TAIWANESE JOURNAL OF MATHEMATICS
Vol. 11, No. 3, pp. 945-958, August 2007
This paper is available online at http://www.math.nthu.edu.tw/tjm/

ON SENSITIVITY ANALYSIS IN VECTOR OPTIMIZATION

G. M. Lee and N. Q. Huy

Abstract. The aim of this paper is to devote to the sensitivity analysis in vector optimization problems. We prove that, under some suitable qualification conditions, the efficient solution map and efficient frontier of a parameterized vector optimization problem are proto-differentiable. It also provide two sufficient conditions for inner and outer approximation of the proto-derivative of the efficient frontier map.

1. INTRODUCTION

Sensitivity analysis of perturbation maps, where the derivative of a perturbation map is defined graphically: the graph of the derivative is certain tangent cone to the graph of a multifunction under consideration, was intensively studied in the past several years. A number of interesting results of sensitivity analysis of perturbation maps have been found in [2-13, 15-18] via the concept of the contingent cone, but a few results are established concerning the adjacent cone (see [6-8, 11, 13]).

The concept of the proto-differentiability of a multifunction, in which the contingent cone and adjacent cone at a point to its graph coincide, was introduced by Rockafellar [13]. For more details, we refer the reader to [13, 14]. Rockafellar [13] showed that for a parametric scalar optimization problem with smooth object and constraint functions, the perturbation map which associates with each parametric vector the corresponding Kuhn-Tucker points and multipliers is often proto-differentiable.

The aim of this paper is to devote to the sensitivity analysis in vector optimization problems. We prove that, under some suitable qualification conditions, the efficient solution map and the efficient frontier map of a parameterized vector optimization

Received March 8, 2007.
Communicated by J. C. Yao.
2000 *Mathematics Subject Classification*: 90C29, 90C30, 90C31.
Key words and phrases: Vector optimization, Multifunction, Efficient solution map, Efficient frontier map, Proto-derivative.

problem are proto-differentiable. It also provide two sufficient conditions for inner and outer approximation of the proto-derivative of the efficient frontier map.

The organization of the paper is as follows. In Section 2, we recall several concepts of derivatives of multifunctions and their properties which are needed in the sequel. In Section 3, we establish the proto-differentiability of the efficient solution map and the efficient frontier map. The sufficient conditions in order to approximate to the proto-derivative of the efficient frontier maps are presented in Section 4.

2. Preliminaries

Throughout this paper let C be a nonempty convex closed pointed cone in the n-dimensional Euclidean space $\mathbb{R}^n$. For a set $\Omega \subset R^n$, denote by $\mathrm{cl}\,\Omega$ and $\mathrm{int}\,\Omega$, the topological closure and the interior of Ω, respectively.

Consider a parametric vector optimization problem:

$$(P_u) \qquad \begin{cases} \text{Minimize } \ f(u,x) := (f_1(u,x),\ldots,f_n(u,x)) \\ \text{subject to } \ x \in \Gamma(u), \end{cases}$$

where $\Gamma : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ is a multifunction and $f_i : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$, $i = 1, 2, \ldots, n$, is a single-valued function.

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$. A vector $\bar{v} \in \Omega$ is said to be an *efficient* (resp. *weakly efficient*) *point* of Ω if there is no $v \in \Omega$ satisfying $v - \bar{v} \in C \setminus \{0\}$ (resp. $v - \bar{v} \in \mathrm{int}\,C$). The set of all the efficient (resp. weakly efficient) points of Ω is denoted by $E(\Omega)$ (resp. $E^w(\Omega)$).

We set

$$\Lambda(u) = f(u,\Gamma(u)) := \{y \in \mathbb{R}^n : \ y = f(u,x), \ x \in \Gamma(u)\}.$$

Definition 2.2. The set $\mathrm{S}(u) = \{x \in \Gamma(u) \ : \ f(u,x) \in E(\Lambda(u))\}$ is said to be the *efficient solution* set of (P_u). The set $E(u) = \{f(u,x) : x \in \mathrm{S}(u)\} \subset R^n$ is called the *efficient frontier* of (P_u).

We recall some basic concepts in Set-valued analysis. For more details, we refer the reader to Ref. 1.

Let $y \in \mathrm{cl}\,\Delta \subset \mathbb{R}^n$. The *contingent cone* $T_\Delta(y)$ and the *adjacent cone* (or the *intermediate tangent cone*) $T^b_\Delta(y)$ of Δ at y are defined by the formulas:

$$T_\Delta(y) = \{\eta \in \mathbb{R}^n \mid \exists\{t^k\} \to 0^+,$$
$$\exists\{\eta^k\} \to \eta \text{ such that } y + t^k\eta^k \in \Delta \text{ for all } k\},$$
$$T^b_\Delta(y) = \{\eta \in \mathbb{R}^n \mid \forall\{t^k\} \to 0^+,$$
$$\exists\{\eta^k\} \to \eta \text{ such that } y + t^k\eta^k \in \Delta \text{ for all } k\}.$$

Definition 2.3. [1]. Let $F : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ be a multifunction with the graph defined by

$$\text{gph}\, F = \{(u, y) \in \mathbb{R}^d \times \mathbb{R}^n \mid y \in F(u)\}.$$

