TAIWANESE JOURNAL OF MATHEMATICS
Vol. 12, No. 6, pp. 1477-1492, September 2008
This paper is available online at http://www.tjm.nsysu.edu.tw/

IN ASPLUND SPACES, APPROXIMATELY CONVEX FUNCTIONS AND REGULAR FUNCTIONS ARE GENERICALLY DIFFERENTIABLE

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Dedicated to Professor Wataru Takahashi on the occasion of his 65th birthday

Abstract. We prove that an approximately convex function on an open subset of an Asplund space is generically Fréchet differentiable, as are genuine convex functions. Thus, we give a positive answer to a question raised by S. Rolewicz. We also prove a more general result of that type for regular functions on an open subset of an Asplund space.

1. Introduction

It is a deep and famous result of Preiss [21] that any locally Lipschitzian real-valued function on an Asplund space is Fréchet differentiable at the points of a dense subset. However, it is known that such a set does not always contain a $\mathcal{G}_\delta$ subset, i.e. a countable intersection of open subsets (we are indebted to Prof. D. Preiss for the information that a full description of such sets is given in [33] and that such a fact was known much before that paper). Thus it is not possible to conclude for instance that given two Lipschitzian functions they have a common point of Fréchet differentiability. It is the purpose of the present article to give a positive answer to such a genericity problem by restricting our attention to well established classes of functions.

The main class of functions we have in view is the class of regular functions in the sense of Clarke ([4, Def. 2.3.4]). For such a class, the main concepts of nonsmooth analysis coincide and better calculus rules are available than for general locally Lipschitzian functions. For that reason, such a class is popular.

The proof of our main result being somewhat involved, we first present a simple proof for the more restricted class of approximately convex functions which has been

Received December 23, 2007.
2000 *Mathematics Subject Classification*: 26E15, 46G05, 58C20.
Key words and phrases: Approximately convex function, Asplund space, Differentiability, Fréchet derivative, Genericity.

recently introduced in [12, Prop. 3.1]. Such a class has obtained a great interest in view of its simplicity and of its generality ([1, 7, 13, 15, 27). It contains the class of paraconvex functions which has been thoroughly studied by S. Rolewicz ([24, 25, 30, 31]) who gave a positive answer to the genericity problem for such a class. It also contains the class of continuously differentiable functions. As recalled in Proposition 2 below, this class satisfies desirable stability properties which make the class large enough. This class retains part of the properties of convex functions, but also of continuously differentiable functions, so that our separate treatment is justified. Moreover, in doing so, we give a positive answer to a question raised by S. Rolewicz in [27] Question 8.

2. Preliminaries

Approximate convexity is defined as follows.

Definition 1. Given $\varepsilon > 0$ and a convex subset C of a Banach space X, a function $f : C \to \mathbb{R}$ is said to be ε-convex if for every $x, y \in C$ and any $t \in [0, 1]$ one has

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon t(1-t) \|x - y\| . \tag{1}$$

A function $f : U \to \mathbb{R}$ defined on an open subset U of X is said to be approximately convex at $x_0 \in U$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that f is ε-convex on the ball $B(x_0, \delta)$. It is approximately convex around $x_0 \in U$ if it is approximately convex at any point of some neighborhood of x_0.

The terminology we use here is slightly different from the one used in [12, Prop. 3.1] and [13] (but coincides with the one of [5]) since we stress the difference between the pointwise property (at x_0) and the local property (around x_0). As mentioned above, the class of approximately convex functions on U (i.e. at each point of U) contains the class of paraconvex functions on U which plays an important role in various fields (Hamilton-Jacobi equations and optimal control [3], duality [19], regularization [10]...). The class of approximately convex functions has been characterized in finite dimensions by a lower C^1 property in the sense of [23], [32], i.e. as suprema of families of C^1 functions. This characterization is extended in [13] (see also [7], [17] for variants). Another characterization uses a subdifferential of the function and an approximate monotonicity property; the locally Lipschitz case is given in [1] and the case of a lower semicontinuous function is presented in [13].

The class of approximately convex functions has interesting stability properties; see for instance [12, Prop. 3.1], [6, Section 6], [1].

Proposition 2. *The set of functions $f : U \to \mathbb{R}$ which are approximately convex at $x_0 \in X$ is a convex cone containing the functions which are strictly differentiable at x_0. It is stable under finite suprema. Moreover, if $f = h \circ g$, where $g : U \to Y$ is strictly differentiable at x_0 and $h : Y \to \mathbb{R}$ is approximately convex at $g(x_0)$, then f is approximately convex at x_0.*

