min \left\{ \frac{1}{v^{\frac{1}{1+\beta}}}, \frac{2^{1-\frac{1}{\beta}} a^{-\frac{1}{\beta}}}{(1 + \beta)(1 + v^{\frac{1}{1+\beta}})} \right\} & \beta = p. \end{cases}$$

*Then the sequence  $\{x_n\}$  generated by the inexact Newton method converges to a solution  $x^*$  of (1.1) satisfying*

$$\|x_n - x^*\| \leq t^* - t_n \quad \text{for each } n \in \mathbb{N}.$$

Furthermore, taking  $v = 0$  in Corollary 3.1, one has the following corollary for Newton's method which includes the corresponding result in [11] for  $\beta = p$  as a special case.

**Corollary 3.2.** *Suppose that  $f'(x_0)^{-1} f'$  satisfies the  $(L, p)$ -Hölder condition*

(3.1) with  $r = \frac{1+\beta}{\beta}\alpha$ . Suppose that

$$\alpha \leq \begin{cases} \frac{\beta}{1+\beta} \min \left\{ 1, 2^{1-\frac{1}{\beta}} L^{-\frac{1}{\beta}} \right\} & 0 < \beta < p, \\ \frac{\beta}{1+\beta} 2^{1-\frac{1}{\beta}} L^{-\frac{1}{\beta}} & \beta = p. \end{cases}$$

Then the sequence  $\{x_n\}$  generated by Newton's method converges to a solution  $x^*$  of (1.1) satisfying

$$\|x_n - x^*\| \leq t^* - t_n \quad \text{for each } n \in \mathbb{N}.$$

Consider the special case  $p = \beta = 1$ . Then  $\{t_n\}$  defined by (2.7) reduces to the Newton sequence with initial point  $t_0 = 0$ ; in particular, one has that (cf. [18])

$$(3.40) \quad t^* - t_n = \frac{\lambda^{2^n-1} - \lambda^{2^n}}{1 - \lambda^{2^n}} t^* \quad \text{for each } n \in \mathbb{N},$$

where

$$(3.41) \quad t^* = \frac{1 - \sqrt{1 - 2ab}}{a}, \quad \lambda = \frac{1 - \sqrt{1 - 2ab}}{1 + \sqrt{1 - 2ab}}.$$

Thus applying Corollary 3.1, we have the following corollary.

**Corollary 3.3.** Suppose that  $f'(x_0)^{-1}f'$  satisfies the Lipschitz condition on  $B(x_0, 2b)$  with the Lipschitz constant  $L$ . Suppose that  $\beta = 1$  and that

$$\alpha \leq \min \left\{ \frac{1}{\sqrt{v}}, \frac{1}{2(1 + \sqrt{v})^2(L + 2v)} \right\}.$$

Then the sequence  $\{x_n\}$  generated by the inexact Newton method converges to a solution  $x^*$  of (1.1) satisfying

$$\|x_n - x^*\| \leq \frac{\lambda^{2^n-1} - \lambda^{2^n}}{1 - \lambda^{2^n}} t^* \quad \text{for each } n \in \mathbb{N},$$

where

$$t^* = \frac{1 - \sqrt{1 - 2(1 + \sqrt{v})^2(L + 2v)\alpha}}{(1 + \sqrt{v})(L + 2v)}, \quad \lambda = \frac{1 - \sqrt{1 - 2(1 + \sqrt{v})^2(L + 2v)\alpha}}{1 + \sqrt{1 - 2(1 + \sqrt{v})^2(L + 2v)\alpha}}.$$

## 4. APPLICATION TO A NONLINEAR INTEGRAL EQUATION OF HAMMERSTEIN TYPE

In this section, we provide an application of the main result to a special nonlinear Hammerstein integral equation of the second kind (cf. [14]). Letting  $\mu \in \mathbb{R}$  and  $p \in (0, 1]$ , we consider

$$(4.1) \quad x(s) = l(s) + \int_a^b G(s, t)[x(t)^{1+p} + \mu x(t)]dt, \quad s \in [a, b],$$

where  $l$  is a continuous function such that  $l(s) > 0$  for all  $s \in [a, b]$  and the kernel  $G$  is a non-negative continuous function on  $[a, b] \times [a, b]$ . This kind of nonlinear Hammerstein integral equation has been already studied by many authors, see for example [6, 7] and etc.