The contingent derivative $DF(u, y)$ and the adjacent derivative $D^b F(u, y)$ of F at $(u, y) \in$ gph F are the multifunctions from $\mathbb{R}^d$ to $\mathbb{R}^n$ given by

$$DF(u, y)(\zeta) = \{\eta \in \mathbb{R}^n \mid (\zeta, \eta) \in T_{\text{gph}\, F}(u, y)\} \quad \forall \zeta \in \mathbb{R}^d$$

and

$$D^b F(u, y)(\zeta) = \{\eta \in \mathbb{R}^n \mid (\zeta, \eta) \in T^b_{\text{gph}\, F}(u, y)\} \quad \forall \zeta \in \mathbb{R}^d, \text{ respectively.}$$

Clearly,

$$\text{gph}\, DF(u, y) = T_{\text{gph}\, F}(u, y) \text{ and } \text{gph}\, D^b F(u, y) = T^b_{\text{gph}\, F}(u, y).$$

Definition 2.4. [13]. Let $G : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ be a multifunction, and let $u \in$ dom$G$ and $y \in G(u)$. Let $\Xi_t : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ be the difference quotient multifunction at u relative to y defined by

$$\Xi_t(\zeta) = \frac{G(u + t\zeta) - y}{t} \quad \text{for } t > 0. \tag{2.1}$$

The multifunction G is said to be proto-differentiable at u relative to y if there is a multifunction $\Xi : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ such that Ξ_t converges in graph to Ξ, i.e., the set gph Ξ_t converges in $\mathbb{R}^d \times \mathbb{R}^n$ to the set gph Ξ in the sense of Kuratowski-Painlevé as $t \to 0^+$. We call Ξ the *proto-derivative* of G at u relative to y and denote it by $G'_{u,y}$.

Definition 2.5. [13]. We shall say that the set Δ is *approximable* at $y \in \Delta$ if the contingent cone and adjacent cone of Δ at y coincide.

Lemma 2.1. [13]. *A multifunction $G : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ is proto-differentiable at u relative to $y \in G(u)$ if and only if the set* gph$G$ *is approximable at (u, y). In this case,*

$$\text{gph}G'_{u,y} = T_{\text{gphG}}(u, y) = T^b_{\text{gphG}}(u, y).$$

Definition 2.6. [11, 13]. Let $G : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ be a multifunction, and let $u \in$ dom$G$ and $y \in G(u)$. We say that G is *semi-differentiable* at u relative to y if there is a multifunction $\Xi : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ such that the quotient multifunctions Ξ_t, $t > 0$, in (2.1) satisfy the condition

$$\lim_{t \to 0^+; \zeta' \to \zeta} \Xi_t(\zeta') = \Xi(\zeta) \quad \text{for all } \zeta \in \mathbb{R}^d,$$

where the convergence of sets is understood in the sense of Kuratowski-Painlevé.

From Theorem 3.1 in [13], it follows that semi-differentiability implies proto-differentiability. Note that the reverse implication is not true in general (see for instance [13]). Obviously, if a single-valued function $G : \mathbb{R}^d \to \mathbb{R}^n$ is continuously Fréchet differentiable at u, then it is semi-differentiable at u relative to $y = G(u)$. Furthermore, the proto-derivative $G'_{u,y}$ of G at u relative to y and the Fréchet derivative G'_u of G at u coincide.

In the sequel, $\mathbb{R}_+$ denotes the set of all the nonnegative real numbers and $\mathbb{R}_{++} := \mathbb{R}_+ \setminus \{0\}$. We shall need the following type of derivative for multifunctions.

Definition 2.7. [3]. For any $(u, y) \in \text{gph}\, F$, the set $RC_{\text{gph}\, F}(u, y) \subset \mathbb{R}^d \times \mathbb{R}^n$ defined by

$$\begin{aligned} &RC_{\text{gph}\, F}(u, y) \\ &= \{(\zeta, \eta) \in \mathbb{R}^d \times \mathbb{R}^n \mid \text{there exist } \{t^k\} \subset \mathbb{R}_{++},\ \{u^k\} \subset \mathbb{R}^d,\ y^k \in F(u^k) \\ &\quad \text{such that } \lim_{k\to\infty} u^k = u,\ \lim_{k\to\infty} [t^k((u^k, y^k) - (u, y))] = (\zeta, \eta)\}. \end{aligned}$$

is called the *closed radial cone* to the graph of F at (u, y).

A straightforward calculation gives an alternative characterization of the closed radial cone as follows:

$$\begin{aligned} &RC_{\text{gphF}}(u, y) \\ &= \{(\zeta, \eta) \in \mathbb{R}^d \times \mathbb{R}^n \mid \exists\{t^k\} \subset \mathbb{R}_{++},\ \exists\{\zeta^k\} \subset \mathbb{R}^d,\ \exists\{\eta^k\} \subset \mathbb{R}^n \text{ such that} \\ &\quad \lim_{k\to\infty} \zeta^k = \zeta,\ \lim_{k\to\infty} \eta^k = \eta,\ \lim_{k\to\infty} (u + t^k\zeta^k) \\ &= u,\ y + t^k\eta^k \in F(u + t^k\zeta^k) \text{ for all } k\}. \end{aligned}$$

It is easy to check that

$$T_{\text{gphF}}(u, y) \subset RC_{\text{gphF}}(u, y)$$

and

$$T_{\text{gphF}}(u, y) = RC_{\text{gphF}}(u, y)$$

if gph$F$ is a convex set.

