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CHEN IDEAL KAEHLER HYPERSURFACES

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Abstract. The concept of *Chen ideal submanifolds* is illustrated by characterizing the *complex hypercylinders* in the complex Euclidean spaces and the *complex hyperquadrics* in the complex projective space forms in terms of equalities involving their intrinsic and normal scalar curvatures and simplest "delta" curvatures.

1. Introduction

The study of the differential geometry of "complex Riemannian manifolds", i.e. of Kaehler geometry, was initiated around 1930 by Kaehler and Schouten and van Dantzig. And after Calabi's first contributions dating from the early 1950's, the differential geometry of complex and other submanifolds of Kaehler and other related ambient spaces became a widely and succesfully studied field of research up till now, starting with the work of Smyth in 1967 [23], at the school of Nomizu, which set up the basic framework for the *study of the Kaehler hypersurfaces in the complex space forms* and illustrated its use in a global classification of such Einstein Kaehler hypersurfaces, and with the works of the late 1960's and early 1970's of among others Nomizu, Kobayashi, Chern, Abe, Ryan, Takahashi, Tanno, Yano, Ishihara, Chen, Kon, Ogiue, ...(see, for instance, several concerning Chapters in [15], most in particular Chapter 3 by B.Y. Chen [3], and their references).

On the other hand, ultimately going back to the inequality $K \leq H^2$ of Euler for surfaces M^2 in Euclidean space $\mathbb{E}^3$ between the *intrinsic Gauss curvature* K and the *extrinsic mean curvature* H of M^2 in $\mathbb{E}^3$ and whereby at every point equality is achieved if and only if M^2 is part of either a plane or a round sphere in $\mathbb{E}^3$, essentially since the works of Chen of the early 1990's [4, 5] (see, e.g., also [3,

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6, 7, 9, 24]), several kinds of similar as well as of new types of such generally holding fundamental inequalities between intrinsic and extrinsic characteristics of arbitrary dimensional submanifolds M^n in arbitrary dimensional Euclidean (and other ambient) spaces $\mathbb{E}^{n+m}$ have been obtained and the submanifolds classified for which actually equalities are achieved. The, in some sense, direct generalizations of the above Euler inequality, giving for general submanifolds an inequality between *the scalar curvature* τ of the Riemannian geometry of M^n and *the squared mean curvature* H^2 of the submanifold M^n in $\mathbb{E}^{n+m}$ (or in real or complex or Sasakian space forms, etc.) will be called *Euler-Chen inequalities*. Those linking the so-called δ *curvatures of Chen* to the squared mean curvature H^2 will be called *Chen inequalities*; at this place we restrict to mentioning just one of the two most simple such curvatures, namely $\delta(2) = \tau - min\ K$, whereby K is *the sectional curvature function* on M^n. The inequalities of the form $\rho \leq H^2 - \rho^{\perp}$, whereby ρ and $\rho^{\perp}$ are *the normalized scalar curvature* of the Riemannian manifold M^n and *the normalized scalar normal curvature* of the submanifold M^n in $\mathbb{E}^{n+m}$ (or other ambient spaces), respectively, will be called *Wintgen inequalities*; since their appearance around 1980 for $n = m = 2$ in [26], Rouxel and Guadalupe and Rodriguez extended them for dimension $n = 2$ and arbitrary codimension m and De Smet, Dillen, Vrancken and one of the authors for codimension $m = 2$ and arbitrary dimension n. Submanifolds which, at every point, actually do realise the equalities in such general inequalities were called *"ideal" submanifolds* by Chen, since, roughly speaking, they can be considered as assuming those special shapes in their ambient spaces, for which the corresponding extrinsic curvatures, which can be interpreted as some stresses created in the submanifolds precisely as a result of the shapes they assume, amongst all the shapes which are possible in principle in view of their fixed intrinsic natures, are as small as can be.

