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COMBINATORIAL STRUCTURES OF PSEUDOMANIFOLDS AND MATROIDS

Chien-Hung Chen, Shyh-Nan Lee and Mau-Hsiang Shih*

Abstract. We prove a multiple combinatorial Stokes' theorem and a multiple Sperner's lemma and formulate their matroid versions. The combinatorial properties of pseudomanifolds with matroid structures are discussed.

1. Introduction

The theory of combinatorics of complexes may be traced back to 1928 [16] when Sperner discovered a combinatorial lemma, that is globally called *Sperner's lemma*, which gave a drastic simplification of proofs of two topological theorems, namely theorems of invariance of domain and invariance of dimension. In 1929, Knaster, Kuratowski and Mazurkiewicz [4] used Sperner's lemma to give a combinatorial proof of Brouwer's fixed-point theorem. In 1967, Scarf [12] used Sperner's lemma to give a constructive proof of Brouwer's fixed-point theorem and in 1974, Kuhn [5] gave a constructive proof of the fundamental theorem of algebra based on the combinatorial Stokes' theorem. In 1973, Shapley [13] generalized Sperner's lemma with balancd structure, and gave a simple proof of Scarf's theorem concerning the nonemptyness of cores of NTU games. On the other hand, in 1945, Tucker [17] proved a combinatorial lemma in the cube which gave a combinatorial proof of Lusternik-Schnirelmann's topological theorem. In 1967, Ky Fan [2] proved a combinatorial theorem which called combinatorial Stokes' theorem, giving a common generalization of Sperner's lemma and Tucker's combinatorial lemma. In 1992, Shih and Lee [14] proved a combinatorial Lefschetz fixed-point formula, put Sperner's lemma into the form of "alternating sum," and showed that Sperner's

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*The corresponding author is Mau-Hsiang Shih.

lemma is the case of the Lefchetz number one for any simplical map on a triangulation of a simplex. In 1989, Bapat [1] proved a multiple Sperner's lemma which gave a combinatorial proof of Gale's theorem [3]. In 1993, Shih and Lee [15] obtained a multiple balanced Sperner's lemma which is a common generalization of Shapley's theorem [13] and Bapat's theorem[1]. In 1998, Lee and Shih [6] proved a multiple Stokes' theorem. In 2008, Meunier [10] gave a different approach of Lee and Shih's result. In 1980, Lovasz [9] gave a matroid version of Sperner's lemma and Lee and Shih [7] gave its completion. The purpose of this paper is to give further generalizations of Lee and Shih's results concerning Stokes' theorem on pseudomanifolds [6] and matroids [7, 8].

2. Definitions and Notations

An (*abstract*) *complex* is a finite collection $\mathcal{K}$ of nonempty finite sets such that

$(\mathbf{K1})$ if σ is a member of $\mathcal{K}$ so is every nonempty subset of σ.

The members of $\mathcal{K}$ are its *simplexes*. A simplex σ of $\mathcal{K}$ is a *d-simplex* of $\mathcal{K}$ if the cardinality $|\sigma|$ of σ is $d+1$, and a subset τ of σ is an *r-face* of σ if τ is an r-simplex of $\mathcal{K}$. The union of all simplexes of $\mathcal{K}$ is the *vertex set* $V(\mathcal{K})$ of $\mathcal{K}$.

A *d-pseudomanifold* is a complex $\mathcal{K}$ having the following two properties:

$(\mathbf{M1})$ Every simplex of $\mathcal{K}$ is a face of at least one d-simplex of $\mathcal{K}$.

$(\mathbf{M2})$ Every $(d-1)$-simplex of $\mathcal{K}$ is a common face of at most two distinct d-simplexes of $\mathcal{K}$.

A *boundary* $(d-1)$*-simplex* of $\mathcal{K}$ is a $(d-1)$-simplex of $\mathcal{K}$ that is a face of exactly one d-simplex of $\mathcal{K}$. The set of all boundary $(d-1)$-simplexes of $\mathcal{K}$ is denoted by $\partial\mathcal{K}$.

Let $\sigma = \{v_0, v_1, \ldots, v_d\}$ be a set of $d+1$ elements. Then there are $(d+1)!$ orderings of the elements of σ. Two orderings $I = (v_{i_0}, v_{i_1}, \ldots, v_{i_d})$ and $J = (v_{j_0}, v_{j_1}, \ldots, v_{j_d})$ *have the same orientation*, denoted by $I \sim J$, if $\binom{v_{i_0} v_{i_1} \cdots v_{i_d}}{v_{j_0} v_{j_1} \cdots v_{j_d}}$ is an even permutation. If $d > 0$, then the $(d+1)!$ orderings fall into two equivalence classes. Each of these classes is called an *orientation* of σ, and if we fix one of them arbitrarily, the other one is called the *opposite orientation*. The orientation of σ determined by the ordering $(v_{i_0}, v_{i_1}, \ldots, v_{i_d})$ is denoted by $(+1)[v_{i_0}, v_{i_1}, \ldots, v_{i_d}]$ and its opposite orientation is denoted by $(-1)[v_{i_0}, v_{i_1}, \ldots, v_{i_d}]$. If $d = 0$, there is only one class $[v_0]$ and hence only one orientation of σ. We call the two symbols $(+1)[v_0]$ and $(-1)[v_0]$ *orientations on the singleton* $\{v_0\}$ and they are defined to be *opposite orientations on the singleton* $\{v_0\}$.

Given an orientation $\omega = \varepsilon[v_0, v_1, \ldots, v_d]$ on the set $\sigma = \{v_0, v_1, \ldots, v_d\}$ where $\varepsilon = \pm 1$ and $d > 0$. For each $k = 0, 1, \ldots, d$, the *induced orientation* on $\sigma \setminus \{v_k\}$ from ω is the well defined orientation $(-1)^k \varepsilon[v_0, \ldots, v_{k-1}, v_{k+1}, \ldots, v_d]$ on $\sigma \setminus \{v_k\}$.

A d-pseudomanifold $\mathcal{K}$ is *orientable* if there is an orientation-valued map ω on the set of all d-simplexes of $\mathcal{K}$ which satisfies the following two conditions:

($\mathbf{C1}$) For each d-simplex σ of $\mathcal{K}$, $\omega(\sigma)$ is an orientation on σ.

($\mathbf{C2}$) If τ is an $(d-1)$-simplex of $\mathcal{K}$ which is a common face of two distinct d-simplexes σ and σ' of $\mathcal{K}$, then $\omega(\sigma)$ and $\omega(\sigma')$ induce opposite orientations on τ.

The pair $(\mathcal{K}, \omega)$ is called a *coherently oriented d-pseudomanifold*.