Definition 2.8. Let $F : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ be a multifunction and $(u, y) \in$ gph$F$. The radial epiderivative $D^p(u, y)$ of F at (u, y) is the multifunction from $\mathbb{R}^d$ to $\mathbb{R}^n$ defined by

$$\text{gph} D^p F(u, y) = RC_{\text{gphF}}(u, y).$$

Note that in [15], the notion “closed radial cone” and “radial epiderivative” was called “TP-cone” and “TP-derivative”, respectively.

Let $\Gamma : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ be a multufunction from $\mathbb{R}^d$ to $\mathbb{R}^m$ and let $\Lambda : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ a multifunction defined by

$$\Lambda(u) = \{y \in \mathbb{R}^n : \ y = f(u, x), \ x \in \Gamma(u)\},$$

where $f : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^n$ is a single-valued function. Let $\widetilde{\Gamma} : \mathbb{R}^d \times \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a multifunction from $\mathbb{R}^d \times \mathbb{R}^n$ to $\mathbb{R}^m$ defined by

$$\widetilde{\Gamma}(u, y) = \{x \in \Gamma(u) \mid y = f(u, x)\}.$$

The following lemma shows the proto-differentiability of the perturbation map for effective domain and valued domain of the parametric vector optimization problem (P_u).

Lemma 2.2. *Let $\hat{u} \in \mathbb{R}^d, \ \hat{x} \in \Gamma(\hat{u})$ and $\hat{y} = f(\hat{u}, \hat{x})$. Suppose that $\Gamma(\cdot)$ is proto-differentiable at $\hat{u}$ relative to $\hat{x}$ and $f(\cdot)$ is continuously Fr échet differentiable at $(\hat{u}, \hat{x})$, and*

$$D^p\widetilde{\Gamma}(\hat{u}, \hat{y})(0, 0) = \{0\}. \tag{2.2}$$

Then $\Lambda(\cdot)$ is proto-differentiable at $\hat{u}$ relative to $\hat{y}$. Moreover,

$$\Lambda'_{\hat{u},\hat{y}}(\zeta) = \{\eta \in \mathbb{R}^n \mid \eta = \nabla_u f(\hat{u}, \hat{x})(\zeta) + \nabla_x f(\hat{u}, \hat{x})(\xi) \text{ and } \xi \in \Gamma'_{\hat{u},\hat{x}}(\zeta)\} \quad \forall \zeta \in \mathbb{R}^d.$$

Proof. Let $L := \{(\zeta, \eta) \in \mathbb{R}^d \times \mathbb{R}^n \mid \eta = f'_{\hat{u},\hat{x}}(\zeta, \xi) \text{ and } \xi \in \Gamma'_{\hat{u},\hat{x}}(\zeta)\}$. By Lemma 2.1, we only need to show that

$$T_{\operatorname{gph}\Lambda}(\hat{u}, \hat{y}) \subset L \subset T^b_{\operatorname{gph}\Lambda}(\hat{u}, \hat{y}).$$

Let $(\zeta, \eta) \in T_{\operatorname{gph}\Lambda}(\hat{u}, \hat{y})$. Then there exist $\{t^k\} \subset \mathbb{R}_{++}$, $t^k \to 0$, $\{(\zeta^k, \eta^k)\} \subset \mathbb{R}^d \times \mathbb{R}^n$, $(\zeta^k, \eta^k) \to (\zeta, \eta)$ such that $\hat{y} + t^k\eta^k \in \Lambda(\hat{u} + t^k\zeta^k)$ for all k. Then there exist $\xi^k \in \mathbb{R}^n$ such that

$$\hat{x} + t^k\xi^k \in \Gamma(\hat{u} + t^k\zeta^k) \text{ and } \hat{y} + t^k\eta^k = f(\hat{u} + t^k\zeta^k, \hat{x} + t^k\xi^k) \quad \forall k. \tag{2.3}$$

Hence $\hat{x} + t^k\xi^k \in \widetilde{\Gamma}(\hat{u} + t^k\zeta^k, \hat{y} + t^k\eta^k)$. We claim that the sequence $\{\xi^k\}$ has a convergent subsequence. Indeed, suppose that $\{\xi^k\}$ has not any convergent subsequence. Then we may assume that $\lim\limits_{k\to\infty} \|\xi^k\| = \infty$. Setting

$$\tilde{\xi}^k = \frac{\xi^k}{\|\xi^k\|}, \ \tilde{\zeta}^k = \frac{\zeta^k}{\|\xi^k\|}, \ \tilde{\eta}^k = \frac{\eta^k}{\|\xi^k\|}, \ \tilde{t}^k = t^k\|\xi^k\|,$$

we have $t^k\zeta^k = \tilde{t}^k\tilde{\zeta}^k$, $t^k\eta^k = \tilde{t}^k\tilde{\eta}^k$, $\lim\limits_{k\to\infty}\tilde{\zeta}^k = \lim\limits_{k\to\infty}\tilde{\eta}^k = 0$, $\lim\limits_{k\to\infty}\tilde{t}^k\tilde{\zeta}^k = \lim\limits_{k\to\infty}\tilde{t}^k\tilde{\eta}^k = 0$,