Our purpose in the present article is twofold. First: to consider some of the above kinds of inequalities for *the Kaehler hypersurfaces of complex Euclidean spaces*; in particular, it turns out that *the complex hypercylinders* appear as the ideal complete such hypersurfaces for the most simple of Chen's inequalities as well as for Wintgen inequality. Second: to introduce, by way of concrete example, namely for *the Kaehler hypersurfaces of complex projective space forms*, further kinds of general inequalities between scalar valued curvatures of submanifolds, namely inequalities involving in particular also the extrinsic analogon, say $\delta^{\perp}$ curvatures, of the intrinsic δ curvatures of Chen, and for which *the complex hyperquadrics* turn out to be ideal; this was announced in [17] and we look forward to see results of more deep studies of such $\delta^{\perp}$ curvatures in the near future.

Finally, we would like to point out that ideal submanifolds in general do manifest basic intrinsic symmetries belonging to the class of *Deszcz symmetries* (pseudo-symmetries for instance of the curvature tensors of Riemann-Christoffel, Ricci or

Weyl), which recently were geometrically characterized in terms of new intrinsic curvature invariants (like the so-called *double sectional curvatures* of Riemannian manifolds), and among the ideal $4-$dimensional Lorentz submanifolds of pseudo-Euclidean spaces do appear some of the more beautiful *space-times* of general relativity [12, 13, 15, 17-19, 25].

For the basic definitions and formulas appearing in this paper we refer to Vol 2 of Nomizu-Kobayashi's "Foundations of Differential Geometry" [21] and to Chen's "Geometry of Submanifolds" [8].

2. The Simplest Chen Invariant and Chen Inequalities

In the present and in the following section we are concerned with *Kaehler hypersurfaces* M^n of arbitrary complex dimension n in a *complex Euclidean space* $\mathbf{C}^{n+1}$. *The Kaehler metric* and *the complex structure* of the ambient space $\mathbf{C}^{n+1}$ as well as the induced Kaehler metric and the induced complex structure on the hypersurface M^n will be denoted by g and J, respectively. *The Levi-Civita connection* on the ambient space $\mathbf{C}^{n+1}$ and on the submanifold M^n will be denoted by $\widetilde{\nabla}$ and ∇, respectively. Let $\eta, \zeta, ...$ be *normal vector fields* and let $X, Y, ...$ be *tangent vector fields* on M^n in $\mathbf{C}^{n+1}$. Then, *the formula of Gauss* reads $\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)$ and *the formula of Weingarten* reads $\widetilde{\nabla}_X \eta = -A_\eta(X) + \nabla^\perp_X \eta$, whereby, via these canonical decompositions of the vector fields $\widetilde{\nabla}_X Y$ and $\widetilde{\nabla}_X \eta$ into their tangent and normal components with respect to the complex hypersurface M^n in $\mathbf{C}^{n+1}$, *the second fundamental form* h, *the shape operator* A_η associated with η and *the normal connection* $\nabla^\perp$ are well defined, and $g(h(X,Y),\eta) = g(A_\eta(X),Y)$. It is always possible locally to choose an adapted orthonormal frame field $e_1, ..., e_n, e_{1^*} = Je_1, ..., e_{n^*} = Je_n, \xi, J\xi$ on M^n such that $e_1, ..., e_n, e_{1^*}, ..., e_{n^*}$ are tangent and such that ξ, $J\xi$ are normal to M^n in $\mathbf{C}^{n+1}$, respectively, and this can be done in such a way that the shape operators $A = A_\xi$ and $A_{J\xi} = JA$ are given by

$$A_\xi = \left(\begin{array}{ccc|ccc} \lambda_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & -\lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & -\lambda_n \end{array}\right),$$

$$A_{J\xi} = \left(\begin{array}{ccc|ccc} 0 & \cdots & 0 & \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \lambda_n \\ \hline \lambda_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n & 0 & \cdots & 0 \end{array}\right),$$