Let m be a positive integer and $\mathcal{K}$ be a d-pseudomanifold. An *m-labelling* in $\mathcal{K}$ is a map φ on $V(\mathcal{K})$ such that for each vertex v of $\mathcal{K}$, $\varphi(v)$ is an m-tuple $(\varphi_1(v), \ldots, \varphi_m(v))$ where $\varphi_k(v) \in \{0, 1, \ldots, d\}$ for $k = 1, \ldots, m$. Given $\sigma \in \mathcal{K}$ and $f : \sigma \to \{1, \ldots, m\}$, the pair (σ, f) is *complete* or *subcomplete* if the set $\{\varphi_{f(v)}(v) \mid v \in \sigma\}$ is $\{0, 1, \ldots, d\}$ or $\{0, 1, \ldots, d-1\}$ respectively. When $\mathcal{K}$ is oriented, there is an orientation-valued map ω on the set of all d-simplexes of $\mathcal{K}$ so that $(\mathcal{K}, \omega)$ is a coherently oriented d-pseudomanifold. Then the pair (σ, f) is *positively* or *negatively complete* if $\varphi_{f(v_k)}(v_k) = k$ for $k = 0, 1, \ldots, d$, and $\omega(\sigma) = (+1)[v_0, v_1, \ldots, v_d]$ or $\omega(\sigma) = (-1)[v_0, v_1, \ldots, v_d]$, respectively. Let τ be a $(d-1)$-face of a d-simplex σ of $\mathcal{K}$ and let $g : \tau \to \{1, \ldots, m\}$. Then the pair (τ, g) is *positively* or *negatively subcomplete* in σ if the induced orientation on τ from $\omega(\sigma)$ is, respectively, $(+1)[v_0, v_1, \ldots, v_{d-1}]$ or $(-1)[v_0, v_1, \ldots, v_{d-1}]$ and $\varphi_{g(v_k)}(v_k) = k$ for $k = 0, 1, \ldots, d-1$. When $\tau \in \partial\mathcal{K}$ such d-simplex σ is unique, we simply call the pair (τ, g) *positively* or *negatively subcomplete*. We define

$$\mathcal{K}(\varphi) = \{(\sigma, f) \mid (\sigma, f) \text{ is complete }\},$$

$$\partial\mathcal{K}(\varphi) = \{(\tau, g) \mid (\tau, g) \text{ is subcomplete and } \tau \in \partial\mathcal{K}\},$$

and, when $(\mathcal{K}, \omega)$ is a coherently oriented d-pseudomanifold,

$$\mathcal{K}^+(\varphi) = \{(\sigma, f) \mid (\sigma, f) \text{ is positively complete }\},$$

$$\mathcal{K}^-(\varphi) = \{(\sigma, f) \mid (\sigma, f) \text{ is negatively complete }\},$$

$$\partial\mathcal{K}^+(\varphi) = \{(\tau, g) \mid (\tau, g) \text{ is positively subcomplete }\},$$

$$\partial\mathcal{K}^-(\varphi) = \{(\tau, g) \mid (\tau, g) \text{ is negatively subcomplete }\}.$$

We finally define $\mathcal{K}(\varphi)_*$, $\mathcal{K}^+(\varphi)_*$ and $\mathcal{K}^-(\varphi)_*$ to be the sets of the pairs (σ, f) of $\mathcal{K}(\varphi)$, $\mathcal{K}^+(\varphi)$ and $\mathcal{K}^-(\varphi)$ such that f is one-to-one and define $\partial\mathcal{K}(\varphi)_*$, $\partial\mathcal{K}^+(\varphi)_*$ and $\partial\mathcal{K}^-(\varphi)_*$ to be the sets of the pairs (τ, g) of $\partial\mathcal{K}(\varphi)$, $\partial\mathcal{K}^+(\varphi)$ and $\partial\mathcal{K}^-(\varphi)$ such that g is one-to-one, respectively.

3. Multiple Combinatorial Stokes' Theorem

Theorem 1. *Let φ be an m-labelling in a d-pseudomanifold $\mathcal{K}$ $(d > 0)$. Then*

$$|\mathcal{K}(\varphi)| \equiv m|\partial\mathcal{K}(\varphi)| \quad \textit{(mod 2)} \tag{3.1}$$

and

$$|\mathcal{K}(\varphi)_*| \equiv (m-d)|\partial\mathcal{K}(\varphi)_*| \quad \textit{(mod 2).} \tag{3.2}$$

Suppose further, $(\mathcal{K}, \omega)$ is a coherently oriented d-pseudomanifold, then

$$(-1)^d\{|\mathcal{K}^+(\varphi)| - |\mathcal{K}^-(\varphi)|\} = m\{|\partial\mathcal{K}^+(\varphi)| - |\partial\mathcal{K}^-(\varphi)|\} \tag{3.3}$$

and

$$(-1)^d\{|\mathcal{K}^+(\varphi)_*| - |\mathcal{K}^-(\varphi)_*|\} = (m-d)\{|\partial\mathcal{K}^+(\varphi)_*| - |\partial\mathcal{K}^-(\varphi)_*|\}. \tag{3.4}$$

Proof. Let

$$S = \{(\sigma, f) \mid \sigma \text{ is a } d\text{-simplex and } f : \sigma \to \{1, \ldots, m\}\}$$

and

$$T = \{(\tau, g) \mid \tau \text{ is a } (d-1)\text{-simplex and } g : \tau \to \{1, \ldots, m\}\}.$$

Define an incidence relation $\prec$ from T to S by $(\tau, g) \prec (\sigma, f)$ if and only if

$(\mathbf{R1})$ $(\sigma, f) \in S$ and $(\tau, g) \in T$,
$(\mathbf{R2})$ (τ, g) is subcomplete,
$(\mathbf{R3})$ $\tau \subset \sigma$ and $g = f|_\tau$ (the restriction of f to τ).

Put

$$\begin{aligned}
S_1 &= \{(\sigma, f) \in S \mid (\sigma, f) \text{ is complete }\},\\
S_2 &= \{(\sigma, f) \in S \mid (\sigma, f) \text{ is subcomplete }\},\\
S_3 &= \{(\sigma, f) \in S \mid (\sigma, f) \text{ is not complete and not subcomplete }\},\\
T_1 &= \{(\tau, g) \in T \mid (\tau, g) \text{ is subcomplete and } \tau \in \partial\mathcal{K}\},\\
T_2 &= \{(\tau, g) \in T \mid (\tau, g) \text{ is subcomplete and } \tau \notin \partial\mathcal{K}\},\\
T_3 &= \{(\tau, g) \in T \mid (\tau, g) \text{ is not subcomplete }\}.
\end{aligned}$$

Then

$$\{S_1, S_2, S_3\} \text{ is a partition of } S \tag{3.5}$$

and

$$\{T_1, T_2, T_3\} \text{ is a partition of } T. \tag{3.6}$$

Let

$$S_t = \{s \in S \mid t \prec s\} \quad (t \in T)$$

and

$$T_s = \{t \in T \mid t \prec s\} \quad (s \in S).$$

We claim that

$$|S_t| = m \ (t \in T_1), \tag{3.7}$$

$$|S_t| = 2m \ (t \in T_2), \tag{3.8}$$

$$|S_t| = 0 \ (t \in T_3). \tag{3.9}$$

To see (3.7), (3.8) and (3.9), let us fix $t = (\tau, g) \in T$, where

$$\tau = \{v_0, v_1, \ldots, v_{d-1}\}. \tag{3.10}$$

Case 1. $t = (\tau, g) \in T_1$. Then $\tau \in \partial\mathcal{K}$, so that τ is a face of exactly one d-simplex σ of $\mathcal{K}$, say

$$\sigma = \{v_0, v_1, \ldots, v_d\}, \tag{3.11}$$

and there are exactly m extensions $f_1, \ldots, f_m$ of g to the set σ into $\{1, \ldots, m\}$, namely,

$$f_j(v_k) = \begin{cases} g(v_k) & \text{if } k = 0, 1, \ldots, d-1 \\ j & \text{if } k = d \end{cases} \tag{3.12}$$

for $j = 1, \ldots, m$, thus

$$S_t = \{(\sigma, f_1), \ldots, (\sigma, f_m)\}$$

and (3.7) follows.