As mentioned above, approximately convex functions retain some of the nice properties of convex functions. In particular they are continuous on segments contained in their domains ([12, Cor. 3.3]) and locally Lipschitzian on the interiors of their domains. Moreover, they have radial derivatives ([12, Cor. 3.5]). Furthermore, the subdifferentials of approximately convex functions all coincide provided they are between the *Fréchet subdifferential* ∂^F and the *Clarke-Rockafellar subdifferential* ∂^C. Recall that

$$x^* \in \partial^F f(x) \Leftrightarrow \forall \varepsilon > 0\ \exists \delta > 0 : \forall u \in B(x, \delta)\ f(u) - f(x) - \langle x^*, u - x \rangle \geq -\varepsilon \|u - x\|$$

and that, for a continuous function f,

$$x^* \in \partial^C f(x) \Leftrightarrow \forall u \in X\ \langle x^*, u \rangle \leq f^C(x, u),$$

where $f^C(x, \cdot)$ is the *Clarke-Rockafellar derivative* given by

$$f^C(x, u) := \sup_{\varepsilon > 0} \limsup_{(t,w) \to (0_+, x)} \inf_{v \in B(u, \varepsilon)} \frac{1}{t} \left(f(w + tv) - f(w) \right).$$

Here a *subdifferential* is a map $\partial : \overline{\mathbb{R}}^X \times X \to \mathcal{P}(X^*)$, where $\overline{\mathbb{R}}^X$ is the set of extended-real valued functions on X, X^* is the dual space of X and $\mathcal{P}(X^*)$ the space of subsets of X^*, such that $\partial f(x) := \partial(f, x)$ is empty if f is not finite at x.

Recall that an *Asplund space* is a Banach space X such that the dual of every separable closed subspace of X is separable. Such spaces have been introduced for their characteristic property: a convex continuous function on a nonempty open convex subset U of an Asplund space X is generically Fréchet differentiable, i.e. Fréchet differentiable on a dense $\mathcal{G}_\delta$-subset of U. Here a subset D is said to be a $\mathcal{G}_\delta$-subset of U if it is the intersection of a countable family of open subsets. It has also been shown by D. Preiss ([21]) that any locally Lipschitz function f on an open subset U of an Asplund space is Fréchet differentiable on a dense subset. Here we make the supplementary assumption that f is approximately convex and we get generic differentiability, i.e. differentiability on a dense $\mathcal{G}_\delta$-subset of U.

Recall that by a *weak* slice* of a nonempty set $A \subset X^*$ one means a subset of A of the form

$$S(x, A, \alpha) = \{x^* \in A : \langle x^*, x \rangle > \sigma_A(x) - \alpha\},$$

where $x \in X\backslash\{0\}, \alpha > 0$ and

$$\sigma_A(x) = \sup\{\langle x^*, x\rangle : x^* \in A\}.$$

The following important characterization of Asplund spaces is well known. It will be used in the proof of the main result below.

Lemma 3. ([20]). *A Banach space X is an Asplund space if and only if its dual space X^* has the Radon-Nikodým property, i.e. every nonempty bounded subset A of X^* admits weak* slices of arbitrary small diameter.*

Let us note the following results.

Lemma 4. *Let U be an open subset of an Asplund space X and let $f : U \to \mathbb{R}$ be a lower semicontinuous function. Let $\partial f : U \rightrightarrows X^*$ be a subdifferential such that $\partial^F f(u) \subset \partial f(u) \subset \partial^C f(u)$ for all $u \in U$. Let $x \in U$ be such that $\partial f(x)$ is nonempty and such that for any $\varepsilon > 0$ there exists some $\delta > 0$ for which $B_\delta := B(x, \delta) \subset U$ and diam$(\partial f(B_\delta)) < \varepsilon$. Then f is (strictly) Fréchet differentiable at x : for every $\varepsilon > 0$ there exists $\delta > 0$ such that $B(x, \delta) \subset U$ and for every $u, v \in B(x, \delta)$ one has*

$$|f(v) - f(u) - \langle x^*, v - u\rangle| \leq \varepsilon \|v - u\|.$$

Proof. Clearly, $\partial f(x)$ is a singleton $\{x^*\}$. Now, given $\varepsilon > 0$, let $\delta > 0$ be such that $B_\delta := B(x, \delta) \subset U$ and diam$(\partial f(B_\delta)) < \varepsilon$. Given $u, v \in B(x, \delta)$, the Mean Value Theorem ([11], [18]) ensures that there exist $w, z \in [u, v]$ and sequences $(w_n) \to w$, $(z_n) \to z$, (w_n^*), (z_n^*) such that $w_n^* \in \partial f(w_n)$, $z_n^* \in \partial f(z_n)$ for all $n \in \mathbb{N}$ and

$$f(v) - f(u) - \langle x^*, v - u\rangle \leq \liminf_n \langle w_n^* - x^*, v - u\rangle \leq \varepsilon \|v - u\|,$$

$$f(u) - f(v) - \langle x^*, u - v\rangle \leq \liminf_n \langle z_n^* - x^*, u - v\rangle \leq \varepsilon \|v - u\|,$$

so that f is strictly Fréchet differentiable at x. ■

Proposition 5. *Let U be an open subset of a normed vector space X. The set of points at which a function $f : U \to \mathbb{R}$ is approximately convex is a $\mathcal{G}_\delta$-set.*

Proof. Let $A \subset U$ be the set of points at which the function f is approximately convex. For $n \in \mathbb{N}\backslash\{0\}$, let A_n be the set of points x for which there exists some $\delta > 0$ such that f is $1/n$-convex on $B(x, \delta)$. Obviously, A_n is open and $\cap_{n=1}^{\infty} A_n = A$. ■

The conclusion of the preceding proposition entail that if f is densely approximately convex on U (i.e. approximately convex at all points of a dense subset of

U), then the set of points of U at which f is approximately convex is a generic subset of U, i.e. it is a dense $\mathcal{G}_\delta$ subset of U. Let us note the following criterion ensuring that the set of points at which f is approximately convex is a generic subset of U. Here we say that a locally Lipschitzian function is *Clarke regular* if it is directionally differentiable at each point, its directional derivative being equal to its Clarke-Rockafellar derivative.

