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A MODIFIED KOHLBERG CRITERION AND A NONLINEAR METHOD TO COMPUTE THE NUCLEOLUS OF A COOPERATIVE GAME

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Dedicated to Professor Wataru Takahashi on the occasion of his 65th birthday

Abstract. We propose a new nonlinear method for finding a coalition array in a cooperative game, a new coherent criterion for concluding whether its array corresponds to the nucleolus or not, and an adjusting process for obtaining the nucleolus strictly from its nonlinear approximation.

1. Introduction

The nucleolus of a cooperative game was first defined by Schmeidler [8] in 1969. Subsequently, Kohlberg [6] proposed two criteria, called property I and property II, for x to be the nucleolus. In particular, property II is a characterization of the nucleolus by the notion of balancedness of a collection of coalitions, which is closely related to the calculation methods of the nucleolus. Thereafter, many authors have developed various types of algorithms to calculate the nucleolus. However, in all approaches, the calculations are performed by solving linear problems. In this paper, we introduce a new approach. Kido [5] has developed a nonlinear approximation method for the nucleolus: by solving a good-natured nonlinear minimization problem, such as minimization with respect to an ℓ_p-norm in a convex cell, we can obtain a sufficiently good approximation of the nucleolus. Next, we modify property II in order to verify the sufficiency of the approximation easily. This check is accomplished through a new type of coalition array, that is, a modified version of the coalition array defined by Kohlberg [6]. Furthermore, the modified criterion, say property IIM, is suitable for this nonlinear approach and is efficient because a rank condition is used (this idea was introduced by Dragan [2] for finding the prenucleolus). If the approximation is judged to be sufficient, as a final step, we

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construct a single small size of linear equations and solve it. This process adjusts the existing small difference between the nucleolus and its nonlinear approximation. In other words, the obtained solution of the system of linear equations is the proper nucleolus.

2. Definitions and Main results

Let $N = \{1, 2, \ldots, n\}$ be a set of n players. Let v be a real valued function of 2^N, where 2^N is the collection of all subsets, called **coalitions**, of N; $v(\emptyset)$ is assumed to be 0 as usual. The pair (N, v) denotes an n-person cooperative game in a characteristic function form with a transferable utility. Let $x = (x_1, \cdots, x_n)$ in $\mathbb{R}^n$ denote a **payoff vector**, where x_i denotes the payoff for player i in N. For any nonempty coalition S of N, we define $x(S) = \sum_{i \in S} x_i$. A payoff vector x is said to be **efficient** if $x(N) = v(N)$, **individually rational** if $x_i \geq v(\{i\})$ for all i in N, and an **imputation** if it is both efficient and individually rational. $I(v)$ denotes the set of imputations, which is clearly a compact convex subset of $\mathbb{R}^n$. Throughout this paper, we assume the following.

Assumption 2.1. $I(v) \neq \emptyset$.

Let $x \in \mathbb{R}^n$ be a payoff vector. We define $e(S, x) = v(S) - x(S)$, called the **excess** of S at x, for any nonempty coalition S.

The following definitions are defined by Kohlberg.

Definition 2.1. ([6]). Let b_0, $b_1, \cdots, b_p$ be collections of coalitions. We call b_0, $b_1, \cdots, b_p$ a **coalition array** if

(i) for every $S \in b_0$, there exists a unique $i \in N$ such that $S = \{i\}$;
(ii) for every nonempty coalition S of N, there exists a unique $j : 1 \leq j \leq p$ such that $S \in b_j$.

Definition 2.2. ([6]). Let x be an imputation. Define a coalition array, called the **coalition array that belongs to** $(\mathbf{v}, \mathbf{x})$, as follows.

$$\begin{cases} b_0(v, x) = \{\{i\} : x_i = v(\{i\})\}, \\ b_1(v, x) = \{S \subset N : S \neq \emptyset, S \text{ attains the maximal excess}\}, \\ b_2(v, x) = \{S \subset N : S \neq \emptyset, S \text{ attains the second maximal excess}\}, \\ \quad \vdots \\ b_p(v, x) = \{S \subset N : S \neq \emptyset, S \text{ attains the } p\text{th maximal excess}\}. \end{cases}$$