i.e. such that $A(e_i) = \lambda_i e_i$ and $A(e_{i^*}) = -\lambda_i e_{i^*}$, $(i, j, ... \in \{1, 2, ..., n\})$, whereby $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \geq 0$ [23] (and this remains possible also when the ambient space is e.g. a non-flat complex space-form). In particular, this shows that as a complex submanifold of a complex ambient space, the hypersurface M^n is *minimal* in $\mathbf{C}^{n+1}$, i.e. has vanishing *mean curvature* $H = \{g(\overrightarrow{H}, \overrightarrow{H})\}^{\frac{1}{2}}$, whereby $\overrightarrow{H} = \frac{1}{n}$ $trace\ h$ is the mean curvature vector field on M^n in $\mathbf{C}^{n+1}$. Further, we remark that the *eigen directions* e_i and e_{i^*} do determine specific *"eigen" planes* $\pi_{ij} = e_i \wedge e_j$, $\pi_{ij^*} = e_i \wedge e_{j^*}$ and $\pi_{i^*j^*} = e_{i^*} \wedge e_{j^*}$, $(i \neq j)$, which are totally real planes π, (i.e. for which $J(\pi) \perp \pi$), and *"eigen" planes* $\pi_{ii^*} = e_i \wedge e_{i^*}$ which are *holomorphic* planes π, (i.e. for which $J(\pi) = \pi$), for the Kaehler M^n in $\mathbf{C}^{n+1}$.

From *the Gauss equation*,

$$R(X, Y, Z, W) = g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)),$$

whereby R denotes the $(0, 4)-$*curvature tensor of Riemann-Christoffel* of M^n, we thus find the following values for *the sectional curvatures* of M^n:

$$\begin{aligned} K_{ij} &= R(e_i, e_j, e_j, e_i) = \lambda_i \lambda_j \,, \\ K_{i^*j^*} &= R(e_{i^*}, e_{j^*}, e_{j^*}, e_{i^*}) = \lambda_i \lambda_j \,, \\ K_{ij^*} &= R(e_i, e_{j^*}, e_{j^*}, e_i) = -\lambda_i \lambda_j \,, \\ K_{ii^*} &= R(e_i, e_{i^*}, e_{i^*}, e_i) = -2{\lambda_i}^2 \,, \end{aligned}$$

$(i \neq j)$, and, hence the following value for *the scalar curvature*

$$\tau = \Sigma_{\alpha<\beta} K_{\alpha\beta} = -2\Sigma_i {\lambda_i}^2$$

of M^n, (whereby α and β run over all tangent indices i and j^*). Consequently, we obtain the following well-known result.

Proposition 2.1. *For every Kaehler hypersurface M^n in $\mathbf{C}^{n+1}$ the scalar curvature τ satisfies $\tau \leq 0$, and a Kaehler hypersurface M^n is totally geodesic in $\mathbf{C}^{n+1}$, i.e. is part of a complex linear hyperplane $\mathbf{C}^n$ in $\mathbf{C}^{n+1}$, if and only if $\tau = 0$ at all points of M^n.*

The $\delta(2)$ *curvature of Chen* is defined by $\delta(2) = \tau - min\ K$, whereby K denotes the sectional curvature function of M^n, and so, in the present situation, is given by

$$\delta(2) = \tau - K_{11^*} = -2\Sigma_{k>1}\lambda_k^2.$$

Thus results the following *Chen inequality*.

Proposition 2.2. *For every Kaehler hypersurface M^n in $\mathbf{C}^{n+1}$, the $\delta(2)$ curvature of Chen satisfies $\delta(2) \leq 0$ and, at any point of M^n, $\delta(2) = 0$ if and only if (real) rank $A \leq 2$ at that point.*

Hence, from Abe's complex version of the Hartman-Nirenberg *cylinder theorem* [1], we have the following.

Theorem 2.1. *The complex hypercylinders $\mathcal{C}^n$ in $\mathbf{C}^{n+1}$, i.e. the products of any complex curve C in a complex $2-$dimensional plane $\mathbf{C}^2$ in $\mathbf{C}^{n+1}$ with complex $(n-1)-$dimensional complex linear subspaces $\mathbf{C}^{n-1}$ of $\mathbf{C}^{n+1}$ which are perpendicular to the plane $\mathbf{C}^2$ of the curve C, are the complete Kaehler hypersurfaces M^n in $\mathbf{C}^{n+1}$ for which the $\delta(2)$ curvature of Chen vanishes identically, $\delta(2) \equiv 0$.*

We remark that the *minimum* K_{11^*} of K for Kaehler M^n in $\mathbf{C}^{n+1}$ coincides with minimum of the sectional curvatures for the holomorphic "eigen" planes of M^n, since $K_{ii^*} = K(\pi_{ii^*}) = H(e_i) = H(e_{i^*})$, whereby H here denotes *the holomorphic sectional curvature function* on the Kaehler manifold M^n.