$$\tilde{\xi}^k \in \frac{\widetilde{\Gamma}(\hat{u}+\tilde{t}^k\tilde{\zeta}^k, \hat{y}+\tilde{t}^k\tilde{\eta}^k)-\hat{x}}{\tilde{t}^k} \tag{2.4}$$

and $\|\tilde{\xi}^k\| = 1$ for all k. By taking a subsequence if necessary, we may assume that $\lim\limits_{k\to\infty}\tilde{\xi}^k = \tilde{\xi}$ and $\|\tilde{\xi}\|$=1. Then, from (2.4) it follows that $\tilde{\xi} \in D^p\widetilde{\Gamma}(\hat{u},\hat{y})(0,0)$, contrary to (2.2) and our claim is proved. There is no loss of generality in assuming that $\lim\limits_{k\to\infty}\xi^k = \xi$. Combining the proto-differentiability of $\Gamma(\cdot)$ at $\hat{u}$ relative to $\hat{x}$ and the continuous Fréchet differentiability of $f(\cdot)$ at $(\hat{u},\hat{x})$ with (2.3), we have $\xi \in \Gamma'_{u,x}(\zeta)$ and $\eta = f'_{\hat{u},\hat{x}}(\zeta,\xi)$. Hence $T_{\mathrm{gph}\,\Lambda}(\hat{u},\hat{y}) \subset L$.

What is left is to show that $L \subset T^b_{\mathrm{gph}\,\Lambda}(\hat{u},\hat{y})$. Let $(\zeta,\eta) \in L$. Then $\eta = f'_{\hat{u},\hat{x}}(\zeta,\xi)$ for some $\xi \in \Gamma'_{\hat{u},\hat{x}}(\zeta)$. Since $\Gamma(\cdot)$ is proto-differentiable at $\hat{u}$ relative to $\hat{x}$ it follows that for any $\{\bar{t}^k\} \subset \mathbb{R}_{++}$, $\bar{t}^k \to 0$ there exists $\{(\bar{\zeta}^k,\bar{\xi}^k)\} \subset \mathbb{R}^d\times\mathbb{R}^m$, $(\bar{\zeta}^k,\bar{\xi}^k) \to (\zeta,\xi)$ such that $\hat{x}+\bar{t}^k\bar{\xi}^k \in \Gamma(\hat{u}+\bar{t}^k\bar{\zeta}^k)$ for all k. Taking

$$\bar{\eta}^k = \frac{f(\hat{u}+\bar{t}^k\bar{\zeta}^k, \hat{x}+\bar{t}^k\bar{\xi}^k)-\hat{y}}{\bar{t}^k},$$

we have $\hat{y}+\bar{t}^k\bar{\eta}^k \in \Lambda(\hat{u}+\bar{t}^k\bar{\zeta}^k)$ for all k and $\lim\limits_{k\to\infty}\bar{\eta}^k = \eta$. Hence $(\zeta,\eta) \in T^b_{\mathrm{gph}\,\Lambda}(\hat{u},\hat{y})$. Thus $L \subset T^b_{\mathrm{gph}\,\Lambda}(\hat{u},\hat{y})$. By the continuous Fréchet differentiability of $f(\cdot)$ at $(\hat{u},\hat{x})$, it is easy to check that

$$f'_{\hat{u},\hat{x}}(\zeta,\xi) = \nabla_u f(\hat{u},\hat{x})(\zeta) + \nabla_x f(\hat{u},\hat{x})(\xi).$$

The proof is complete. ∎

The semi-differentiability of $\Lambda(\cdot)$ is established by our next lemma.

Lemma 2.3. *In Lemma 2.2, if $\Gamma(\cdot)$ is semi-differentiable at $\hat{u}$ relative to $\hat{x}$ then $\Lambda(\cdot)$ is semi-differentiable at $\hat{u}$ relative to $\hat{y}$.*

Proof. Let $\zeta \in \mathbb{R}^d$ and $\eta \in \limsup\limits_{\substack{t\to0^+\\ \zeta'\to\zeta}} \dfrac{\Lambda(\hat{u}+t\zeta')-\hat{y}}{t}$. From Proposition 2.3 in [13] it follows that $\eta \in \Lambda'_{\hat{u},\hat{y}}(\zeta)$. Analysis similar to that in the proof of Lemma 2.2 shows that $\xi \in \Gamma'_{u,x}(\zeta)$ and $\eta = f'_{\hat{u},\hat{x}}(\zeta,\xi)$. From the semi-differentiability of $\Gamma(\cdot)$ at $\hat{u}$ relative to $\hat{x}$ it follows that for any $t^k \to 0^+$ and $\zeta^k \to \zeta$ there exists $\xi^k \to \xi$ such that $\hat{x}+t^k\xi^k \in \Gamma(\hat{u}+t^k\zeta^k)$ for all k. Setting

$$\eta^k = \frac{f(\hat{u}+t^k\zeta^k, \hat{x}+t^k\xi^k)-\hat{y}}{t^k},$$

we have $\lim_{k\to\infty} \eta^k = \eta$ and $\hat{y} + t^k\eta^k \in \Lambda(\hat{u} + t^k\zeta^k)$ for all k. Hence $\eta \in \liminf_{\substack{t\to 0^+\\ \zeta'\to\zeta}} \dfrac{\Lambda(\hat{u}+t\zeta') - \hat{y}}{t}$. Thus $\Lambda(\cdot)$ is semi-differentiable at $\hat{u}$ relative to $\hat{y}$. ■