Note that if  $G$  is the Green function defined by

$$(4.2) \quad G(s, t) = \begin{cases} \frac{(b-s)(t-a)}{b-a} & t \leq s, \\ \frac{(s-a)(b-t)}{b-a} & s \leq t, \end{cases}$$

equation (4.1) is equivalent to the following boundary value problem (cf. [16]):

$$\begin{cases} x'' = -x^{1+p} - \mu x \\ x(a) = v(a), \quad x(b) = v(b). \end{cases}$$

To apply Theorem 3.1, let  $X = Y = C[a, b]$ , the Banach space of real-valued continuous functions on  $[a, b]$  with the uniform norm. Let  $\mathbf{Q}$  denote the set of all rationals  $p \in (0, 1]$  such that  $p = \frac{u}{q}$  for some odd number  $q$  and positive integer  $u$ . Let

$$\Omega_p = \begin{cases} \{x \in C[a, b] : x(s) > 0, s \in [a, b]\} & p \in (0, 1] \setminus \mathbf{Q}, \\ C[a, b] & p \in \mathbf{Q}. \end{cases}$$

Define  $f : \Omega_p \rightarrow C[a, b]$  by

$$(4.3) \quad [f(x)](s) = x(s) - l(s) - \int_a^b G(s, t)[x(t)^{1+p} + \mu x(t)]dt, \quad s \in [a, b].$$

Then solving equation (4.1) is equivalent to solving equation (1.1) with  $f$  being defined by (4.3).

We start by calculating the parameter  $\alpha$  in the study. Firstly, we have

$$[f'(x)u](s) = u(s) - \int_a^b G(s, t)[(1+p)x(t)^p + \mu]u(t)dt, \quad s \in [a, b].$$

Let  $x_0 \in \Omega_p$  be fixed. Then

$$\|\mathbf{I} - f'(x_0)\| \leq M((1+p)\|x_0\|^p + \mu),$$

where

$$M = \max_{s \in [a, b]} \int_a^b |G(s, t)| dt.$$

By Banach's Lemma, if  $M((1+p)\|x_0\|^p + \mu) < 1$ , one has

$$(4.4) \quad \|f'(x_0)^{-1}\| \leq \frac{1}{1 - M((1+p)\|x_0\|^p + \mu)}.$$

Since

$$\|f(x_0)\| \leq \|x_0 - l\| + M(\|x_0\|^{1+p} + \mu\|x_0\|),$$

it follows that

$$(4.5) \quad \|f'(x_0)^{-1}f(x_0)\| \leq \frac{\|x_0 - l\| + M(\|x_0\|^{1+p} + \mu\|x_0\|)}{1 - M((1+p)\|x_0\|^p + \mu)}.$$

Therefore,  $\alpha$  is estimated. On the other hand, for  $x, y \in \Omega_p$ ,

$$[(f'(x) - f'(y))u](s) = - \int_a^b G(s, t)[(1+p)(x(t)^p - y(t)^p)]u(t)dt, \quad s \in [a, b]$$

and consequently,

$$\|f'(x) - f'(y)\| \leq M(1+p)\|x - y\|^p, \quad x, y \in \Omega_p.$$

This together with (4.4) implies

$$(4.6) \quad \begin{aligned} & \|f'(x_0)^{-1}(f'(x) - f'(y))\| \\ & \leq \frac{M(1+p)}{1 - M((1+p)\|x_0\|^p + \mu)}\|x - y\|^p, \quad x, y \in \Omega_p. \end{aligned}$$

Hence  $L$  is estimated and so are the corresponding  $a$  and  $b$ . Consequently, one can use Theorems 3.1 and its corollaries to function  $f$  defined by (4.3) to establish the convergence criterions of the inexact methods for solving the nonlinear Hammerstein integral Eq. (4.1). As examples, below we list the results corresponding to Corollary 3.1 for the inexact Newton method to find the approximative solution of Eq. (4.1).