Definition 2.3. ([6]). A coalition array $b_0, \cdots, b_p$ has **property II** if, for each $k = 1, \cdots, p$, there exists a 2^n-dimensional vector $w : w \geq 0$ such that

$$\sum_{S \in b_0 \cup \cdots \cup b_k} w_S e_S = e_N \text{ and } w_S > 0 \text{ for } S \in b_1 \cup \cdots \cup b_k.$$

Our aim in this paper is to construct an efficient criterion to judge whether a nonlinear approximation to the nucleolus of a given cooperative game is achieved sufficiently. For this purpose, we introduce some slightly modified definitions. (Additionally, we remark that our definitions are easily generalized to the corresponding definitions for coalition structures).

Definition 2.4. Let $I_0, I_1, \cdots, I_p$ be collections of coalitions. We call I_0, $I_1, \cdots$, I_p a **modified coalition array** if (i) I_0 is the union of $\{N\}$ and a set of some one-element coalitions; and (ii) for every nonempty proper subset S of N, there exists a unique $j : 1 \leq j \leq p$ such that $S \in I_j$.

Definition 2.5. Let x be an imputation. Define a modified coalition array $I_0, I_1, \cdots, I_p$ as follows.

$$\begin{cases} I_0 = \{N\} \cup b_0, \text{ where } b_0 = \{\{i\} : i \in N, x_i = v(\{i\})\}, \\ I_1 = \{S \subset N : \emptyset \neq S \neq N, S \text{ attains the maximal excess}\}, \\ I_2 = \{S \subset N : \emptyset \neq S \neq N, S \text{ attains the second maximal excess}\}, \\ \qquad \vdots \\ I_p = \{S \subset N : \emptyset \neq S \neq N, S \text{ attains the } p\text{th maximal excess}\}, \end{cases}$$

where all types of maximal excesses are collected over all nonempty proper subsets of N. We call $I_0, \cdots, I_p$ the **modified coalition array that belongs to (*v*, *x*)**.

Definition 2.6. We say that a collection of coalitions $\mathbb{S}$ is **(*v*, *x*)-balanced** (with weights $\{\delta_S\}$) if (i) there exist an index $i : 0 \leq i \leq p$ and a nonempty subset J_i of I_i, maybe $J_i = I_i$, such that $\mathbb{S} = \{I_0, \cdots, I_{i-1}, J_i\}$; and (ii) there exist nonnegative weights $\{\delta_S : S \in \cup\, \mathbb{S}\}$ such that $\delta_S > 0$ for each $S \in \cup\, \mathbb{S} \setminus b_0$ and $e_N = \sum_{S \in \cup\, \mathbb{S}} \delta_S e_S$, where e_S is a characteristic function of S for any subset S of N. Here, we call i the **degree** of $\mathbb{S}$ and define $\mathbb{S}^+ = \{S \in \cup\mathbb{S} : \delta_S > 0\}$, $b_0^+ = \mathbb{S}^+ \cap\ b_0$ and $b_0^0 = b_0 \setminus b_0^+$. Since all notions depend on the collection $\mathbb{S}$ and its weights $\{\delta_S\}$, we may denote this dependency explicitly by, for example, $i(\mathbb{S})$, $\delta_S(\mathbb{S})$, $\mathbb{S}^+(\{\delta_S\})$, or $\mathbb{S}^+(\{\delta_S(\mathbb{S})\})$.

Definition 2.7. Let $\mathbb{S}$ and $\mathbb{S}'$ be two (v, x)-balanced collections of coalitions such that (i) $\cup\mathbb{S} \subsetneq \cup\mathbb{S}'$; (ii) there is no other (v, x)-balanced collection of coalitions