Lemma 2.4. *Suppose that $G : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ is proto-differentiable at u relative to $y \in G(u)$ and the following constraint qualification holds:*

$$D^pG(u, y)(0) \cap (-C) = \{0\}. \tag{2.5}$$

Then the multifunction $G + C : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ defined by $(G + C)(\cdot) = G(\cdot) + C$ is proto-differentiable at u relative to x and

$$(G + C)'_{u,y}(\zeta) = G'_{u,y}(\zeta) + C \quad \forall\zeta \in \mathbb{R}^d. \tag{2.6}$$

Proof. Let $u \in \mathbb{R}^d$ and $y \in G(u)$. Obviously, $y \in (G + C)(u)$. We shall establish the lemma if we prove the following:

$$D(G + C)(u, y)(\zeta) \subset DG(u, y)(\zeta) + C \quad \forall\zeta \in \mathbb{R}^d \tag{2.7}$$

and

$$DG^b(u, y)(\zeta) + C \subset D^b(G + C)(u, y)(\zeta), \quad \forall\zeta \in \mathbb{R}^d \tag{2.8}$$

Let $\xi \in D(G + C)(u, y)(\zeta)$. Then there exist $\{t^k\} \subset \mathbb{R}_{++}$, $t^k \to 0$, $\{(\zeta^k, \xi^k)\} \subset \mathbb{R}^d \times \mathbb{R}^n$, $(\zeta^k, \xi^k) \to (\zeta, \xi)$ such that $y + t^k\xi^k \in G(u + t^k\zeta^k) + C$ for all k. It follows that

$$\xi^k = \eta^k + c^k, \tag{2.9}$$

where $\eta^k \in \dfrac{G(u + t^k\zeta^k) - y}{t^k}$ and $c^k \in \dfrac{C}{t^k} \subset C$. We claim that the sequence $\{\eta^k\}$ has a convergent subsequence. Indeed, suppose that $\{\eta^k\}$ has not any convergent subsequence. Then we may assume that $\lim_{k\to\infty} \|\eta^k\| = \infty$. Setting

$$\tilde{\eta}^k = \frac{\eta^k}{\|\eta^k\|},\ \tilde{\zeta}^k = \frac{\zeta^k}{\|\eta^k\|},\ \tilde{t}^k = t^k\|\eta^k\|,$$

we have $t^k\zeta^k = \tilde{t}^k\tilde{\zeta}^k$, $\lim_{k\to\infty} \tilde{\zeta}^k = 0$, $\lim_{k\to\infty} \tilde{t}^k\tilde{\zeta}^k = 0$, $\tilde{\eta}^k \in \dfrac{G(u + \tilde{t}^k\tilde{\zeta}^k) - y}{\tilde{t}^k}$ and $\|\tilde{\eta}^k\| = 1$ for all k. By taking a subsequence if necessary, we may assume that $\lim_{k\to\infty} \tilde{\eta}^k = \tilde{\eta}$ and $\|\tilde{\eta}\|$ =1. Then $\tilde{\eta} \in D^p\tilde{G}(u, y)(0)$. From (2.9) it follows that $\lim_{k\to\infty} \dfrac{c^k}{\|\eta^k\|} = -\tilde{\eta}$.

By the closedness of C, $-\tilde{\eta} \in C$. Hence $\tilde{\eta} \in D^p G(u,y)(0) \cap (-C)$, contrary to (2.5), and hence our claim is proved. There is no loss of generality in assuming that $\lim_{k\to\infty} \eta^k = \eta$. It follows that $\eta \in DG(u,y)(\zeta)$. By (2.9) and the closedness of C, $\lim_{k\to\infty} c^k = \xi - \eta$ and $\xi - \eta \in C$. Hence $\xi \in DG(u,y)(\zeta) + C$. Therefore (2.7) is fulfilled.

It remains to prove that (2.8) holds. Let $\zeta \in \mathbb{R}^d$ and $\eta \in D^b G(x,y)(\zeta) + C$. Set

$$\xi = \eta + c, \tag{2.10}$$

where $\eta \in D^b G(u,y)(\zeta)$ and $c \in C$. Since $\eta \in D^b G(u,y)(\zeta)$, it follows that for any $\{t^k\} \subset \mathbb{R}_{++}$, $t^k \to 0$, there exist $\{(\zeta^k, \eta^k)\} \subset \mathbb{R}^d \times \mathbb{R}^n$, $(\zeta^k, \eta^k) \to (\zeta, \eta)$ such that $y + t^k \eta^k \in G(u + t^k \zeta^k)$ for all k. Taking $\xi^k = \eta^k + c$, by (2.10), we have

$$\lim_{k\to\infty} \xi^k = \xi \ \text{ and } \ y + t^k \xi^k \in (G + C)(u + t^k \zeta^k) \ \text{ for all } k.$$

Thus $\xi \in D^b(G + C)(u,y)(\zeta)$ and (2.8) is proved. Combining (2.7) and (2.8) with the proto-differentiability of G at u relative to y we obtain $D^b(G + C)(u,y)(\zeta) = D(G + C)(u,y)(\zeta)$ for all $\zeta \in \mathbb{R}^d$. Thus $(G + C)(\cdot)$ is proto-differentiable at x relative to y and the formula (2.6) is fulfilled. ■

The following example shows that the condition (2.5) in Lemma 2.4 is essential for the validity of the conclusion (2.6).