Lemma 6. ([5, Prop. 5]). *Every Clarke regular function on an Asplund space is generically approximately convex.*

We will use the following subdifferential characterization of approximate convexity.

Proposition 7. ([13, Theorem 10]). *Let X be an Asplund space, let $f : X \to \overline{\mathbb{R}}$ be lower semicontinuous, let $x_0 \in \operatorname{dom} f$, and let ∂ be a subdifferential such that $\partial^F f \subset \partial f \subset \partial^C f$. Then the following assertions are equivalent:*

- (a) *f is approximately convex at x_0;*
- (b) *$\partial f : X \rightrightarrows X^*$ is approximately monotone at x_0 in the following sense: for every $\varepsilon > 0$ there exists some $\rho > 0$ such that for all $u, v \in B(x_0, \rho)$, $u^* \in \partial f(u)$, $v^* \in \partial f(v)$ one has*

$$\langle u^* - v^*, u - v \rangle \geq -\varepsilon \|u - v\| .$$

An elementary differentiability result of independent interest will be used in the last part of the paper.

Proposition 8. *Let W and X be normed vector spaces and let $f : W \times X \to \mathbb{R} \cup \{+\infty\}$ be a convex function which is Fréchet (resp. Gâteaux) differentiable at some point $(\overline{w}, \overline{x}) \in W \times X$. Let $p : W \to \overline{\mathbb{R}}$ be the performance function defined by $p(w) = \inf_{x \in X} f(w, x)$. If $p(\overline{w}) = f(\overline{w}, \overline{x})$, then p is Fréchet (resp. Gâteaux) differentiable at $\overline{w}$.*

In particular, if X is a vector subspace of W, if the norm on W is Fréchet (resp. Gâteaux) differentiable off 0 and if $\overline{w} \in W \backslash X$ has a best approximation in X, then the distance function d_X to X is Fréchet (resp. Gâteaux) differentiable at $\overline{w}$.

Proof. Let $r : \mathbb{R}_+ \to \mathbb{R}_+$ be a remainder, i.e. a function such that $r(0) = 0$ and $r(t)/t \to 0$ as $t \to 0_+$ such that

$$|f(\overline{w} + w, \overline{x} + x) - f(\overline{w}, \overline{x}) - \langle (\overline{w}^*, \overline{x}^*), (w, x) \rangle| \leq r(\|(w, x)\|)$$

for $\|(w,x)\|$ small enough, where $(\overline{w}^*, \overline{x}^*)$ is the derivative of f at $(\overline{w}, \overline{x})$. Since f is continuous at $(\overline{w}, \overline{x})$, p is bounded above on a neighborhood of $\overline{w}$, hence is subdifferentiable at $\overline{w}$. Let $\overline{z}^* \in \partial p(\overline{w})$. Then, for $\|w\|$ small enough one has

$$\begin{aligned}0 &\le p(\overline{w}+w) - p(\overline{w}) - \langle \overline{z}^*, w\rangle \\ &\le f(\overline{w}+w, \overline{x}) - f(\overline{w}, \overline{x}) - \langle (\overline{w}^*, \overline{x}^*), (w, 0)\rangle + \langle \overline{w}^* - \overline{z}^*, w\rangle \\ &\le r(\|w\|) + \langle \overline{w}^* - \overline{z}^*, w\rangle.\end{aligned}$$

This shows that $\overline{w}^* = \overline{z}^*$ and that p is differentiable at $\overline{w}$.

The last assertion is obtained by taking $f(w,x) := \|w - x\|$ for $(w,x) \in W \times X$. ■

3. Generic Differentiability of Approximately Convex Functions

The main result of the present section will be deduced from the following statement which is not as striking. The interest of the refinement will appear in the last section.

Proposition 9. *Let X be an Asplund space, let U be an open subset of X and let A be a dense subset of U. Suppose that $f : U \to \mathbb{R}$ is lower semicontinuous on U and approximately convex around each point of A. Then f is (strictly) Fréchet differentiable on a dense $\mathcal{G}_\delta$-subset of U.*

Proof. The proof is inspired by the one of [9] (see also [20, Thm 2.30]). We set $T = \partial^F f$. For each $n \in \mathbb{N}$, $n > 1$, we introduce the set

$$G_n := \{x \in U : \exists \delta > 0,\ B(x,\delta) \subset U,\ \operatorname{diam} T(B(x,\delta)) < \frac{1}{n}\}.$$

Obviously, G_n is open since for all $x \in G_n$ we have $B(x,\delta) \subset G_n$ whenever $\delta > 0$ is such that $B(x,\delta) \subset U$, $\operatorname{diam} T(B(x,\delta)) < 1/n$. Thus $G := \cap_{n=1}^{\infty} G_n$ is a $\mathcal{G}_\delta$-set and, by Lemma 4, f is Fréchet differentiable at every $x \in G$. Thus, it suffices to prove that, for each $n \in \mathbb{N}$, G_n is dense in U.