$\mathbb{S}'' : \cup\mathbb{S} \subsetneq \cup\mathbb{S}'' \subsetneq \cup\mathbb{S}'$; and (iii) $i(\mathbb{S}) = i(\mathbb{S}')$ or $\cup\mathbb{S} = I_0 \cup \cdots \cup I_{i(\mathbb{S}')-1}$. Then, we say that $\mathbb{S}'$ is a **(v, x)-balanced 1-extension** of $\mathbb{S}$ and denote it by $\mathbb{S} \overset{B_1}{\prec} \mathbb{S}'$. Next, if $\mathbb{S}$ and $\mathbb{S}'$ are two (v, x)-balanced collections of coalitions such that

$$\mathbb{S} \overset{B_1}{\prec} \mathbb{S}_1 \overset{B_1}{\prec} \cdots \overset{B_1}{\prec} \mathbb{S}_m \overset{B_1}{\prec} \mathbb{S}'$$

for some (v, x)-balanced collections of coalitions $\mathbb{S}_1, \cdots, \mathbb{S}_m$, then, we say that $\mathbb{S}'$ is a **(v, x)-balanced extension** of $\mathbb{S}$ and denote it by $\mathbb{S}\overset{B}{\prec}\mathbb{S}'$. We call this a maximal balanced collection of coalitions with respect to the ordering $\overset{B}{\prec}$ a **(v, x)-maximal balanced** collection of coalitions. We also say that a maximal balanced collection of coalitions $\mathbb{S}'$ is a **(v, x)-maximal balanced extension** of a non-maximal (v, x)-balanced collection of coalitions $\mathbb{S}$ if $\mathbb{S} \overset{B}{\prec} \mathbb{S}'$.

Since I_0 is a (v, x)-balanced collection of coalitions, I_0 always has a (v, x)-maximal balanced extension with respect to the ordering $\overset{B}{\prec}$.

Definition 2.8. Let $\mathbb{S}$ be a (v, x)-maximal balanced extension of I_0 (with non- negative weights $\{\delta_S\}$). Then, we say that (v, x) has **property IIM** if rank $(e_S : S \in \mathbb{S}^+) = n$.

Although the above definition appears to depend on $\mathbb{S}$ and $\{\delta_S\}$, we can observe the well-definedness from Corollary 2.1. The next lemma shows a characterization of (v, x)-balancedness.

Lemma 2.1. *Let $\mathbb{S}$ be a collection of coalitions such that $\cup\mathbb{S} = I_0 \cup I_1 \cup \cdots \cup I_{j-1} \cup J_j$, where J_j is a nonempty subset of I_j for some $j : 1 \leq j \leq p$. Then, $\mathbb{S}$ is a (v, x)-balanced collection of coalitions if and only if there exists a (v, x)-balanced collection of coalitions $\bar{\mathbb{S}}$ with weights $\{\bar{\delta}_S : S \in \cup\bar{\mathbb{S}}\}$ such that*

(i) *$\cup\bar{\mathbb{S}} \subsetneq \cup\mathbb{S}$;*

(ii) *there exists $w_S \in \mathbb{R}$ for each $S \in \bar{\mathbb{S}}^+(\{\bar{\delta}_S\})$, $\delta_T > 0$ for each $T \in \mathbb{T} = \cup\mathbb{S} \setminus \cup\bar{\mathbb{S}}$, and $\delta_T \geq 0$ for each $T \in b_0^0(\{\bar{\delta}_S\})$ such that*

$$\sum_{S \in \bar{\mathbb{S}}^+(\{\bar{\delta}_S\})} w_S e_S = \sum_{T \in \mathbb{T}} \delta_T e_T + \sum_{T \in b_0^0(\{\bar{\delta}_S\})} \delta_T e_T.$$

Furthermore, if $I_0 \cup \cdots \cup I_{j-1} \subset \cup\bar{\mathbb{S}}$, $\mathbb{S}$ is a (v, x)-balanced extension of $\bar{\mathbb{S}}$.