Case 2. $t = (\tau, g) \in T_2$. Then $\tau \notin \partial\mathcal{K}$, so that τ is a face of exactly two distinct d-simplexes σ and σ' of $\mathcal{K}$, say

$$\sigma = \tau \cup \{v_d\} \text{ and } \sigma' = \tau \cup \{v'_d\}. \tag{3.13}$$

For each $j = 1, \ldots, m$, let f_j and f'_j be the functions on σ and σ' into $\{1, \ldots, m\}$ defined by

(3.14) $$f_j|_\tau = f'_j|_\tau = g \text{ and } f_j(v_d) = f'_j(v'_d) = j,$$

we have

$$S_t = \{(\sigma, f_1), \ldots, (\sigma, f_m)\} \cup \{(\sigma', f'_1), \ldots, (\sigma', f'_m)\}$$

and (3.8) follows.

Case 3. $t = (\tau, g) \in T_3$. Then (τ, g) is not subcomplete, so that, by **(R2)**, $S_t = \emptyset$ and (3.9) follows.

We next claim that

(3.15) $$|T_s| = 1 \ \ (s \in S_1)$$

(3.16) $$|T_s| = 2 \ \ (s \in S_2)$$

(3.17) $$|T_s| = 0 \ \ (s \in S_3)$$

To see (3.15), (3.16) and (3.17), let us fix $s = (\sigma, f) \in S$, where σ is given by (3.11). Let

(3.18) $$\tau = \sigma \setminus \{v_d\}$$

and

(3.19) $$g = f\ |_\tau\ .$$

Then, by **(K1)**, (3.10), (3.18) and (3.19),

(3.20) $$(\tau, g) \in T.$$

Case 1'. $s = (\sigma, f) \in S_1$. Then (σ, f) is complete, we may assume that

(3.21) $$\varphi_{f(v_k)}(v_k) = k \text{ for } k = 0, 1, \ldots, d.$$

It follows from (3.18), (3.19) and (3.21) that

(3.22) $$(\tau, g) \text{ is subcomplete,}$$

so that, by comparing (3.18), (3.19), (3.20) and (3.22) with **(R1)**, **(R2)** and **(R3)**, we have

(3.23) $$T_s = \{(\tau, g)\}$$

and (3.15) follows.

Case 2'. $s = (\sigma, f) \in S_2$. Then (σ, f) is subcomplete, so that

$$\{\varphi_{f(v)}(v) \mid v \in \sigma\} = \{0, 1, \ldots, d-1\}. \tag{3.24}$$

By (3.11) and (3.24), we may assume that

$$\varphi_{f(v_k)}(v_k) = k \text{ for } k = 0, 1, \ldots, d-1 \tag{3.25}$$

and

$$\varphi_{f(v_d)}(v_d) = i \text{ for some } i \in \{0, 1, \ldots, d-1\}. \tag{3.26}$$

Put

$$\tau' = \sigma \setminus \{v_i\} \text{ and } g' = f\mid_{\tau'} . \tag{3.27}$$

We have

$$(\tau', g') \text{ is subcomplete,} \tag{3.28}$$

it follows that

$$T_s = \{(\tau, g), (\tau', g')\} \tag{3.29}$$

and (3.16) follows.

Case 3'. $s = (\sigma, f) \in S_3$. Then (σ, f) is not complete and not subcomplete, so that

$$\{0, 1, \ldots, d-1\} \not\subset \{\varphi_{f(v)}(v) \mid v \in \sigma\} \tag{3.30}$$

thus

$$T_s = \emptyset \tag{3.31}$$

and (3.17) follows.

Define $\lambda : T \times S \to \{0, 1\}$ by

$$\lambda(t, s) = \begin{cases} 1, & \text{if } t \prec s \\ 0, & \text{otherwise.} \end{cases}$$

Then, by (3.6), (3.7), (3.8) and (3.9),

$$\begin{aligned}\sum_{t\in T}\sum_{s\in S}\lambda(t,s) &= \sum_{t\in T_1}\sum_{s\in S}\lambda(t,s)+\sum_{t\in T_2}\sum_{s\in S}\lambda(t,s)+\sum_{t\in T_3}\sum_{s\in S}\lambda(t,s)\\ &= \sum_{t\in T_1}|S_t|+\sum_{t\in T_2}|S_t|+\sum_{t\in T_3}|S_t|\\ &= m|T_1|+2m|T_2|+0|T_3|\end{aligned}$$

and, by (3.5), (3.15), (3.16) and (3.17),

$$\begin{aligned}\sum_{s\in S}\sum_{t\in T}\lambda(t,s) &= \sum_{s\in S_1}\sum_{t\in T}\lambda(t,s)+\sum_{s\in S_2}\sum_{t\in T}\lambda(t,s)+\sum_{s\in S_3}\sum_{t\in T}\lambda(t,s)\\ &= \sum_{s\in S_1}|T_s|+\sum_{s\in S_2}|T_s|+\sum_{s\in S_3}|T_s|\\ &= |S_1|+2|S_2|+0|S_3|.\end{aligned}$$

It follows that

(3.32) $$m|T_1|+2m|T_2|=|S_1|+2|S_2|.$$

As

(3.33) $$S_1=\mathcal{K}(\varphi) \text{ and } T_1=\partial\mathcal{K}(\varphi),$$

from (3.32) and (3.33), (3.1) is proved.