Let $\overline{u} \in U$ be given. Given $\rho > 0$ such that $B(\overline{u}, \rho) \subset U$, let us show that $G_n \cap B(\overline{u}, \rho)$ is nonempty. Since A is dense in U, we can find some $a \in A \cap B(\overline{u}, \rho)$. Let $\delta \in (0, \rho - d(\overline{u}, a))$ be such that f is approximately convex on $B(a,\delta)$. Since an approximately convex function is locally Lipschitzian, shrinking δ if necessary, we may assume that f is Lipschitzian on $B(a,\delta)$. Thus $T(B(a,\delta))$ is bounded and since $\partial^F f(u) = \partial^C f(u)$ for all $u \in B(a,\delta)$ by [12, Thm 3.6], T has nonempty values on $B(a,\delta)$. We shall show that $G_n \cap B(a,\delta)$ is nonempty, what will imply that $G_n \cap B(\overline{u}, \rho)$ is nonempty too. According to Lemma 3, we can find $\alpha > 0$, $z \in S_X$, the unit sphere of X, such that the diameter of the weak* slice

$$S := S(z, T(B(a,\delta)), \alpha) = \{u^* \in T[B(a,\delta)] : \langle u^*, z\rangle > \sigma_{T(B(a,\delta))}(z) - \alpha\}$$

is less than $\frac{1}{n}$. Let us take $u \in B(a,\delta)$, $u^* \in T(u)$ such that

$$\langle u^*, z\rangle > \sigma_{T(B(a,\delta))}(z) - \alpha/2.$$

By the approximate monotonicity of T at u exhibited in Proposition 7, there exists $\varepsilon \in (0,\delta)$ such that

$$\langle v^* - u^*, v - u\rangle \geq -\frac{\alpha}{2}\|v - u\|, \qquad \forall v \in B(u,\varepsilon), \forall v^* \in T(v).$$

Taking $t > 0$ such that $v := u + tz \in B(a,\delta) \cap B(u,\varepsilon)$, one has

$$\langle v^* - u^*, (u + tz) - u\rangle \geqslant -(\alpha/2)t, \qquad \forall v^* \in T(v).$$

Thus,

$$\langle v^*, z\rangle \geqslant \langle u^*, z\rangle - \alpha/2 > \sigma_{T(B(a,\delta))}(z) - \alpha \qquad \forall v^* \in T(v).$$

Since T is norm to weak* upper semicontinuous at v (because f is locally Lipschitz around v and T coincides with the Clarke subdifferential [4, Prop. 2.1.5]), there exists $\gamma > 0$ such that $B(v,\gamma) \subset B(a,\delta)$ and

$$\langle w^*, z\rangle \geqslant \sigma_{T(B(a,\delta))}(z) - \alpha \qquad \forall w \in B(v,\gamma),\ w^* \in T(w).$$

That is $T[B(v,\gamma)] \subset S$. Hence $\mathrm{diam} T[B(v,\gamma)] < \frac{1}{n}$. That is $v \in G_n \cap B(a,\delta)$. The proof is complete. ■

The preceding result is close to Corollary 2.2 (ii) of [8]. There f is just lower semicontinuous and it is shown that the set of points where f is Fréchet subdifferentiable but not Fréchet directionally differentiable is of first category in U. However, the linearity of the derivative is not obtained on the complement of this set.

Taking for A the set U itself, we get the following consequence.

Theorem 10. *Let $f : U \to \mathbb{R}$ be a lower semicontinuous, approximately convex function on an open subset U of an Asplund space. Then f is Fréchet differentiable on a dense $\mathcal{G}_\delta$-subset of U.*

This theorem can be deduced from a delicate result of Zajícek [34] asserting that a lower semicontinuous function f on an open subset U of an Asplund space X

has the property that the set of points where f is Fréchet subdifferentiable but not Fréchet differentiable is first category in U. Here we avoid the use of such a result but we rely on characterizations of Asplund spaces which are not simple either but which can be considered as classical. In order to get the result via [34], as above, one uses the facts that an approximately convex function on an open subset U of X is locally Lipschitzian and that its Fréchet subdifferential coincides with the Clarke subdifferential, hence is nonempty valued.

4. An Extension to Regular Functions

By using the well known notion of Banach-Mazur game ([16], [20, Thm 4.23], [22]), we can prove a general version of Proposition 9 and of its consequences. Let us recall that a *Banach-Mazur game* on a nonempty open set U of X with objective a subset G of U is a decreasing sequence of nonempty open subsets of $U : U_1 \supseteq V_1 \supseteq U_2 \supseteq V_2 \supseteq \cdots \supseteq U_n \supseteq V_n \supseteq \cdots$, where (U_n) and (V_n) have been chosen by player A and by player B alternatively. Player B is said to be the *winner* if $\cap_{n=1}^{\infty} V_n \subseteq G$. We say that player B has a winning strategy if using it, B wins for any choice of A.