Example 2.1. [6]. Let $C = \mathbb{R}_+$ and let the multifunctions $G : \mathbb{R} \rightrightarrows \mathbb{R}$ be given by the formula

$$G(u) = \{y \mid -u^2 \leq y \leq u^2\} \cup \{y \mid y \leq -1\}.$$

We can check that $G(\cdot)$ and $(G + \mathbb{R}_+)(\cdot)$ are proto-differentiable at 0 relative to 0, $G'_{0,0} = \{0\}$ and $(G + \mathbb{R}_+)_{0,0}(0) = \mathbb{R}_+$. Thus

$$(G + C)'_{0,0}(0) \neq G'_{0,0}(0) + \mathbb{R}_+.$$

We check at one that

$$D^p G(0,0)(0) \cap (-C) = \mathbb{R}_+,$$

and the condition (2.5) is not fulfilled.

3. Proto-differentiability of the Efficient Solution Map and Efficient Frontier Map

In this section, we consider a family of parameterized vector optimization problems (P_u) defined by in Section 2. We recall that $f : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^n$ is a single-valued function, $\Gamma : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ is a multifunction from $\mathbb{R}^d$ to $\mathbb{R}^m$ and the multifunction $\Lambda : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ has the form $\Lambda(u) = \{y \in \mathbb{R}^n : \ y = f(u,x), \ x \in \Gamma(u)\}$. The efficient frontier (resp. efficient solution) map of a family of parameterized vector optimization problem (P_u) is a multifunction $E : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ (resp. $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$) defined by $E(u) = E(\Lambda(u))$ (resp. $S(u) = \{x \in \Gamma(u) \ : \ f(u,x) \in E(u)\}$) for all $u \in \mathbb{R}^d$.

Let $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ be single-valued function. φ is said to be monotone if for any $x_1, \ x_2 \in \mathbb{R}^m$ one has $\langle f(x_2) - f(x_1), x_2 - x_1\rangle \geq 0$. φ is said to be strictly monotone if for any $x_1, \ x_2 \in \mathbb{R}^m$ and $x_1 \neq x_2$ imply $\langle f(x_2) - f(x_1), x_2 - x_1\rangle > 0$.

Two sufficient conditions for the proto-differentiability of the efficient solution map and the efficient frontier map are established by our next theorem.

Theorem 3.1. *Let $\hat{u} \in \mathbb{R}^d, \ \hat{x} \in S(\hat{u})$ and $\hat{y} = f(\hat{u}, \hat{x})$. Suppose that the following properties hold:*

(*i*) *f is continuously Fréchet differentiable at ($\hat{u}$,$\hat{x}$);*

(*ii*) *$\nabla_x f(\hat{u}, \hat{x})(\cdot)$ is strictly monotone on $\mathbb{R}^m$;*

(*iii*) $D^p\widetilde{\Gamma}(\hat{u}, \hat{y})(0,0) = \{0\}$.

Then the efficient solution map $S(\cdot)$ is proto-differentiable at $\hat{u}$ relative to $\hat{x}$ whenever the efficient frontier map $E(\cdot)$ is proto-differentiable at $\hat{u}$ relative to $\hat{y}$. Moreover

$$S'_{\hat{u},\hat{x}}(\zeta) = \{\xi \in D\Gamma(\hat{u}, \hat{x})(\zeta) \mid \nabla_u f(\hat{u}, \hat{x})(\zeta) + \nabla_x f(\hat{u}, \hat{x})(\xi) \in E'_{\hat{u},\hat{y}}(\zeta)\} \quad \forall \zeta \in \mathbb{R}^d. \tag{3.1}$$

Proof. Let $L = \{(\zeta, \xi) \in \mathbb{R}^d \times \mathbb{R}^m \mid \xi \in D\Gamma(\hat{u}, \hat{x})(\zeta)$ and $f'_{\hat{u},\hat{x}}(\zeta, \xi) \in E'_{\hat{u},\hat{y}}(\zeta)\}$. By Lemma 2.1, we only need to show that

$$T_{\operatorname{gph} S}(\hat{u}, \hat{x}) \subset L \subset T^b_{\operatorname{gph} S}(\hat{u}, \hat{x}). \tag{3.2}$$

Let $(\zeta, \xi) \in T_{\operatorname{gph} S}(\hat{u}, \hat{x})$. Then there exist $\{t^k\} \subset \mathbb{R}_{++}$, $t^k \to 0^+$, $\{(\zeta^k, \xi^k)\} \subset \mathbb{R}^d \times \mathbb{R}^m$, $(\zeta^k, \xi^k) \to (\zeta, \xi)$ such that $\hat{x} + t^k\xi^k \in \Gamma(\hat{u} + t^k\zeta^k)$ and $f(\hat{u} + t^k\zeta^k, \hat{x} + t^k\xi^k) \in E(\hat{u} + t^k\zeta^k)$ for all k. Hence $\xi \in D\Gamma(\hat{u}, \hat{x})(\zeta)$. Taking $\eta^k = \dfrac{f(\hat{u} + t^k\zeta^k, \hat{x} + t^k\xi^k) - \hat{y}}{t^k}$, we have $\hat{y} + t^k\eta^k \in E(\hat{u} + t^k\zeta^k)$ and $\lim\limits_{k\to\infty} \eta^k =$