To see (3.2), let the sets, respectively, S_*, S_{1*}, S_{2*}, S_{3*}, T_*, T_{1*}, T_{2*} and T_{3*} be the sets of the pairs in S, S_1, S_2, S_3, T, T_1, T_2 and T_3 such that all the functions f or g in the pairs (σ, f) or (τ, g) are one-to-one. As before, we have

(3.34) $$\{S_{1*}, S_{2*}, S_{3*}\} \text{ is a partition of } S_*$$

and

(3.35) $$\{T_{1*}, T_{2*}, T_{3*}\} \text{ is a partition of } T_*.$$

Let

$$S_{t*} = \{s\in S_* \mid t\prec s\} \quad (t\in T_*)$$

and

$$T_{s*} = \{t\in T_* \mid t\prec s\} \quad (s\in S_*).$$

We claim that, in case $m > d$,

$$|S_{t*}| = m - d \quad (t \in T_{1*}) \tag{3.36}$$

$$|S_{t*}| = 2(m - d) \quad (t \in T_{2*}) \tag{3.37}$$

$$|S_{t*}| = 0 \quad (t \in T_{3*}) \tag{3.38}$$

$$|T_{s*}| = 1 \quad (s \in S_{1*}) \tag{3.39}$$

$$|T_{s*}| = 2 \quad (s \in S_{2*}) \tag{3.40}$$

$$|T_{s*}| = 0 \quad (s \in S_{3*}). \tag{3.41}$$

To see (3.36), (3.37) and (3.38), let us fix $t = (\tau, g) \in T_*$. Since g is one-to-one, the cardinality of the image of τ under g is $|g(\tau)| = d$. If $t = (\tau, g) \in T_{1*}$, then there are exactly $m - d$ injective extentions of g to σ into $\{1, \ldots, m\}$, namely,

$$S_{t*} = \{(\sigma, f_j) \mid j \in \{1, \ldots, m\} \setminus g(\tau)\}(t \in T_{1*}) \tag{3.42}$$

where τ, σ and f_j are the same as in (3.10), (3.11) and (3.12) respectively. Similarly, if we define f_j and f'_j as in (3.14), then we have

$$\begin{aligned} S_{t*} = \ & \{(\sigma, f_j) \mid j \in \{1, \ldots, m\} \setminus g(\tau)\} \\ & \cup\{(\sigma', f'_j) \mid j \in \{1, \ldots, m\} \setminus g(\tau)\} \ \ (t \in T_{2*}). \end{aligned} \tag{3.43}$$

It is clear that

$$S_{t*} = \emptyset(t \in T_{3*}). \tag{3.44}$$

This proves (3.36), (3.37) and (3.38). The same argument in the proof of (3.15), (3.16) and (3.17) shows that (3.39), (3.40) and (3.41) are true. It follows from (3.34) $\sim$ (3.41) that

$$\sum_{t \in T_*} \sum_{s \in S_*} \lambda(t, s) = (m - d)|T_{1*}| + 2(m - d)|T_{2*}|$$

and

$$\sum_{s \in S_*} \sum_{t \in T_*} \lambda(t, s) = |S_{1*}| + 2|S_{2*}|,$$

so that

$$(m-d)|T_{1*}| + 2(m-d)|T_{2*}| = |S_{1*}| + 2|S_{2*}|. \tag{3.45}$$

As

$$S_{1*} = \mathcal{K}(\varphi)_* \text{ and } T_{1*} = \partial\mathcal{K}(\varphi)_*, \tag{3.46}$$

from (3.45) and (3.46), (3.2) is proved.

Suppose further, $(\mathcal{K}, \omega)$ is a coherently oriented d-pseudomanifold. Put

$$\begin{aligned}
S_t^+ &= \{(\sigma, f) \in S_t \mid (\tau, g) \text{ is positively subcomplete in } \sigma\} \quad (t = (\tau, g) \in T),\\
S_t^- &= \{(\sigma, f) \in S_t \mid (\tau, g) \text{ is negatively subcomplete in } \sigma\} \quad (t = (\tau, g) \in T),\\
T_s^+ &= \{(\tau, g) \in T_s \mid (\tau, g) \text{ is positively subcomplete in } \sigma\} \quad (s = (\sigma, f) \in S),\\
T_s^- &= \{(\tau, g) \in T_s \mid (\tau, g) \text{ is negatively subcomplete in } \sigma\} \quad (s = (\sigma, f) \in S).
\end{aligned}$$

Note that

$$\{\mathcal{K}^+(\varphi), \mathcal{K}^-(\varphi)\} \text{ partitions } S_1 \tag{3.47}$$

and

$$\{\partial\mathcal{K}^+(\varphi), \partial\mathcal{K}^-(\varphi)\} \text{ partitions } T_1. \tag{3.48}$$

We claim that

$$|S_t^+| = m \text{ and } |S_t^-| = 0 \quad (t \in \partial\mathcal{K}^+(\varphi)), \tag{3.49}$$

$$|S_t^+| = 0 \text{ and } |S_t^-| = m \quad (t \in \partial\mathcal{K}^-(\varphi)), \tag{3.50}$$

$$|S_t^+| = |S_t^-| = m \quad (t \in T_2), \tag{3.51}$$

$$|S_t^+| = |S_t^-| = 0 \quad (t \in T_3). \tag{3.52}$$

If $t = (\tau, g) \in \partial\mathcal{K}^+(\varphi)$, then (τ, g) is positively complete, so that

$$S_t^+ = \{(\sigma, f_1), \ldots, (\sigma, f_m)\} \text{ and } S_t^- = \emptyset,$$

where τ, σ and f_j are given by (3.10), (3.11) and (3.12), this proves (3.49). Similarly, if $t = (\tau, g) \in \partial\mathcal{K}^-(\varphi)$ then $S_t^+ = \emptyset$ and $S_t^- = \{(\sigma, f_1), \ldots, (\sigma, f_m)\}$ and (3.50) follows. It follows from **(M2)**, **(C2)** that if $t \in T_2$ then S_t^+ is one of the two sets $\{(\sigma, f_1), \ldots, (\sigma, f_m)\}$ and $\{(\sigma', f_1'), \ldots, (\sigma', f_m')\}$ and S_t^- is the other, where

σ, σ', f_j and f_j' are given by (3.13) and (3.14), this proves (3.51). If $t = (\tau, g) \in T_3$ then (τ, g) is not subcomplete, so that $S_t^+ = S_t^- = \emptyset$ and (3.52) follows.

We next claim that

$$|T_s^+| = \frac{1 + (-1)^d}{2} \text{ and } |T_s^-| = \frac{1 - (-1)^d}{2} \quad (s \in \mathcal{K}^+(\varphi)), \tag{3.53}$$

$$|T_s^+| = \frac{1 - (-1)^d}{2} \text{ and } |T_s^-| = \frac{1 + (-1)^d}{2} \quad (s \in \mathcal{K}^-(\varphi)), \tag{3.54}$$

$$|T_s^+| = |T_s^-| = 1 \quad (s \in S_2), \tag{3.55}$$

$$|T_s^+| = |T_s^-| = 0 \quad (s \in S_3). \tag{3.56}$$

Let $s = (\sigma, f)$ and $\omega(\sigma) = \varepsilon[v_0, v_1, \ldots, v_d]$ ($\varepsilon = \pm 1$). If $s = (\sigma, f) \in \mathcal{K}^+(\varphi)$, then $\omega(\sigma) = (+1)[v_0, v_1, \ldots, v_d]$ with the assumption (3.21), so that (3.22) holds and $\omega(\sigma)$ induces $(-1)^d[v_0, v_1, \ldots, v_{d-1}]$ on τ, thus