Lemma 11. ([16], [20, Thm 4.23]). *The player B has a winning strategy if and only if the objective set G contains a dense $\mathcal{G}_\delta-$set.*

The following technical lemma will be needed in the proof of the main result of this section.

Lemma 12. *Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of equivalent norms on X such that there exist $m, m' > 0$ such that*

$$m\|x\| \le p_n(x) \le m'\|x\|, \text{ for all } n \in \mathbb{N},\ x \in X.$$

Let $(e_n)_{n\in\mathbb{N}}$ be a sequence of elements in X satisfying $p_n(e_n) = p_n(e_{n-1}) = 1$ for every $n \in \mathbb{N}$ and such that there are $x^* \in X^*$ and $\lambda > 0$ satisfying $\langle x^*, e_n\rangle > \lambda$ for all $n \in \mathbb{N}$. If $\sum_{n=0}^{\infty} d(e_{n+1}, \mathbb{R}e_n)$ is finite then the sequence $(e_n)_{n\in\mathbb{N}}$ is convergent.

Proof. For each $n \in \mathbb{N}$, let $t_n \in \mathbb{R}$ be such that $\|e_{n+1} - t_n e_n\| = d(e_{n+1}, \mathbb{R}e_n)$. Set $u_n = e_{n+1} - t_n e_n$. Then $\sum_{n=0}^{\infty} \|u_n\|$ is finite. Hence $\|u_n\| \to 0$ as $n \to \infty$. Thus, from the relation

$$\langle x^*, e_{n+1}\rangle = t_n\langle x^*, e_n\rangle + \langle x^*, u_n\rangle,$$

and since $\langle x^*, e_n\rangle > \lambda$ for all $n \in \mathbb{N}$, then $t_n > 0$ when n is sufficiently large. We can assume that $t_n > 0$ for all n. Since $\sum_{n=0}^{\infty} \|u_n\|$ is finite, then $\sum_{n=0}^{\infty} p_{n+1}(u_n)$ is finite. Hence, $\sum_{n=0}^{\infty} |1 - t_n| < +\infty$ by the following relation

$$|1 - t_n| = |p_{n+1}(e_{n+1}) - t_n p_{n+1}(e_n)| \le p_{n+1}(e_{n+1} - t_n e_n) = p_{n+1}(u_n).$$

Thus

$$\sum_{n=0}^{\infty}\|e_{n+1}-e_n\| \le \sum_{n=0}^{\infty}\|e_n\||1-t_n|+\sum_{n=0}^{\infty}\|u_n\| \le m^{-1}\sum_{n=0}^{\infty}|1-t_n|+\sum_{n=0}^{\infty}\|u_n\| < +\infty.$$

This implies that $(e_n)_{n\in\mathbb{N}}$ is convergent. ■

Let us say that a set-valued mapping $T : U \rightrightarrows X^*$ is *Fréchet continuous at* $x \in U$ if $T(x)$ is a singleton and T is norm to norm upper semicontinuous at x, i.e., for every $\varepsilon > 0$ there exists a neighborhood V of x such that

$$T(V) \subseteq T(x) + \varepsilon B^*.$$

Theorem 13. *Suppose that X admits a Fréchet smooth equivalent norm and let U be an open subset of X. Suppose that $f : U \to \mathbb{R}$ is lower semicontinuous on U and approximately convex at each point on a dense subset of U. Then f is (strictly) Fréchet differentiable on a dense $\mathcal{G}_\delta$-subset of U.*

Proof. Assume that the norm $\|\cdot\|$ is Fréchet smooth. Set $T := \partial^C f$. By Lemma 4, it suffices to show that the set G of $x \in U$ such that T is Fréchet continuous at x is a generic subset in U. To prove this, we use the Banach-Mazur game on U with the objective set G. Let A and B be two players in this Banach-Mazur game. By Lemma 11, it suffices to prove that B has a winning strategy. This means that for any choice (U_n) of A we will construct a choice (V_n) of B such that $\cap_{n=0}^{\infty} V_n \subseteq G$.