$f'_{\hat{u},\hat{x}}(\zeta,\xi)$. Hence $f'_{\hat{u},\hat{x}}(\zeta,\xi) \in E'_{\hat{u},\hat{y}}(\zeta)$. Since $f(\cdot)$ is continuously Fréchet differentiable at $(\hat{u},\hat{x})$ it follows that

$$f'_{\hat{u},\hat{x}}(\zeta,\xi) = \nabla_u f(\hat{u},\hat{x})(\zeta) + \nabla_x f(\hat{u},\hat{x})(\xi).$$

Therefore the first inclusion in (3.2) is fulfilled.

It remains to prove that the second inclusion in (3.2) holds. Let $(\zeta,\xi) \in L$. Then $\xi \in D\Gamma(\hat{u},\hat{x})(\zeta)$ and $f'_{\hat{u},\hat{x}}(\zeta,\xi) \in E'_{\hat{u},\hat{y}}(\zeta)$. Since $E(\cdot)$ is proto-differentiable at $\hat{u}$ relative to $\hat{y}$ it follows that for any $\tilde{t}^k \to 0^+$ there exists $\{(\tilde{\zeta}^k,\tilde{\eta}^k)\} \subset \mathbb{R}^d \times \mathbb{R}^n$, $(\tilde{\zeta}^k,\tilde{\eta}^k) \to (\zeta, f'_{\hat{u},\hat{x}}(\zeta,\xi))$ such that

$$\hat{y} + \tilde{t}^k\tilde{\eta}^k \in E(\hat{u} + \tilde{t}^k\tilde{\zeta}^k) \quad \forall k. \tag{3.3}$$

Hence there exists $\{\tilde{\xi}^k\} \subset \mathbb{R}^m$ such that

$$\hat{x} + \tilde{t}^k\tilde{\xi}^k \in \Gamma(\hat{u} + \tilde{t}^k\tilde{\zeta}^k) \text{ and } \hat{y} + \tilde{t}^k\tilde{\eta}^k = f(\hat{u} + \tilde{t}^k\tilde{\zeta}^k, \hat{x} + \tilde{t}^k\tilde{\xi}^k) \quad \forall k. \tag{3.4}$$

From (iii) and analysis similar to that in the proof of Lemma 2.2, the sequence $\{\tilde{\xi}^k\}$ has a convergent subsequence. Without loss of generality we can assume that $\lim\limits_{k\to\infty} \tilde{\xi}^k = \tilde{\xi}$. Then $\tilde{\xi} \in D^b\Gamma(\hat{u},\hat{x})(\zeta) \subset D\Gamma(\hat{u},\hat{x})(\zeta)$ and $\lim\limits_{k\to\infty} \tilde{\eta}^k = f'_{\hat{u},\hat{x}}(\zeta,\tilde{\xi})$. Hence $f'_{\hat{u},\hat{x}}(\zeta,\tilde{\xi}) = f'_{\hat{u},\hat{x}}(\zeta,\xi)$. It follows that $\nabla_x f(\hat{u},\hat{x})(\xi) = \nabla_x f(\hat{u},\hat{x})(\tilde{\xi})$. By (ii), we have $\tilde{\xi} = \xi$, and hence $\xi \in D\Gamma(\hat{u},\hat{x})(\zeta)$. Combining this with (3.3) and (3.4) we obtain $(\zeta,\xi) \in T^b_{\mathrm{gph}\, S}$. Thus the second inclusion in (3.2) is fulfilled. The proof is complete. ■

Definition 3.1. [18]. Let $\Omega \subset \mathbb{R}^n$. A vector $\bar{v} \in \Omega$ is said to be a *normally efficient* if $\bar{v} \in E(\Omega)$ and $N_{\Omega+C}(\bar{v}) \subset \operatorname{int} C \cup \{0\}$, where $N_\Delta(v)$ denotes the normal cone to Δ at v. The set of all the normally efficient points of Ω is denoted by $E^N(\Omega)$. A vector $x \in \Gamma(u)$ is said to be a normally efficient solution of the problem (P_u) if $f(u,x) \in E^N(\Lambda(u))$. The set of all the normally efficient solutions of (P_u) is denoted by $S^N(u)$.

We say that the multifunction $G : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ is convex (resp. C-convex) if for any $u^1,\ u^2 \in \mathbb{R}^d$ and any $t \in [0,1]$,

$$(1-t)G(u^1) + tG(u^2) \subset G((1-t)u^1 + tu^2)$$
$$(\text{resp.} \quad (1-t)G(u^1) + tG(u^2) \subset G((1-t)u^1 + tu^2)) + C).$$

It is easy to check that if the function $f : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^n$ is C-convex then the multifunction $\Lambda(\cdot)$ is $C-$convex.