$$\begin{aligned} &T_s^+ = \{(\tau, g)\} \text{ and } T_s^- = \emptyset && \text{if } d \text{ is even,} \\ &T_s^+ = \emptyset \text{ and } T_s^- = \{(\tau, g)\} && \text{if } d \text{ is odd,} \end{aligned}$$

where τ and g are given in (3.18) and (3.19) respectively. This proves (3.53). Similarly, if $s = (\sigma, f) \in \mathcal{K}^-(\varphi)$, then

$$\begin{aligned} &T_s^+ = \emptyset \text{ and } T_s^- = \{(\tau, g)\} && \text{if } d \text{ is even,} \\ &T_s^+ = \{(\tau, g)\} \text{ and } T_s^- = \emptyset && \text{if } d \text{ is odd,} \end{aligned}$$

and (3.54) follows. Next, if $s = (\sigma, f) \in S_2$, then by (3.18) and (3.27) we have

$$\begin{aligned} &\omega(\sigma) \text{ induces } (-1)^d \varepsilon[v_0, v_1, \ldots, v_{d-1}] \text{ on } \tau, \\ &\omega(\sigma) \text{ induces } (-1)^{d+1} \varepsilon[v_0, v_1, \ldots, v_{i-1}, v_d, v_{i+1}, \ldots, v_{d-1}] \text{ on } \tau', \end{aligned}$$

so that by (3.25), (3.26) and (3.27), one of the two pairs (τ, g) and (τ', g') is positively subcomplete in σ and the other one is negatively subcomplete in σ, thus (3.55) is true. Finally, if $s = (\sigma, f) \in S_3$, then (3.17) implies that $|T_s^+| = |T_s^-| = 0$, so that (3.56) is true.

Define $\Lambda : T \times S \to \{-1, 0, 1\}$ by

$$\Lambda(t, s) = \begin{cases} 1, & \text{if } t \prec s \text{ and } (\tau, g) \text{ is positively subcomplete in } \sigma \\ -1, & \text{if } t \prec s \text{ and } (\tau, g) \text{ is negatively subcomplete in } \sigma \\ 0, & \text{otherwise.} \end{cases}$$

where $t = (\tau, g) \in T$ and $s = (\sigma, f) \in S$. Then by (3.48) $\sim$ (3.52),

$$\begin{aligned}\sum_{t\in T}\sum_{s\in S}\Lambda(t,s) &= \sum_{t\in T}(|S_t^+| - |S_t^-|)\\ &= (\sum_{t\in\partial\mathcal{K}^+(\varphi)} + \sum_{t\in\partial\mathcal{K}^-(\varphi)} + \sum_{t\in T_2} + \sum_{t\in T_3})(|S_t^+| - |S_t^-|)\\ &= |\partial\mathcal{K}^+(\varphi)|(m-0) + |\partial\mathcal{K}^-(\varphi)|(0-m) + |T_2|(m-m) + |T_3|(0-0)\\ &= m\{|\partial\mathcal{K}^+(\varphi)| - |\partial\mathcal{K}^-(\varphi)|\}\end{aligned}$$

and, by (3.47) and (3.53) $\sim$ (3.56),

$$\begin{aligned}\sum_{s\in S}\sum_{t\in T}\Lambda(t,s) &= \sum_{s\in S}(|T_s^+| - |T_s^-|)\\ &= (\sum_{s\in\mathcal{K}^+(\varphi)} + \sum_{s\in\mathcal{K}^-(\varphi)} + \sum_{s\in S_2} + \sum_{s\in S_3})(|T_s^+| - |T_s^-|)\\ &= |\mathcal{K}^+(\varphi)|\{\frac{1+(-1)^d}{2} - \frac{1-(-1)^d}{2}\} +\\ &\quad |\mathcal{K}^-(\varphi)|\{\frac{1-(-1)^d}{2} - \frac{1+(-1)^d}{2}\} + |S_2|(1-1) + |S_3|(0-0)\\ &= (-1)^d\{|\mathcal{K}^+(\varphi)| - |\mathcal{K}^-(\varphi)|\},\end{aligned}$$

thus (3.3) holds. If $m > d$, by a similar argument as in the proof of (3.2) and (3.3), the equality (3.4) holds. We mention that if $m \le d$, then both sides of (3.2) and (3.4) are zeros, thus (3.1) $\sim$ (3.4) hold for any positive integers m and d. This completes the proof of Theorem 1.

4. Multiple Sperner's Lemma

A subset $\sigma = \{v_0, v_1, \ldots, v_d\}$ of a Euclidean space is *affinely independent* if

(A1) $\sum\limits_{k=0}^{d}\lambda_k v_k = 0$ and $\sum\limits_{k=0}^{d}\lambda_k = 0$ imply each $\lambda_k = 0$,

the convex hull *conv* σ of the affinely independent set σ is called a (geometric) d-simplex with the vertices $v_0, v_1, \ldots, v_d$, sometimes we denote this simplex by $\overline{v_0 v_1 \ldots v_d}$, thus

$$\overline{v_0 v_1 \ldots v_d} = \left\{\sum_{k=0}^{d}\lambda_k v_k \mid \sum_{k=0}^{d}\lambda_k = 1,\ \text{each } \lambda_k \ge 0\right\},$$

for $0 \le r \le d$ and $0 \le k_0 < k_1 < \ldots < k_r \le d$, the simplex $\overline{v_{k_0}v_{k_1}\ldots v_{k_r}}$ is called an r-face of $\overline{v_0v_1\ldots v_d}$.

A finite collection T of (geometric) simplexes is called a *triangulation* of a d-simplex $\overline{a_0a_1\ldots a_d}$ if it satisfies the following three conditions:

(T1) $\overline{a_0a_1\ldots a_d} = \bigcup_{s\in T} s.$

(T2) If $s \in T$ and t is a face of s then $t \in T$.

(T3) If $s, t \in T$ and $s \cap t \neq \emptyset$, then $s \cap t$ is a common face of s and t.

A point $v \in \overline{a_0a_1\ldots a_d}$ is a *vertex* of T if v is a vertex of some simplex of T. The set of all vertices of T is denoted by $V(T)$. The collection $\widetilde{T}$ of all subsets $\{v_0, v_1, \ldots, v_r\}$ of $V(T)$ such that $\overline{v_0v_1\ldots v_r} \in T$ is the *vertex scheme* of T and which is a d-pseudomanifold. For each d-simplex $\sigma = \{v_0, v_1, \ldots, v_d\}$ of $\widetilde{T}$, the *canonical orientation* $\omega(\sigma)$ on σ is $(+1)[v_0, v_1, \ldots, v_d]$ or $(-1)[v_0, v_1, \ldots, v_d]$ according as $det(\lambda_{ij}) > 0$ or $det(\lambda_{ij}) < 0$ respectively, where (λ_{ij}) is the $d+1$ square matrix satisfying

$$v_i = \sum_{j=0}^{d} \lambda_{ij} a_j \quad (\sum_{j=0}^{d} \lambda_{ij} = 1) \quad \text{for } i = 0, 1, \ldots, d.$$

Then $(\widetilde{T}, \omega)$ becomes a coherently oriented d-pseudomanifold.
An m-labelling φ in $\widetilde{T}$ is *Sperner* if it satisfies the following facial condition:

(F1) For each $v \in V(T)$ and each $j \in \{1, \ldots, m\}$,

$$v \in \overline{a_{k_0}a_{k_1}\ldots a_{k_r}} \quad \text{implies} \quad \varphi_j(v) \in \{k_0, k_1, \ldots, k_r\}$$

whenever $0 \le r \le d$ and $0 \le k_0 < k_1 < \ldots < k_r \le d$.