Let us choose a decreasing sequence of positive numbers (ε_n) such that $\varepsilon_0 = 1$, $\sum_{n=1}^{\infty}\sqrt{\varepsilon_n} < 1$. For each n, let us define D_n as the set of $x \in U$ for which there exists some $\delta > 0$ such that $B(x,\delta) \subset U$, T is bounded on $B(x,\delta)$ and

$$\langle y^* - x^*, y - x\rangle > -\frac{\varepsilon_n^2}{2}\|y - x\| \qquad \forall y \in B(x,\delta),\ x^* \in T(x),\ y^* \in T(y).$$

By Proposition 3.2 in [12] and Theorem 10 in [13], D_n is a dense open subset in U. First, suppose that U_0 has been chosen by A. Player B can choose an open set $V_0 \subseteq U_0 \cap D_0$ such that $T(V_0)$ is bounded (for example, $V_0 := B(x,\delta)$ with $x \in U_0 \cap D_0$ and $\delta > 0$ such that T is bounded on $B(x,\delta)$ and $B(x,\delta)$ is included in $U_0 \cap V_0$). Set $e_0 := 0$, $p_0 := 0$, $p_1 := \varepsilon_0 d(\cdot, \mathbb{R}e_0) = \|\cdot\|$ and denote by p_1^* the dual norm of p_1 on X^*. For $U_1 \subseteq V_0$ chosen by A, set

$$s_1 := \sup\{p_1^*(x^*) :\ x^* \in T(U_1 \cap D_1)\} < +\infty.$$

Let $x_1 \in U_1 \cap D_1$, $x_1^* \in T(x_1)$ and $e_1 \in X$ be such that

$$p_1(e_1) = 1 \quad \text{and} \quad \langle x_1^*, e_1\rangle > s_1 - \varepsilon_1^2/2.$$

Then player B can choose

$$V_1 := \{x \in U_1 \cap D_1 : \ \langle x^*, e_1 \rangle > s_1 - \varepsilon_1^2 \ \forall x^* \in T(x)\}.$$

Indeed, since f is locally Lipschitzian at each point in D_1, then $T = \partial^C f$ is norm to weak* upper semicontinuous at each point of D_1. Hence V_1 is an open set. Let us prove that V_1 is nonempty. Since $x_1 \in U_1 \cap D_1$, there is some $\delta > 0$ such that $B(x_1, \delta) \subseteq U_1$ and

$$\langle y^* - x_1^*, y - x_1 \rangle > -\frac{\varepsilon_1^2}{2}\|x_1 - y\|, \text{ for all } y \in B(x_1, \delta),\, y^* \in T(y),\ x_1^* \in T(x_1).$$

Taking $y := x_1 + \delta e_1/2$ in this relation, one obtains

$$\langle y^*, e_1 \rangle > \langle x_1^*, e_1 \rangle - \frac{\varepsilon_1^2}{2}\|e_1\| > s_1 - \varepsilon_1^2 \text{ for all } y^* \in T(y),\ x_1^* \in T(x_1).$$

Thus, since D_1 is dense in U and $T = \partial^C f$ is norm to weak* upper semicontinuous on D_1, we can find $z \in U_1 \cap D_1$ such that

$$\langle z^*, e_1 \rangle > s_1 - \varepsilon_1^2 \text{ for all } z^* \in T(z).$$

That is, V_1 is nonempty.

Suppose defined for $k = 1, ..., n$ nonempty open subsets U_k, V_k, real numbers s_k, norms p_k on X, $x_k \in U_k \cap D_k$, $x_k^* \in T(x_k)$, $e_k \in X$ such that, for $k = 1, ..., n$, one has

$$p_k^2(x) := p_{k-1}^2(x) + \varepsilon_{k-1} d^2(x, \mathbb{R}e_{k-1}), \tag{2}$$

$$s_k := \sup\{p_k^*(x^*) : \ x^* \in T(U_k \cap D_k)\}, \tag{3}$$

$$p_k(e_k) = 1, \quad \langle x_k^*, e_k \rangle > s_k - \varepsilon_k^2/2, \tag{4}$$

$$V_k := \{x \in U_k \cap D_k : \ \langle x^*, e_k \rangle > s_k - \varepsilon_k^2 \ \forall x^* \in T(x)\}. \tag{5}$$

Given $U_{n+1} \subset V_n$, set

$$\begin{aligned} p_{n+1}^2(x) &:= p_n^2(x) + \varepsilon_n d^2(x, \mathbb{R}e_n), \\ s_{n+1} &:= \sup\{p_{n+1}^*(x^*) : \ x^* \in T(U_{n+1} \cap D_{n+1})\}. \end{aligned}$$

Take $x_{n+1} \in U_{n+1} \cap D_{n+1}$, $x_{n+1}^* \in T(x_{n+1})$ and $e_{n+1} \in X$ such that

$$p_{n+1}(e_{n+1}) = 1 \quad \text{and} \quad \langle x_{n+1}^*, e_{n+1} \rangle > s_{n+1} - \varepsilon_{n+1}^2/2.$$

Then B can choose

$$V_{n+1} := \{x \in U_{n+1} \cap D_{n+1} : \ \langle x^*, e_{n+1}\rangle > s_{n+1} - \varepsilon_{n+1}^2 \ \forall x^* \in T(x)\},$$

As above, one can show that V_{n+1} is a nonempty and open set. Thus, for any choice (U_n) of A, player B has a strategy to obtain a sequence (V_n) by constructing the sequences (s_n), (e_n) and the sequence of norms (p_n) as above. To complete the proof, we need to prove that $\cap_{n=1}^{\infty} V_n \subseteq G$.