Proof. First, we show the if part. Define $\bar{\delta} = \frac{1}{2}\min_{S \in \bar{\mathbb{S}}^+} \bar{\delta}_S > 0$. Take $K > 0$ sufficiently large so that $\bar{\delta} > \frac{1}{K} w_S$ for every $S \in \bar{\mathbb{S}}^+$. Since $e_N = \sum_{S \in \bar{\mathbb{S}}^+} \bar{\delta}_S e_S$, we have

$$e_N = \sum_{S \in \bar{\mathbb{S}}^+} (\bar{\delta}_S - \frac{1}{K} w_S) e_S + \sum_{T \in \mathbb{T}} \frac{1}{K} \delta_T e_T + \sum_{T \in b_0^0(\{\bar{\delta}_S\})} \frac{1}{K} \delta_T e_T.$$

This shows that $\mathbb{S}$ is (v, x)-balanced. Next, we prove the only if part. Let $\{\delta_S : S \in \cup\mathbb{S}\}$ be weights for $\mathbb{S}$, and $\bar{\mathbb{S}}$ be any (v, x)-balanced collection of coalitions with weights $\{\bar{\delta}_S : S \in \cup\bar{\mathbb{S}}\}$ such that $\cup\bar{\mathbb{S}} \subsetneq \cup\mathbb{S}$. Since $e_N = \sum_{S \in \bar{\mathbb{S}}^+(\{\bar{\delta}_S\})} \bar{\delta}_S e_S = \sum_{S \in \mathbb{S}^+} \delta_S e_S$, we have the desired equation:

$$\sum_{S \in \bar{\mathbb{S}}^+(\{\bar{\delta}_S\})} (\bar{\delta}_S - \delta_S) e_S = \sum_{T \in \mathbb{T}} \delta_T e_T + \sum_{T \in b_0^0(\{\bar{\delta}_S\})} \delta_T e_T.$$

The last statement is clear. ∎

Lemma 2.2. *Let $\mathbb{S}$ be a (v, x)-balanced extension of I_0 such that $\mathbb{S} = \{I_0, \cdots, I_{j-1}, J_j\}$, where J_j is a nonempty proper subset of I_j for some $j : 1 \leq j \leq p$. Then, if $\mathbb{S}$ is maximal (with respect to the ordering $\overset{B}{\prec}$), rank$(e_S : S \in \mathbb{S}^+) < n$ and (v, x) do not have property IIM.*

Proof. Assume that rank$(e_S : S \in \mathbb{S}^+) = n$. Fix $T \in I_j \setminus J_j$ arbitrarily. Then, there exists $\{w_S \in \mathbb{R} : S \in \mathbb{S}^+\}$ such that $e_T = \sum_{S \in \mathbb{S}^+} w_S e_S$. From Lemma 2.1, this implies that $\{I_0, \cdots, I_{j-1}, I_j \cup \{T\}\}$ is a (v, x)-balanced extension of $\mathbb{S}$. This contradicts the maximality of $\mathbb{S}$. Next, assume that $\mathbb{S}'$ is another (v, x)-maximal balanced extension of I_0 with rank$(e_S : S \in \mathbb{S}'^+) = n$. Then, from the maximality of $\mathbb{S}'$ and Lemma 2.1, we easily obtain that $I_0 \cup \cdots \cup I_k$ is (v, x)-balanced for every $k = 0, \cdots, p$. This contradicts the maximality of $\mathbb{S}$. Thus, (v, x) cannot have property IIM. ∎

Corollary 2.1. *The following statements are all equivalent:*

(i) *There exists a (v, x)-(maximal)balanced extension $\mathbb{S}$ of I_0 with rank$(e_S : S \in \mathbb{S}^+) = n$, that is, (v, x) has property IIM.*

(ii) *For every (v, x)-maximal balanced extension $\mathbb{S}$ of I_0, rank$(e_S : S \in \mathbb{S}^+) = n$.*

(iii) *$\{I_0, \cdots, I_j\}$ is (v, x)-balanced for $j = 0, \cdots, p$.*

(iv) *$\{I_0, \cdots, I_p\}$ is the unique (v, x)-maximal balanced extension of I_0.*

Corollary 2.2. *Let $\mathbb{S}$ be a (v, x)-balanced extension of I_0 such that $\mathbb{S} = \{I_0, \cdots, I_{j-1}, J_j\}$ for some $j : 1 \leq j \leq p$ and some nonempty proper subset J_j of I_j. Set $\mathbb{S}' = \{I_0, \cdots, I_j\}$ and $\mathbb{T} = I_j \setminus J_j$. If*

$$\mathrm{rank}(e_S : S \in \mathbb{S}^+) + \mathrm{rank}(e_T : T \in \mathbb{T} \cup b_0^0) = \mathrm{rank}(e_S : S \in \cup\mathbb{S}'),$$

then, (v, x) *does not have property IIM.*

Proof. From this assumption, we have

$$[e_S : S \in \mathbb{S}^+] \oplus [e_T : T \in \mathbb{T} \cup b_0^0] = [e_S : S \in \cup \mathbb{S}'].$$