Theorem 2. *Let φ be an m-labelling in the vertex scheme $\widetilde{T}$ of a triangulation T of a given d-simplex $\overline{a_0a_1\ldots a_d}$. If φ is Sperner, then, with the canonical orientation ω, we have*

$$|\widetilde{T}^{+}(\varphi)| - |\widetilde{T}^{-}(\varphi)| = m^{d+1}, \tag{4.1}$$

and if $m > d$, we have

$$|\widetilde{T}^{+}(\varphi)_*| - |\widetilde{T}^{-}(\varphi)_*| = m(m-1)\ldots(m-d). \tag{4.2}$$

Proof. For each $k = 0, 1, \ldots, d$, let T_k be the restricted triangulation of T to the k-simplex $\overline{a_0a_1\ldots a_k}$, that is,

$$T_k = \{s \in T \mid s \subset \overline{a_0a_1\ldots a_k}\}. \tag{4.3}$$

Then $(\widetilde{T}_k, \omega_k)$ is a coherently oriented k-pseudomanifold where ω_k is the canonical orientation on the set of all k-simplexes of $\widetilde{T}_k$. Precisely,

$$\omega_k(\{v_0, v_1, \ldots, v_k\}) = \varepsilon[v_0, v_1, \ldots, v_k] \quad (\varepsilon = \pm 1) \tag{4.4}$$

if and only if

$$det(\lambda_{ij})_{(k+1)\times(k+1)} = \varepsilon|det(\lambda_{ij})_{(k+1)\times(k+1)}| \tag{4.5}$$

where

$$v_i = \sum_{j=0}^{k} \lambda_{ij} a_j \quad (\sum_{j=0}^{k} \lambda_{ij} = 1) \quad \text{for } i = 0, 1, \ldots, k. \tag{4.6}$$

It follows from **(F1)** that the restriction of φ to $V(T_k)$ is a Sperner m-labelling in $\widetilde{T}_k$. We shall show that

$$\widetilde{T}_0^+(\varphi)| = m, \quad |\widetilde{T}_0^-(\varphi)| = 0, \tag{4.7}$$

$$|\partial\widetilde{T}_k^+(\varphi)| - |\partial\widetilde{T}_k^-(\varphi)| = (-1)^k\{|\widetilde{T}_{k-1}^+(\varphi)| - |\widetilde{T}_{k-1}^-(\varphi)|\}, \tag{4.8}$$

$$|\widetilde{T}_0^+(\varphi)_*| = m, \quad |\widetilde{T}_0^-(\varphi)_*| = 0, \tag{4.9}$$

$$|\partial\widetilde{T}_k^+(\varphi)_*| - |\partial\widetilde{T}_k^-(\varphi)_*| = (-1)^k\{|\widetilde{T}_{k-1}^+(\varphi)_*| - |\widetilde{T}_{k-1}^-(\varphi)_*|\}, \tag{4.10}$$

where $0 < k \leq d$.
Observe that Theorem 1, (4.8) and (4.10) will imply

$$|\widetilde{T}_k^+(\varphi)| - |\widetilde{T}_k^-(\varphi)| = m\{|\widetilde{T}_{k-1}^+(\varphi)| - |\widetilde{T}_{k-1}^-(\varphi)|\} \tag{4.11}$$

and

$$|\widetilde{T}_k^+(\varphi)_*| - |\widetilde{T}_k^-(\varphi)_*| = (m-k)\{|\widetilde{T}_{k-1}^+(\varphi)_*| - |\widetilde{T}_{k-1}^-(\varphi)_*|\}, \tag{4.12}$$

so that (4.1) will follow from (4.7) and (4.11) and (4.2) will follow from (4.9) and (4.12).

By (4.3), $T_0 = \{\overline{a_0}\}$, so that, by **(F1),**

$$\varphi(a_0) = (\varphi_1(a_0), \ldots, \varphi_m(a_0)) = (0, \ldots, 0) \tag{4.13}$$

it follows from (4.4), (4.5), (4.6) and (4.13) that

$$\widetilde{T}_0^+(\varphi) = \{(\{a_0\}, f_j) \mid j = 1, \ldots, m\} \text{ and } \widetilde{T}_0^-(\varphi) = \emptyset \tag{4.14}$$

where $f_j(a_0) = j$ for $j = 1, \ldots, m$, thus (4.7) is true.

As $\{a_0\}$ is a singleton, each f_j in (4.14) is one-to-one, so that

$$\widetilde{T}_0^+(\varphi)_* = \widetilde{T}_0^+(\varphi) \text{ and } \widetilde{T}_0^-(\varphi)_* = \widetilde{T}_0^-(\varphi), \tag{4.15}$$

thus (4.9) is also true.

To see (4.8) and (4.10), let $g : \tau \to \{1, \ldots, m\}$ where

$$\tau = \{v_0, v_1, \ldots, v_{k-1}\} \tag{4.16}$$

and

$$\varphi_{g(v_j)}(v_j) = j \text{ for } j = 0, 1, \ldots, k-1. \tag{4.17}$$

Then (4.3) and **(F1)** imply the following (4.18) and (4.19) are equivalent:

$$(\tau, g) \in \partial \widetilde{T}_k(\varphi), \tag{4.18}$$

$$(\tau, g) \in \widetilde{T}_{k-1}(\varphi). \tag{4.19}$$

We claim that

$$\partial \widetilde{T}_k^\pm(\varphi) = \widetilde{T}_{k-1}^\pm(\varphi) \text{ if } k \text{ is even} \tag{4.20}$$

$$\partial \widetilde{T}_k^\pm(\varphi) = \widetilde{T}_{k-1}^\mp(\varphi) \text{ if } k \text{ is odd} \tag{4.21}$$

and

$$\partial \widetilde{T}_k^\pm(\varphi)_* = \widetilde{T}_{k-1}^\pm(\varphi)_* \text{ if } k \text{ is even} \tag{4.22}$$

$$\partial \widetilde{T}_k^\pm(\varphi)_* = \widetilde{T}_{k-1}^\mp(\varphi)_* \text{ if } k \text{ is odd.} \tag{4.23}$$

It is clear that (4.8) will follow from (4.20) and (4.21) and (4.10) will follow from (4.22) and (4.23).