Set

$$p^2(x) = \|x\|^2 + \sum_{n=1}^{\infty} \varepsilon_n d^2(x, \mathbb{R}e_n).$$

Then p is an equivalent norm on X since $\|x\| \le p(x) \le \sqrt{2}\|x\|$ for all $x \in X$. Obviously, the sequence (p_n^2) uniformly converges to p^2 on every bounded set of X.

By Proposition 8, p_n^2 is Fréchet differentiable, too. By using the Weierstrass $M-$test, we can show that p_n' is uniformly convergent on every bounded set of X. Hence, p^2 is Fréchet differentiable on X. Consequently, the norm p is Fréchet smooth.

Since (p_n) is an increasing sequence, the sequence (p_n^*) is decreasing. Therefore, the sequence (s_n) is a decreasing sequence of nonnegative numbers. Thus it is convergent, say, to $s \ge 0$. Let us consider the following two cases.

Case 1. $s = 0$. Obviously, for any $x \in \cap_{n=1}^{\infty} V_n$ (if it is nonempty), by (3) and the equivalence of p_{n+1} with p_1 we have $T(x) = \{0\}$ and T is norm to norm upper semicontinuous at x. That is, $\cap_{n=1}^{\infty} V_n \subseteq G$.

Case 2. $s > 0$. Let $x \in \cap_{n=1}^{\infty} V_n$, $x^* \in T(x)$. By our construction, for each n, one has $p_{n+1}(e_{n+1}) = p_{n+1}(e_n) = 1$ and $\langle x^*, e_n\rangle > s_n - \varepsilon_n^2 > s/2$ when n is sufficiently large. Moreover, since $p_{n+1}(e_n) = 1$

$$s_{n+1} \ge p_{n+1}^*(x^*) \ge \langle x^*, e_n\rangle > s_n - \varepsilon_n^2.$$

Since $s_n := \sup p_n^*(T(U_n \cap D_n))$ and $x_{n+1}^* \in T(U_n \cap D_n)$, we have $s_n \ge p_n^*(x_{n+1}^*)$ and, by (4) with $k := n+1$,

$$\begin{aligned} p_n(e_{n+1}) &\ge p_n(e_{n+1}) p_n^*(x_{n+1}^*)/s_n \ge \langle x_{n+1}^*, e_{n+1}\rangle / s_n \\ &> s_{n+1}/s_n - \varepsilon_{n+1}^2/2s_n > 1 - \varepsilon_n^2/s_n - \varepsilon_{n+1}^2/2s_n > 1 - 3\varepsilon_n^2/2s. \end{aligned}$$

Consequently,

$$d^2(e_{n+1}, \mathbb{R}e_n) = \varepsilon_n^{-1}(1 - p_n^2(e_{n+1})) \le \varepsilon_n^{-1}(1 - (1 - 3\varepsilon_n^2/2s)^2) \le 3\varepsilon_n/s.$$

Therefore, $d(e_{n+1}, \mathbb{R}e_n) \leq \sqrt{3/s}\sqrt{\varepsilon_n}$, which implies $\sum_{n=1}^{\infty} d(e_{n+1}, \mathbb{R}e_n) < +\infty$. According to Lemma 12, the sequence (e_n) converges to some $e \in X$. Hence $p(e) = \lim_{n\to\infty} p_n(e_n) = 1$.

From the relations

$$s_n \geq p_n^*(x^*)) \geq \langle x^*, e_n \rangle \geq s_n - \varepsilon_n^2,$$

by letting $n \to \infty$, one obtains $\langle x^*, e \rangle = s = p^*(x^*)$ or $\langle x^*/s, e \rangle = 1 = p^*(x^*/s)$, $p(e) = 1$. Thus $x^*/s = p'(e)$ and since p is Fréchet smooth, then $T(x)$ is a singleton.

It remains to show that T is Fréchet continuous at x. Let $\varepsilon > 0$ be given. Take $\delta \in (0,1)$ such that $\delta^2 p'(e) + 2\delta B^* \subseteq \varepsilon B^*$ and that

$$\|p'(u) - p'(e)\| \leq \frac{\varepsilon}{2(s+1)} \qquad \text{for all } u \in B(e, \delta).$$

For every $n \geq 2$ by (5), one has

$$\langle y^*, e_{n-1} \rangle > s_{n-1} - \varepsilon_{n-1}^2 \qquad \text{for all } y \in V_n,\ y^* \in T(y).$$

Hence there exists an index k such that for all $n > k$, one has

$$\langle y^*, e \rangle > s - \delta^2/2 \text{ for all } \quad y \in V_n,\ y^* \in T(y). \tag{6}$$

On the other hand, by our construction, $p^*(y^*) \leq p_{n-1}^*(y^*) \leq s_{n-1}$ for all $y \in V_n$, $y^* \in T(y)$. Therefore, when n is sufficiently large, say, $n > k$, then

$$p^*(y^*) \leq s + \delta^2/2 \quad \text{for all } \ y \in V_n,\ y^* \in T(y). \tag{7}$$

Since $p(e) = 1$ and by the definition of p^*, the inequalities (6), (7) imply that for each $y \in V_n$, $y^* \in T(y)$, one has

$$\langle y^*, u - e \rangle \leq (s + \delta^2/2)(p(u) - p(e)) + \delta^2, \text{ for all } u \in X. \tag{8}$$