We remark that, whereas *the complex hyperplanes* $\mathbf{C}^n$ of $\mathbf{C}^{n+1}$ which occur in Proposition 2.1 can also be characterised as those Kaehler hypersurfaces M^n in $\mathbf{C}^{n+1}$ which are flat, from Ryan's work it is known that *the complex hypercylinders* $\mathcal{C}^n$ in $\mathbf{C}^{n+1}$ can be characterized as the Kaehler hypersurfaces M^n in $\mathbf{C}^{n+1}$ which are *semi-symmetric* [22], i.e. geometrically and roughly speaking, as those M^n for which the sectional curvature $K(p, \pi)$ at any point p of M^n for any real $2-$dimensional plane section π of the tangent space of M^n at p is invariant under the parallel translation of π all the way around any infinitesimal co-ordinate parallellogram cornered at p [19, 20].

3. The Wintgen Inequality

The Ricci equation of M^n in $\mathbf{C}^{n+1}$ is given by

$$R^{\perp}(X, Y, \eta, \zeta) = g([A_\eta, A_\zeta](X), Y),$$

whereby $[A_\eta, A_\zeta] = A_\eta A_\zeta - A_\zeta A_\eta$ and $R^{\perp}$ denotes *the normal curvature tensor* of M^n in $\mathbf{C}^{n+1}$, i.e. the curvature tensor of the normal bundle of M^n in $\mathbf{C}^{n+1}$ associated with the normal connection $\nabla^{\perp}$. Calculated in the above "eigen" frame $e_1, ..., e_n, e_{1^*}, .., e_{n^*}, \xi, J\xi$ we thus find the following absolute values for the components of $R^{\perp}$ (the eventual positive or negative signs of these components themselves basically depending on arbitrary choices of orientations, we will rather not take them into consideration here):

$$|K^{\perp}_{ii^*}| = |R^{\perp}(e_i, e_{i^*}, \xi, J\xi)| = 2{\lambda_i}^2\,,$$
$$|K^{\perp}_{ij}| = R^{\perp}(e_i, e_j, \xi, J\xi) = 0\,,$$
$$|K^{\perp}_{i^*j^*}| = R^{\perp}(e_{i^*}, e_{j^*}, \xi, J\xi) = 0\,,$$
$$|K^{\perp}_{ij^*}| = R^{\perp}(e_i, e_{j^*}, \xi, J\xi) = 0\,,$$

$(i \neq j)$, (whereby in the notation of those components we did drop reference to the normals ξ and $J\xi$ because they are essentially the only possible choice), and hence the following value for *the scalar normal curvature*:

$$\tau^{\perp} = \{\Sigma_{\alpha<\beta}(K^{\perp}_{\alpha\beta})^2\}^{\frac{1}{2}} = 2(\Sigma_i {\lambda_i}^4)^{\frac{1}{2}}.$$

This then gives the following *Wintgen inequality*.

Proposition 3.1. *For every Kaehler hypersurface M^n in $\boldsymbol{C}^{n+1}$ the scalar curvature τ and the normal scalar curvature $\tau^{\perp}$ satisfy $\tau \leq -\tau^{\perp}$, and, at any point of M^n, $\tau = -\tau^{\perp}$ if and only if (real) rank $A \leq 2$ at that point.*

Consequently, we also have the following.