Now suppose (4.18) and (4.19) hold, let

$$\sigma = \{v_0, v_1, \ldots, v_k\} \in \widetilde{T}_k. \tag{4.24}$$

Then τ is a face of σ, by (4.18), τ is a boundary $(k-1)$-simplex of $\widetilde{T}_k$, so that such a k-simplex σ is unique. By (4.3) and (4.24), we may assume that (4.6) holds, and by (4.16), (4.19) and the affine independence of $\{a_0, a_1, \ldots, a_k\}$, we have

$$\lambda_{ik} = 0 \text{ for } i = 0, 1, \ldots, k-1 \tag{4.25}$$

and

$$\lambda_{kk} > 0, \tag{4.26}$$

so that, by (4.25),

$$det(\lambda_{ij})_{(k+1)\times(k+1)} = \lambda_{kk} det(\lambda_{ij})_{k\times k}. \tag{4.27}$$

Let

$$\omega_k(\sigma) = \varepsilon[v_0, v_1, \ldots, v_k]. \tag{4.28}$$

Then $\omega_k(\sigma)$ induces

$$(-1)^k \varepsilon[v_0, v_1, \ldots, v_{k-1}] \tag{4.29}$$

on τ. By (4.4), (4.5), (4.6), (4.26), (4.27) and (4.28), we have

$$det(\lambda_{ij})_{k\times k} = \varepsilon|det(\lambda_{ij})_{k\times k}|, \tag{4.30}$$

so that, by replacing k by $k-1$ in (4.4), (4.5) and (4.6), we have

$$\omega_{k-1}(\tau) = \varepsilon[v_0, v_1, \ldots, v_{k-1}]. \tag{4.31}$$

It follows from (4.17), (4.29) and (4.31) that (4.20) $\sim$ (4.23) hold. This completes the proof.

5. Combinatorial Formulae and Matroids

An ordered pair $(E, \mathcal{I})$ is a *matroid* if E is a finite set and $\mathcal{I}$ is a collection of subsets of E such that the following three conditions are satisfied:

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I \in \mathcal{I}$ and $I' \subset I$, then $I' \in \mathcal{I}$.

(I3) If $I_1 \in \mathcal{I}$ and $I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$, then $I_1 \cup \{e\} \in \mathcal{I}$ for some $e \in I_2 \setminus I_1$.

In the matroid $(E, \mathcal{I})$, a subset of E is *independent* if it is a member of $\mathcal{I}$ and *dependent* if it is not independent, a maximal independent subset of E is a *basis*, a minimal dependent subset of E is a *circuit*, $e \in E$ is a *loop* if the singleton $\{e\}$ is a circuit, the number

$$r(X) = max\{|I| \mid I \subset X, I \in \mathcal{I}\}$$

is the *rank* of X, where $X \subset E$, and the *closure* (or *span*) of X is the set

$$cl(X) = \{e \in E \mid r(X \cup \{e\}) = r(X)\}.$$

The number $r(E)$ is called the *rank* of the matroid $(E, \mathcal{I})$. It is well known that all bases are equicardinal. For all necessary background materials we refer to Oxley [11].

Let $B = (b_0, b_1, \ldots, b_d)$ be an ordered basis of a matroid $(E, \mathcal{I})$ of rank $d + 1$ and

(5.1) $$F_j = cl(\{b_0, b_1, \ldots, b_j\}) \text{ for } j = 0, 1, \ldots, d.$$

An ordering $(e_0, e_1, \ldots, e_k)$ of $k + 1$ $(0 \leq k \leq d)$ elements of E is a *B-sequence* if

(5.2) $$e_0 \in F_0 \text{ and } e_j \in F_j \setminus F_{j-1} \text{ for } j = 1, \ldots, k.$$

Let $\psi_B : E \to \{0, 1, \ldots, d\}$ be the function defined by

(5.3) $$\psi_B(e) = 0 \text{ if } e \in F_0$$

(5.4) $$\psi_B(e) = j \text{ if } e \in F_j \setminus F_{j-1} \text{ for } j = 1, \ldots, d.$$

Then we have the following properties:

(B1) $cl(\emptyset) \subset F_0 \subset F_1 \subset \ldots \subset F_d = E$ ($cl(\emptyset)$ is the set of all loops in E),

(B2) if $(e_0, e_1, \ldots, e_k)$ is a B-sequence, then the rank $r(\{e_0, e_1, \ldots, e_k\})$ is k or $k + 1$ provided e_0 is a loop or not,

(B3) $(e_0, e_1, \ldots, e_k)$ is a B-sequence if and only if

$$\psi_B(e_j) = j \quad \text{for} \quad j = 0, 1, \ldots, k.$$

Let $\mathcal{K}$ be a d-pseudomanifold, $B = (b_0, b_1, \ldots, b_d)$ an ordered basis of a matroid $(E, \mathcal{I})$ of rank $d + 1$. If ϕ is a map from $V(\mathcal{K})$ into E^m, the Cartesian product $E \times \cdots \times E$ of m factors, we shall write

(5.5) $$\phi(v) = (\phi_1(v), \ldots, \phi_m(v))$$

and

(5.6) $$(\psi_B \circ \phi)(v) = ((\psi_B \circ \phi_1)(v), \ldots, (\psi_B \circ \phi_m)(v))$$

for $v \in V(\mathcal{K})$, where ψ_B is given by (5.3) and (5.4), so that $\varphi = \psi_B \circ \phi$ is an m-labelling in $\mathcal{K}$ and we may consider a given pair (σ, f) which is complete, subcomplete or not by using the map ϕ and the ordered basis B instead of the m-labelling φ. We may consider the orientations or the induced orientations when $(\mathcal{K}, \omega)$ is coherently oriented. We call a pair (σ, f) *B-complete*, *positively B-complete* or *negatively B-complete*, (relative to ϕ and B) if it is complete, positively complete or negatively complete (relative to φ) and a pair (τ, g) *B-subcomplete*, *positively B-subcomplete* or *negatively B-subcomplete* if it is subcomplete, positively subcomplete or negatively subcomplete, and we write $\mathcal{K}_B(\phi)$, $\mathcal{K}_B^+(\phi)$, $\mathcal{K}_B^-(\phi)$, $\partial\mathcal{K}_B(\phi)$, $\partial\mathcal{K}_B^+(\phi)$, and $\partial\mathcal{K}_B^-(\phi)$, in place of $\mathcal{K}(\varphi)$, $\mathcal{K}^+(\varphi)$, $\mathcal{K}^-(\varphi)$, $\partial\mathcal{K}(\varphi)$, $\partial\mathcal{K}^+(\varphi)$, and $\partial\mathcal{K}^-(\varphi)$ respectively. The notations such as $\mathcal{K}_B(\phi)_*, \ldots, \partial\mathcal{K}_B^-(\phi)_*$ are defined by a similar way.

The following Theorem 3 is a direct consequence of Theorem 1.