For each $n > k$, $y \in V_n$, $y^* \in T(y)$, let us consider the function g defined by

$$f(u) := (s + \delta^2/2)p(u) - \langle y^*, u \rangle,\ u \in X.$$

Then, by relation (8), $f(e) \leq \inf_{u\in X} f(u) + \delta^2$. By the Ekeland variational principle, there exists $z \in B(e, \delta)$ such that

$$f(z) \leq f(u) + \delta\|u - z\|, \text{ for all } u \in X.$$

Consequently,

$$y^* \in (s + \delta^2/2)p'(z) + \delta B^* \subseteq (s + \delta^2/2)p'(e) + \varepsilon/2B^* + \delta B^* \subseteq x^* + \varepsilon B^*.$$

Thus $T(V_n) \subseteq T(x) + \varepsilon B^*$. That is, T is Fréchet continuous at x. The proof is complete. ■

By an analogous argument, we can obtain the following result for the case of Gâteaux differentiability. The detailed proof is omitted.

Theorem 14. *Suppose that X admits a Gâteaux smooth equivalent norm and let U be an open subset of X. Suppose that $f : U \to \mathbb{R}$ is lower semicontinuous on U and approximately convex at each point of a dense subset of U. Then g is Gâteaux differentiable on a dense $\mathcal{G}_\delta$-subset of U.*

In order to extend the preceding results to regular functions, we will use the following lemmas due to Zajícek ([34], Lemma 1 and Lemma 2). Here X is an arbitrary normed vector space and $\mathcal{S}(X)$ denotes the family of closed separable subspaces of X.

Lemma 15. *Let U be an open subset of X and let G be a generic subset of U. Then there exists a mapping $S : \mathcal{S}(X) \to \mathcal{S}(X)$ satisfying $Z \subset S(Z)$ for all $Z \in \mathcal{S}(X)$ and such that the following assertion holds: if Y is a closed subspace of X for which the set $B(Y) := \bigcup\{Z : S(Z) \subset Y\}$ is dense in Y, then the set $G \cap Y$ is dense in $U \cap Y$.*

Lemma 16. *Let U be an open subset of X and let $f : U \to \mathbb{R}$ be an arbitrary function. Then there exists a mapping $T : \mathcal{S}(X) \to \mathcal{S}(X)$ satisfying $Z \subset T(Z)$ for all $Z \in \mathcal{S}(X)$ and such that the following assertion holds: if Y is a closed subspace of X for which the set $C(Y) := \bigcup\{Z : T(Z) \subset Y\}$ is dense in Y, then g is strictly differentiable at each point of $U \cap Y$ at which $f|_{U\cap Y}$ is strictly Fréchet differentiable.*

Corollary 17. *Let X be an Asplund space and let U be an open subset of X. Suppose that $f : U \to \mathbb{R}$ is a continuous function which is approximately convex at each point of a dense subset A of U. Then g is (strictly) Fréchet differentiable at each point of a dense $\mathcal{G}_\delta$-subset of U.*

Proof. Note that the set F of points at which f is strictly Fréchet differentiable is a $\mathcal{G}_\delta$-set ([34, Thm A]). Thus, it suffices to show that F is dense in U. Let $u \in U$ and $\varepsilon > 0$ be given. We use the mappings $S, T : \mathcal{S}(X) \to \mathcal{S}(X)$ of Lemmas 15, 16 to construct an increasing sequence (Z_n) of $\mathcal{S}(X)$ with $Z_0 := \mathbb{R}u$ by setting $Z_{n+1} := S(Z_n) + T(Z_n)$. Let Y be the closure of the union of the Z_n's. Then $B(Y)$ and $C(Y)$ are dense in Y since for all $n \in \mathbb{N}$ we have $S(Z_n) \subset Z_{n+1} \subset Y$, $T(Z_n) \subset Z_{n+1} \subset Y$. Since by Proposition 5 and our assumption the set G at which

g is approximately convex is a dense $\mathcal{G}_\delta$ set, Lemma 15 ensures that $G \cap Y$ is dense in $U \cap Y$. Now, since X is an Asplund space, the closed separable subspace Y of X has a separable dual. Hence Y admits a Fréchet smooth renorm ([2, Thm 4.13]). According to Theorem 13, the partial function $f|_{U \cap Y}$ is strictly Fréchet differentiable at each point of a dense $\mathcal{G}_\delta$-subset of $U \cap Y$. By Lemma 16, f itself is strictly differentiable at each such point. Hence, there exists a point $y \in B(u, \varepsilon)$ at which f is strictly Fréchet differentiable. ■

Corollary 18. *Let U be an open subset of an Asplund space X and let $f : U \to \mathbb{R}$ be a locally Lipschitzian regular function. Then f is Fr échet differentiable at each point of a dense $\mathcal{G}_\delta$-subset of U.*

Proof. By [5, Prop. 5] f is approximately convex at each point of a dense subset of U, so that the preceding corollary applies. ■

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