Theorem 3.1. *The complex hypercylinders $\mathcal{C}^n$ in $\boldsymbol{C}^{n+1}$ are the complete Kaehler hypersurfaces M^n in $\boldsymbol{C}^{n+1}$ for which $\tau + \tau^{\perp} \equiv 0$.*

4. An Inequality Involving Also "Normal delta Curvatures"

The previous inequalities are in fact merely some explicitations in some special cases of very general optimal inequalities for submanifolds of Euclidean spaces or of much wider classes of ambient spaces, and further the above results consist in giving the geometrical interpretations of the characteristic expressions of the second fundamental forms of *the corresponding ideal submanifolds*, i.e. of the submanifolds which at all their points actually do realize the equality in these inequalities. As such, for instance, with respect to the above Wintgen inequality, one has the following.

Theorem 4.1. (DDVV, [11]). *Let M^m be a real $m-$dimensional submanifold of real codimension 2 in a real space form $N^{m+2}(c)$ of sectional curvature c. Then, at every point of N^m:*

$$\tau \leq \frac{m(m-1)}{2}H^2 - \tau^{\perp} + \frac{m(m-1)}{2}c,$$

and equality holds at some point p *of* N^m *if and only if there exists an orthonormal basis* $e_1, ..., e_m$ *of the tangent space and an orthonormal basis* ξ_1, ξ_2 *of the normal space of* N^m *in* $N^{m+2}(c)$ *at* p *such that*

$$A_{\xi_1} = \left(\begin{array}{cc|ccc} \lambda & \mu & 0 & \cdots & 0 \\ \mu & \lambda & 0 & \cdots & 0 \\ \hline 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{array} \right),$$

$$A_{\xi_2} = \left(\begin{array}{cc|ccc} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{array} \right).$$

Specialization to the case of Kaehler hypersurfaces $M^n = N^{2n} = N^m$ in $\mathbf{C}^{n+1} = \mathbb{E}^{2n+2} = \mathbb{E}^{m+2}$ readily gives Proposition 3.1; in particular, in this special case the ambient space is flat $(c = 0)$ and the submanifolds are minimal $(H = 0)$, the inequality thus reducing to $\tau \leq -\tau^{\perp}$. Moreover we remark that amongst the examples of ideal submanifolds N^m in $N^{m+2}(c)$ for this inequality which are listed in [11], also the complex hypercylinders $\mathcal{C}^n$ in $\mathbf{C}^{n+1}$ of course were discussed already (in the context of superminimal surfaces in $\mathbf{E}^4$). After finishing the present work, we learned, and here want to point out, that Dillen and coworkers succeeded in proving the conjecture $\tau \leq \frac{m(m-1)}{2}H^2 - \tau^{\perp}$ from [11] for the invariant submanifolds of complex Euclidean spaces and of Sasakian spheres [13].

Since the non-flat complex space forms do not admit Kaehler hypersurfaces like the complex hypercylinders $\mathcal{C}^n$ in $\mathbf{C}^{n+1}$, above we could restrict to flat ambient spaces. And, for the result we have in mind next, concerning complex hyperspheres as ideal Kaehler hypersurfaces in complex space forms, it similarly suffices at present to restrict to the study of complex hypersurfaces M^n of *complex projective spaces* $\mathbf{CP}^{n+1}(c)$ of constant holomorphic sectional curvature $c\ (> 0)$. The curvature tensor $\bar{R}$ of *the Study-Fubini metric* g on $\mathbf{CP}^{n+1}(c)$ is given by

$$\begin{aligned} \bar{R}(A, B, C, D) = \frac{c}{4}\{ & g(B, C)g(A, D) - g(A, C)g(B, D) \\ & + g(JB, C)g(JA, D) - g(JA, C)g(JB, D) \\ & - 2g(JA, B)g(JC, D)\}, \end{aligned}$$

whereby A, B, C and D denote arbitrary vector fields on $\mathbf{CP}^{n+1}(c)$. *The equations of Gauss and Ricci* for a Kaehler hypersurface M^n in $\mathbf{CP}^{n+1}(c)$ are respectively given by

$$R(X,Y,Z,W) = g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)) + \bar{R}(X,Y,Z,W)\,,$$

$$R^{\perp}(X,Y,\eta,\zeta) = g([A_\eta, A_\zeta](X),Y) + \bar{R}(X,Y,\eta,\zeta)\,.$$