**Theorem 4.1.** *Let  $x_0 \in \Omega_p$  be a point such that  $M((1+p)\|x_0\|^p + \mu) < 1$ . Let  $v$  be such that*

$$\sup_{n \geq 0} \theta_n \|f'(x_0)^{-1} P_n^{-1}\| \|P_n f'(x_0)\|^{1+\beta} \leq v,$$

and let

$$a = (1 + v^{\frac{1}{1+\beta}})(L + (1 + \beta)v), \quad b = (1 + v^{\frac{1}{1+\beta}})\alpha.$$

Suppose that

$$(4.7) \quad \alpha \leq \begin{cases} \frac{\beta}{(1 + \beta)(1 + v^{\frac{1}{1+\beta}})} \min \left\{ 1, 2^{1-\frac{1}{\beta}} a^{-\frac{1}{\beta}} \right\} & 0 < \beta < p, \\ \beta \min \left\{ \frac{1}{v^{\frac{1}{1+\beta}}}, \frac{2^{1-\frac{1}{\beta}} a^{-\frac{1}{\beta}}}{(1 + \beta)(1 + v^{\frac{1}{1+\beta}})} \right\} & \beta = p. \end{cases}$$

Then the sequence  $\{x_n\}$  generated by the inexact Newton method converges to a solution  $x^*$  of Eq. (4.1).

We end with an example to illustrate the concrete applications of the results of this section.

**Example 4.1.** Let  $G$  be Green's function on  $[0, 1] \times [0, 1]$  defined by (4.1). Consider the following particular case of (4.1):

$$x(s) = \frac{1}{32} + \int_0^1 G(s, t) \left( x(t)^{4/3} + x(t) \right) dt, \quad s \in [0, 1].$$

The corresponding operator  $f : \Omega_p \rightarrow C[0, 1]$  is equivalent to

$$[f(x)](s) = x(s) - \frac{1}{32} - \int_0^1 G(s, t) \left( x(t)^{4/3} + x(t) \right) dt, \quad s \in [0, 1].$$

Clearly  $p = \frac{1}{3}$ ,  $\mu = 1$  and  $\Omega_p = C[0, 1]$ . Furthermore,  $M = \frac{1}{8}$ . Let  $x_0 \in \Omega_p$ . Then, we can take

$$L = \frac{1}{5.25 - x_0^{1/3}} \quad \text{and} \quad \alpha = \frac{|96x_0 - 3| + 12(x_0^{\frac{4}{3}} + x_0)}{84 - 16x_0^{1/3}}.$$

In order to apply Theorems 4.1, we choose, for each  $n \in \mathbb{N}$ ,  $P_n = I$ . Then we estimate  $v$  to get

$$v = \frac{\left(\frac{9}{8} + \frac{1}{6}x_0^{1/3}\right)^{1+\beta}}{\frac{7}{8} - \frac{1}{6}x_0^{1/3}}\theta_n.$$

For different choices of  $\beta$ ,  $\theta_n$  and  $x_0$ , the  $TF$  values of (4.7) (that is, “ $T$ ” and “ $F$ ” represent that (4.7) holds and fails, respectively) are illustrated in the following Table 1.



Table 1. The TF values of (4.7) for different  $\beta$  and  $\theta_n$ 

$\beta$	$x_0 = 0$					$x_0 = 0.1$				
	$\theta_n=0.01$	0.05	0.1	0.2	0.5	$\theta_n=0.01$	0.05	0.1	0.2	0.5
0.001	F	F	F	F	F	F	F	F	F	F
0.005	F	F	F	F	F	F	F	F	F	F
0.1	T	T	T	F	F	F	F	F	F	F
0.2	T	T	T	T	F	T	T	T	F	F
0.3	T	T	T	T	F	T	T	T	F	F
1/3	T	T	T	T	F	T	T	T	F	F

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