Theorem 3. *Let $\phi : V(\mathcal{K}) \to E^m$ where $\mathcal{K}$ is a d-pseudomanifold $(d > 0)$ and $(E, \mathcal{I})$ is a matroid of rank $d+1$. Then for each ordered basis B of $(E, \mathcal{I})$,*

$$|\mathcal{K}_B(\phi)| \equiv m|\partial\mathcal{K}_B(\phi)| \quad \textit{(mod 2)}$$

and

$$|\mathcal{K}_B(\phi)_*| \equiv (m-d)|\partial\mathcal{K}_B(\phi)_*| \quad \textit{(mod 2).}$$

Suppose further, $(\mathcal{K}, \omega)$ is a coherently oriented d-pseudomanifold, then

$$(-1)^d\{|\mathcal{K}_B^+(\phi)| - |\mathcal{K}_B^-(\phi)|\} = m\{|\partial\mathcal{K}_B^+(\phi)| - |\partial\mathcal{K}_B^-(\phi)|\}$$

and

$$(-1)^d\{|\mathcal{K}_B^+(\phi)_*| - |\mathcal{K}_B^-(\phi)_*|\} = (m-d)\{|\partial\mathcal{K}_B^+(\phi)_*| - |\partial\mathcal{K}_B^-(\phi)_*|\}.$$

The following Theorem 4 is corresponding to Theorem 2.

Theorem 4. *Let $\phi : V(\widetilde{T}) \to E^m$ where $\widetilde{T}$ is the vertex scheme of a triangulation T of a d-simplex $\overline{a_0 a_1 \ldots a_d}$ and $(E, \mathcal{I})$ is a matroid of rank $d+1$ with an ordered basis B. If ϕ satisfies the facial condition (relative to the ordered basis B):*
for each $v \in V(\widetilde{T})$ and each $j \in \{1, \ldots, m\}$,

$$v \in \overline{a_{k_0} a_{k_1} \ldots a_{k_r}} \quad \textit{implies} \quad (\psi_B \circ \phi_j)(v) \in \{k_0, k_1, \ldots, k_r\}$$

whenever $0 \le r \le d$ and $0 \le k_0 < k_1 \ldots < k_r \le d$, then, with the canonical orientation ω, we have

$$|\widetilde{T}_B^+(\phi)| - |\widetilde{T}_B^-(\phi)| = m^{d+1}$$

and

$$|\widetilde{T}_B^+(\phi)_*| - |\widetilde{T}_B^-(\phi)_*| = m(m-1)\ldots(m-d).$$

Proof. By the facial condition (relative to B), the m-labelling $\varphi = \psi_B \circ \phi$ is Sperner. Thus the theorem follows from Theorem 2.

Sometimes we are interested in the combinatorics of a pseudomanifold with an independence structure. This is indeed a special case of Theorem 3 with $m = 1$, $E = V(\mathcal{K})$ and $\phi = id_{V(\mathcal{K})}$ the identity map on $V(\mathcal{K})$. Precisely, let $\mathcal{K}$ be a d-pseudomanifold and $(V(\mathcal{K}), \mathcal{I})$ be a matroid of rank $d+1$ with an ordered basis B. Let $\mathcal{K}_B$ (resp. $\partial\mathcal{K}_B$) be the collection of all those d-simplexes $\sigma = \{v_0, v_1, \ldots, v_d\}$ (resp. boundary $(d-1)$-simplexes) of $\mathcal{K}$ such that $(v_0, v_1, \ldots, v_d)$ (resp. $(v_0, v_1, \ldots, v_{d-1})$) is a B-sequence. When $\mathcal{K}$ is orientable with an coherent orientation-valued function ω, let $\mathcal{K}_B^+$ (resp. $\mathcal{K}_B^-$) be the collection of all those d-simplexes $\sigma = \{v_0, v_1, \ldots, v_d\}$ of $\mathcal{K}$ such that $(v_0, v_1, \ldots, v_d)$ is a B-sequence and $\omega(\sigma) = (+1)[v_0, v_1, \ldots, v_d]$ (resp. $(-1)[v_0, v_1, \ldots, v_d]$) and let $\partial\mathcal{K}_B^+$ (resp. $\partial\mathcal{K}_B^-$) be the collection of all those boundary $(d-1)$-simplexes $\tau = \{v_0, v_1, \ldots, v_{d-1}\}$ of $\mathcal{K}$ such that $(v_0, v_1, \ldots, v_{d-1})$ is a B-sequence and the induced orientation on τ from the orientation $\omega(\sigma)$ on the unique d-simplex σ of $\mathcal{K}$ having τ as a $(d-1)$-face is $(+1)[v_0, v_1, \ldots, v_{d-1}]$ (resp. $(-1)[v_0, v_1, \ldots, v_{d-1}]$). Then, by Theorem 3 with the explanation above, the following Theorem 5 is true.

Theorem 5. *Let $\mathcal{K}$ be a d-pseudomanifold and $(V(\mathcal{K}), \mathcal{I})$ be a matroid of rank $d+1$. Then for each ordered basis B of $(V(\mathcal{K}), \mathcal{I})$,*

$$|\mathcal{K}_B| \equiv |\partial\mathcal{K}_B| \quad \textit{(mod 2)}$$

Suppose further, $(\mathcal{K}, \omega)$ is coherently oriented, then

$$(-1)^d\{|\mathcal{K}_B^+| - |\mathcal{K}_B^-|\} = |\partial\mathcal{K}_B^+| - |\partial\mathcal{K}_B^-|.$$

An analogous discussion about the special case of Theorem 4 is the notion of Sperner Matroid, in which the matroid dependence and the affine dependence are compatible in a triangulation of a simplex as stated in Theorem 6.

Theorem 6. *Let T be a triangulation of a d-simplex $\overline{a_0 a_1 \ldots a_d}$ and $(V(T), \mathcal{I})$ be a Sperner matroid over T, that is, for each $v \in V(T)$,*

$$v \in conv(\{a_{k_0}, a_{k_1}, \ldots, a_{k_r}\}) \quad \textit{implies} \quad v \in cl(\{a_{k_0}, a_{k_1}, \ldots, a_{k_r}\})$$

whenever $0 \leq r \leq d$ and $0 \leq k_0 < k_1 \ldots < k_r \leq d$. If $B = (a_0, a_1, \ldots, a_d)$ is an ordered basis and if the $(d-1)$-simplex $\overline{a_1 \ldots a_d}$ contains no loops of the matroid $(V(T), \mathcal{I})$, then, with the canonical orientation ω,

$$|\widetilde{T}_B^+| - |\widetilde{T}_B^-| = 1$$

where $\widetilde{T}$ is the vertex scheme of T.

Theorem 6 was proved by Lee and Shih [7]. We conclude by remarking that the hypothesis "$\overline{a_1 \ldots a_d}$ contains no loops" in Theorem 6 implies the corresponding labelling is Sperner, hence Theorem 6 holds.

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Chien-Hung Chen
Department of Business Administration,
Transworld Institute of Technology,
Yunlin 640, Taiwan.
E-mail: hung@tit.edu.tw

Shyh-Nan Lee
Department of Applied Mathematics,
Chung Yuan Christian University,
Chung Li 320, Taiwan.
E-mail: nan@math.cycu.edu.tw

Mau-Hsiang Shih
Department of Mathematics,
National Taiwan Normal University,
Taipei 116, Taiwan.
E-mail: mhshih@math.ntnu.edu.tw