Then, using Lemma 2.1, we obtain that $\mathbb{S}'$ cannot be a (v, x)-balanced extension of $\mathbb{S}$. Thus, from Corollary 2.1, we know that (v, x) does not have property IIM. ■

Theorem 2.1. *The following three conditions are equivalent.*

(i) (v, x) *has property IIM.*

(ii) *The coalition array that belongs to* (v, x) *has property II.*

(iii) x *is the nucleolus.*

Proof. We know that (ii) and (iii) are equivalent from Theorem 5 of Kohlberg [6]. Therefore, we have to prove only the equivalence of (i) and (ii).

First, we prove that (i) implies (ii). By Corollary 2.1, if (v, x) has property IIM, $\{I_0, \cdots, I_j\}$ is (v, x)-balanced for every $j = 0, 1, \cdots, p$. Fix j and assume that $N \in b_1 \cup \cdots \cup b_j$. In this case, we know $b_0 \cup \cdots \cup b_j = I_0 \cup \cdots \cup I_j$ or $b_0 \cup \cdots \cup b_j = I_0 \cup \cdots \cup I_{j-1}$. In either case, we obtain that there exists $b_0^j \subset b_0$ such that $b_0^j \cup b_1 \cup \cdots \cup b_j$ is balanced. On the contrary, assume that $N \notin b_1 \cup \cdots \cup b_j$. Then, from the (v, x)-balancedness of $\{I_0, \cdots, I_j\}$, we have $e_N = \sum_{S \in I_0 \cup \cdots \cup I_j} \delta_S e_S$, where $\delta_S > 0$ for every $S \in I_1 \cup \cdots \cup I_j$ and $\delta_S \geq 0$ for every $S \in I_0$. Since $I_0 = \{N\} \cup b_0$, we obtain

$$(1 - \delta_N) e_N = \sum_{S \in b_0 \cup \cdots \cup b_j} \delta_S e_S,$$

where $(1 - \delta_N) > 0$, $\delta_S > 0$ for each $S \in b_1 \cup \cdots \cup b_j$ and $\delta_S \geq 0$ for each $S \in b_0$. Since $j = 0, \cdots, p$ is fixed arbitrarily, condition (ii) is proved.

Next, we show that (ii) implies (i). Fix $j = 1, 2, \cdots, p$ arbitrarily. If $N \in b_1 \cup \cdots \cup b_j$, then $I_0 \cup \cdots \cup I_j = b_0 \cup \cdots \cup b_j$ or $I_0 \cup \cdots \cup I_{j-1} = b_0 \cup \cdots \cup b_j$. This shows that $\{I_0, \cdots, I_j\}$ or $\{I_0, \cdots, I_{j-1}\}$ is (v, x)-balanced. In the case: $N \notin b_1 \cup \cdots \cup b_j$, since $e_N = \sum_{S \in b_0 \cup \cdots \cup b_j} \delta_S e_S$, $e_N = \frac{1}{2} e_N + \sum_{S \in b_0 \cup \cdots \cup b_j} \frac{\delta_S}{2} e_S$. This shows the (v, x)-balancedness of $\{I_0, \cdots, I_j\}$. Inductively, $\{I_0, \cdots, I_p\}$ or $\{I_0, \cdots, I_{p-1}\}$ is the (v, x)-maximal balanced extension of I_0. Remark that $I_0 \cup \cdots \cup I_{j-1} = b_0 \cup \cdots \cup b_j$ for some j implies $I_p = \emptyset$. Then, from Corollary 2.1, condition (i) is proved. ■

3. Calculating the Nucleolus

In this section, we introduce a new algorithm to obtain the nucleolus of an n-

person cooperative game (N, v) such that $I(v) \neq \emptyset$. This algorithm depends completely on the nonlinear approximation result of the nucleolus established in Kido [5]:

Theorem 3.1. (Theorem 2 [5]). *There exists $\bar{k} \in \mathbb{R}$ such that for every $k < \bar{k}$, $\bar{x}_{p,k}$ converges to $\bar{x}$ in $(R^n, ||\cdot||_1)$ as $p \to \infty$, where $\bar{x}$ is the nucleolus and $\bar{x}_{p,k} = \mathrm{argmin}_{x\in I(v)} \sum_{\emptyset\neq S\subset N} |e(S,x)-k|^p$ is the ℓ_p-k-nucleolus defined in* [5].

Let $\mathbb{S}(x) = \{S_1(x), \cdots, S_{2^n-1}(x)\}$ denote a sequence of all nonempty coalitions such that $e(S_1(x), x) \geq e(S_2(x), x) \geq \cdots \geq e(S_{2^n-1}(x), x)$. Fix k to be sufficiently small. Then, from Theorem 3.1 and continuity of excesses with respect to x, $\mathbb{S}(\bar{x}_{p,k})$ approximates $\mathbb{S}(\bar{x})$ as $p \to \infty$. Recalling that every numerical calculation by a computer has some numerical error, let us judge $e(S_j(x), x) = e(S_{j+1}(x), x)$ by $|e(S_j(x), x) - e(S_{j+1}(x), x)| < \epsilon$ for sufficiently small $\epsilon > 0$. (However, ϵ must be slightly larger than $\epsilon(\bar{x}) = \min\{e(S_j(\bar{x}), \bar{x}) - e(S_{j+1}(\bar{x}), \bar{x}) : j = 1, \cdots, 2^n-2, e(S_j(\bar{x}), \bar{x}) - e(S_{j+1}(\bar{x}), \bar{x}) > 0\}$ when $\epsilon(\bar{x})$ is sufficiently small.) Then, if p is sufficiently large, $\mathbb{S}(\bar{x}_{p,k})$ is equivalent to $\mathbb{S}(\bar{x})$. Therefore, the modified coalition array, say $I_0, \cdots, I_q$, that belongs to $(v, \bar{x}_{p,k})$ is equal to that belonging to $(v, \bar{x})$. Let $e_j = e(S, \bar{x}_{p,k})$ for any $S \in I_j$ for $j = 1, \cdots, q$. Set $\mathbb{S} = I_0$ and make a (v, x)-balanced 1-extension of $\mathbb{S}$ repeatedly until (i) rank$(e_S : S \in \mathbb{S}^+) = n$ holds; (ii) Lemma 2.2 is applied; or (iii) Corollary 2.2 is applied. In the case of (ii) or (iii), $\mathbb{S}(\bar{x}_{p,k})$ is not a good approximation of $\mathbb{S}(\bar{x})$; therefore, increase p, recalculate, reconstruct, and recheck $I_0, \cdots, I_q$. In the case of (i), as a final step, we only solve a single system of linear equations to obtain the strict solution of $\bar{x}$ (not $\bar{x}_{p,k}$!). The entire system of linear equations is

$$\begin{cases} (1)\ \displaystyle\sum_{j=1}^{n} \bar{x}_j = \bar{x}(N) = v(N); \\ (2)\ \bar{x}_j = v(\{j\}) \text{ for each } j \in b_0; \\ (3)\ e(S_{k_1}, \bar{x}) = \cdots = e(S_{k_{n_j}}, \bar{x}), \text{ where } \{S_{k_1}, \cdots, S_{k_{n_j}}\} = I_j \text{ for each} \\ \qquad j = 1, \cdots, q. \end{cases}$$

However, we can select only a subsystem with rank n in order to calculate $\bar{x}$.

Example 3.1. The following list shows a cooperative game (N, v) for $N = \{1, \cdots, 5\}$ and $x = (0.0979,\ 0.2779,\ 0.0992,\ 0.1379,\ 0.3871)$ as a numerical approximation of $\bar{x}_{p,k}$ for $p = 800$ and $k = -2$.