Calculated in an "eigen" frame $e_1, ..., e_n, e_{1^*}, .., e_{n^*}, \xi, J\xi$ adapted to the Kaehler hypersurface M^n in $\mathbf{CP}^{n+1}(c)$, we thus find the following values for *the sectional curvatures* of M^n:

$$K_{ij} = \lambda_i\lambda_j + \frac{c}{4}\,,$$

$$K_{i^*j^*} = \lambda_i\lambda_j + \frac{c}{4}\,,$$

$$K_{ij^*} = -\lambda_i\lambda_j + \frac{c}{4}\,,$$

$$K_{ii^*} = -2{\lambda_i}^2 + c\,,$$

$(i \neq j)$, (which, of course, corresponds to the calculations made before in the case of the flat ambient space $\mathbf{C}^{n+1}$, taking into account now that the planes π_{ii^*} and the planes π_{ij}, $\pi_{i^*j^*}$ and π_{ij^*} $(i \neq j)$ are *holomorphic* and *totally real* planes, respectively, on M^n in $\mathbf{CP}^{n+1}(c)$). Hence *the scalar curvature* of M^n is given by

$$\tau = -2\Sigma_i\lambda_i^2 + n(2n-1)c.$$

Now, we recall that when Chen introduced his new scalar valued Riemannian curvature invariants in the 1990's, later called *Chen's δ curvatures* [4-7], actually two different kinds of such curvatures were launched. Later on, most attention so far was focussed on the first one of those two kinds, this probably being caused naturally by the types of studies undertaken in their respect till now. And, from this first series of δ curvatures, the simplest is $\delta(2) = \tau - min\ K$. Similarly, from the second series of δ curvatures, the simplest is $\hat{\delta}(2) = \tau - max\ K$. From the intrinsic Riemannian geometrical point of view, $\delta(2)$ and $\hat{\delta}(2)$, (as well as the more sophisticated $\delta(n_1, n_2, ..., n_k)$ and $\hat{\delta}(n_1, n_2, ..., n_k)$ curvatures), a priori both enjoy an equal status . At this stage we would like to observe that, in our opinion, whereas general studies on the geometry of Riemannian manifolds with respect to min K and max K are rare (see e.g. Berger's "A panoramic view of Riemannian geometry" [2]), Chen's idea of studying min K and max K as while incorporated in his curvatures $\delta(2)$ and $\hat{\delta}(2)$, rather than "on their own", made it possible to obtain significant and general results in this direction. In this connection, and recalling the above made observation that $\delta(2) = \tau - min\ K = \tau - min\ H(e) = \kappa(2)$, $H(e)$ denoting the sectional curvature function on a Kaehler hypersurface M^n in a complex space form for its *holomorphic "eigen" planes* $\pi = e \wedge Je$, in our study of the Kaehler M^n in $\mathbf{CP}^{n+1}(c)$, we would further also like to consider the scalar curvature function defined by $\hat{\kappa}(2) = \tau - max\ H(e)$. In the present situation, we find that

$$\hat{\kappa}(2) = -2\Sigma_{l<n}{\lambda_l}^2 + (n-1)(2n+1)c.$$

Since $c > o$, we obtain the following absolute values of the components of the normal curvature tensor $R^\perp$ of M^n in $\mathbf{CP}^{n+1}(c)$:

$$|K^\perp_{ij}| = |K^\perp_{i^*j^*}| = |K^\perp_{ij^*}| = 0\,,$$
$$|K^\perp_{ii^*}| = |-2{\lambda_i}^2 - \frac{c}{2}| = 2{\lambda_i}^2 + \frac{c}{2}\,,$$

$(i \neq j)$ and whereby we use the same notational conventions as before, namely we put $K^\perp_{\alpha\beta} = R^\perp(e_\alpha, e_\beta, \xi, J\xi)$. In view of the special character of the orthonormal frame $e_1, ..., e_n, e_{1^*}, .., e_{n^*}$ it makes sense to define, as an alternative scalar normal curvature of M^n in $\mathbf{CP}^{n+1}(c)$, the quantity

$$\delta^\perp = \Sigma_{\alpha<\beta}|K^\perp_{\alpha\beta}| = 2\Sigma_i{\lambda_i}^2 + \frac{n}{2}c.$$

(If one would like to proceed along these lines for general submanifolds N^m in arbitrary Riemannian manifolds N^{m+q} of any dimensions m and codimensions q, when then α, β and u, v would run over the tangent and normal indices of an orthonormal frame $e_1, ..., e_m, \xi_1, ..., \xi_q$, respectively, for properly defining a scalar normal curvature invariant in the above sense, one should for instance consider quantities like suprema or infima of $\Sigma_{\alpha<\beta}\Sigma_{u<v}|R^\perp(e_\alpha, e_\beta, \xi_u, \xi_v)|$ to be taken over all such frames, but we will not go into this any further here and now). And, similarly as in Chen's δ curvatures, one may then define e.g. the following *scalar normal δ curvature*: $\hat{\delta}^\perp(2) = \delta^\perp - max\ |K^\perp|$. For Kaehler hypersurfaces M^n in $\mathbf{CP}^{n+1}(c)$ we see from the above that

$$\hat{\delta}^\perp(2) = 2\Sigma_{k>1}\lambda_k^2 + \frac{n-1}{2}c.$$

Thus results the following general inequality for Kaehler hypersurfaces M^n in $\mathbf{CP}^{n+1}(c)$:

$$\hat{\kappa}(2) \leq -\hat{\delta}^\perp(2) + \frac{(n-1)(4n+3)}{2}c.$$

Moreover equality holds at a point of M^n if and only if $\lambda_1^2 = \lambda_n^2$, i.e. if and only if $\lambda_1 = \lambda_n$, and so, by the choice of "eigen" frame, if and only if $\lambda_1 = \lambda_2 = ... = \lambda_{n-1} = \lambda_n = \lambda \in \mathbf{R}^+$, and this holds at all points of M^n if and only if either

M^n is *totally geodesic* in $\mathbf{CP}^{n+1}(c)$, $(\lambda = 0)$, or M^n is a *complex hypersphere* in $\mathbf{CP}^{n+1}(c)$, $(\lambda^2 = \frac{c}{4})$. In the first case, M^n is part of a complex projective space $\mathbf{CP}^n(c)$ of the same constant holomorphic sectional curvature c as $\mathbf{CP}^{n+1}(c)$ and which can be described as an hypersurface given in $\mathbf{CP}^{n+1}(c)$ by putting one of the complex homogeneous co-ordinates $z_0, z_1, ..., z_{n+1}$ equal to zero. In the second case, as known from the local study by Chern of the Einstein Kaehler hypersurfaces of complex space forms [10], M^n is locally isometric to *the complex quadric* Q^n in $\mathbf{CP}^{n+1}(c)$, which is the complex variety defined in $\mathbf{CP}^{n+1}(c)$ by the quadratic equation ${z_0}^2 + {z_1}^2 + .. + {z_{n+1}}^2 = 0$. Summarizing, we thus obtain the following.

Theorem 4.2. *For every Kaehler hypersurface M^n in the complex projective space $\mathbf{CP}^{n+1}(c)$ the scalar valued curvatures $\hat{\kappa}(2)$ and $\hat{\delta}^{\perp}(2)$ always satisfy the following inequality:*

$$\hat{\kappa}(2) \leq -\hat{\delta}^{\perp}(2) + \frac{(n-1)(4n+3)}{2}c.$$

And the only Kaehler hypersurfaces M^n in $\mathbf{CP}^{n+1}(c)$ which are ideal in this respect, i.e. for which the equality holds at all of their points, are parts either of the totally geodesic complex space forms $\mathbf{CP}^n(c)$ or of the complex quadrics Q^n in $\mathbf{CP}^{n+1}(c)$.

In relation with a comment made in the introduction, we finally remark that these ideal complex hypersurfaces M^n of $\mathbf{CP}^{n+1}(c)$, which are of constant holomorphic sectional curvature and Einstein, respectively, are both *locally symmetric*.

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