Set $I_0 = \{N\}$, $I_1 = \{S_1, S_2\} = \{S : e(S, x) \risingdotseq 0.355\}$, and $I_2 = \{S_3, S_4, S_5, S_6\} = \{S : e(S, x) \risingdotseq 0.325\}$. Since $e_{S_1} + e_{S_2} = e_N$, $\mathbb{S} = \{I_0, I_1\}$ is a (v, x)-balanced extension of I_0. However, rank$(e_N, e_{S_1}, e_{S_2}) = 2 < 5$. Therefore, the process

coalition	members of coalition S						
S	player1	player2	player3	player4	player5	$v(S)$	$e(S,x)$
					$*$	0	-0.387
				$*$		0	-0.138
S_1				$*$	$*$	0.88	0.355
			$*$			0	-0.099
S_3			$*$		$*$	0.8125	0.326
			$*$	$*$		0.46	0.223
			$*$	$*$	$*$	0.89	0.266
		$*$				0	-0.278
		$*$			$*$	0.56	-0.105
S_4		$*$		$*$		0.74	0.324
		$*$		$*$	$*$	0.91	0.107
		$*$	$*$			0.68	0.303
		$*$	$*$		$*$	0.48	-0.284
		$*$	$*$	$*$		0.57	0.055
		$*$	$*$	$*$	$*$	0.63	-0.272
	$*$					0	-0.098
	$*$				$*$	0.35	-0.135
S_5	$*$			$*$		0.56	0.324
	$*$			$*$	$*$	0.71	0.087
	$*$		$*$			0.4925	0.295
	$*$		$*$		$*$	0.69	0.106
	$*$		$*$	$*$		0.55	0.215
	$*$		$*$	$*$	$*$	0.37	-0.352
S_6	$*$	$*$				0.7	0.324
	$*$	$*$			$*$	0.72	-0.043
	$*$	$*$		$*$		0.64	0.126
	$*$	$*$		$*$	$*$	0.44	-0.461
S_2	$*$	$*$	$*$			0.83	0.355
	$*$	$*$	$*$		$*$	0.95	0.088
	$*$	$*$	$*$	$*$		0.61	-0.003
N	$*$	$*$	$*$	$*$	$*$	1	0

is continued. Next, since $2e_{S_3} + e_{S_4} + e_{S_5} + e_{S_6} = 2e_N$, we easily know that $\mathbb{S} = \{I_0, I_1, I_2\}$ is a (v,x)-balanced extension of I_0 and $\text{rank}(e_S : S \in \mathbb{S}) = 5$, that is,

(v, x) has property IIM. Lastly, by solving a system of linear equation:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = v(N), \\ v(S_1) - (x_4 + x_5) = v(S_2) - (x_1 + x_2 + x_3), \\ v(S_3) - (x_3 + x_5) = v(S_4) - (x_2 + x_4), \\ v(S_3) - (x_3 + x_5) = v(S_5) - (x_1 + x_4), \\ v(S_3) - (x_3 + x_5) = v(S_6) - (x_1 + x_2), \end{cases}$$

we obtain the solution $x = (0.0975, 02775, 0.1, 0.1375, 0.3875)$. This is the nucleolus of the given game.

4. Conclusions

We summarize the procedure introduced in this paper to obtain the nucleolus by a nonlinear method:

(1) Let k be sufficiently small;

(2) Let p be so large that $\bar{x}_{p,k}$ approximates $\bar{x}$ well; then calculate $\bar{x}_{p,k}$;

(3) Set the modified coalition array that belongs to (v, x), where excesses are to be identified if the difference is less than a sufficiently small positive number ϵ;

(4) Make a (v, x)-balanced extension $\mathbb{S}$ of I_0;

(5) Check the rank condition of $\mathbb{S}$;

(6) If $\mathbb{S}$ does not have property IIM, let ϵ be smaller and return to step (3), or let p be larger and return to step (2);

(7) If rank$(e_S : S \in \mathbb{S}) < n$, extend $\mathbb{S}$ more and return to step (5);

(8) Construct a system of linear equations with a regular coefficient matrix; and solve this;

(9) The solution is the nucleolus of (N, v).

Recalling the Karmarkar method for linear programmings, I expect that this nonlinear method is efficient especially for large-scale games.

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