Theorem 3.2. *Let* $\hat{u} \in \mathbb{R}^d,\ \hat{u} \in \operatorname{int}(\operatorname{dom}\Lambda),\ \hat{x} \in S^N(\hat{u})$ *and* $\hat{y} = f(\hat{u}, \hat{x})$. *Suppose that the following properties hold:*

(i) $\Gamma(\cdot)$ *is convex*;
(ii) f *is* $C-$*convex and continuously Fréchet differentiable at* $(\hat{u}, \hat{x})$;
(iii) $D^p\widetilde{\Gamma}(\hat{u}, \hat{y})(0,0) = \{0\}$.

Then the efficient frontier map $E(\cdot)$ *is proto-differentiable at* $\hat{u}$ *relative to* $\hat{y}$. *Moreover,*

$$E'_{\hat{u},\hat{y}}(\zeta) = E(\Lambda'_{\hat{u},\hat{y}}(\zeta)) \quad \forall \zeta \in \mathbb{R}^d,$$

where $\Lambda'_{\hat{u},\hat{y}}(\zeta) = \{\eta \in \mathbb{R}^n \mid \eta = \nabla_u f(\hat{u}, \hat{x})(\zeta) + \nabla_x f(\hat{u}, \hat{x})(\xi)$ *and* $\xi \in \Gamma'_{\hat{u},\hat{x}}(\zeta)\}$.

Proof. Since $\Gamma(\cdot)$ is a convex multifunction it follows that $\Gamma(\cdot)$ is proto-differentiable at $\hat{u}$ relative to $\hat{x}$. By Lemma 2.2, we have $\Lambda(\cdot)$ is proto-differentiable at $\hat{u}$ relative to $\hat{y}$. Let $\zeta \in \mathbb{R}^d$. It suffices to prove that

$$DE(\hat{u}, \hat{y})(\zeta) \subset E(\Lambda'_{\hat{u},\hat{y}}(\zeta)) \subset D^b E(\hat{u}, \hat{y})(\zeta). \tag{3.5}$$

Let $\eta \in E(\Lambda'_{\hat{u},\hat{y}}(\zeta))$. Then $\eta \in \Lambda'_{\hat{u},\hat{y}}(\zeta)$. By the proto-differentiability of $\Lambda(\cdot)$ at $\hat{u}$ relative to $\hat{y}$, for any $t^k \to 0^+$ there exists $\{(\zeta^k, \eta^k)\} \subset \mathbb{R}^d \times \mathbb{R}^n$, $(\zeta^k, \eta^k) \to (\zeta, \eta)$ such that $\hat{y} + t^k\eta^k \in \Lambda(\hat{u} + t^k\zeta^k)$ for all k. From (ii) it follows that $\Lambda(\cdot)$ is $C-$convex. Consequently, $\Lambda(\hat{u} + t^k\zeta^k) \subset E(\hat{u} + t^k\zeta^k) + C$ for all k. Hence there exists $\{\tilde{\eta}^k\} \subset \mathbb{R}^n$ such that

$$\hat{y} + t^k\tilde{\eta}^k \in E(\hat{u} + t^k\zeta^k) \text{ and } \eta^k - \tilde{\eta}^k \in C \quad \forall k. \tag{3.6}$$

It follows that there exists $\{\tilde{\xi}^k\} \subset \mathbb{R}^m$ such that

$$\hat{x} + t^k\tilde{\xi}^k \in \Gamma(\hat{u} + t^k\zeta^k) \text{ and } \hat{y} + t^k\tilde{\eta}^k = f(\hat{u} + t^k\zeta^k, \hat{x} + t^k\tilde{\xi}^k) \quad \forall k. \tag{3.7}$$

From (iii) and analysis similar to that in the proof of Lemma 2.2, the sequence $\{\tilde{\xi}^k\}$ has a convergent subsequence. Without loss of generality we can assume that $\lim\limits_{k\to\infty} \tilde{\xi}^k = \tilde{\xi}$. Hence, by (3.7), $\tilde{\xi} \in \Gamma'_{\hat{u},\hat{x}}(\zeta)$ and $\lim\limits_{k\to\infty} \tilde{\eta}^k = \tilde{\eta}$ with $\tilde{\eta} = f'_{\hat{u},\hat{x}}(\zeta, \tilde{\xi})$. From (3.6) it follows that $\tilde{\eta} \in D^b E(\hat{u}, \hat{y})(\zeta) \subset D^b\Lambda(\hat{u}, \hat{y})(\zeta)$. By the closedness of C, we have $\eta - \tilde{\eta} \in C$. Since $\eta \in E(\Lambda'_{\hat{u},\hat{y}}(\zeta))$ and $D^b\Lambda(\hat{u}, \hat{y})(\zeta) = \Lambda'_{\hat{u},\hat{y}}(\zeta)$ it may be concluded that $\tilde{\eta} = \eta$. Therefore the second inclusion in (3.5) is fulfilled. Obviously, the first inclusion in (3.5) is fulfilled by Theorem 5.2 in [18]. Thus $E(\cdot)$ is proto-differentiable at $\hat{u}$ relative to $\hat{y}$ and $E'_{\hat{u},\hat{y}}(\zeta) = E(\Lambda'_{\hat{u},\hat{y}}(\zeta))$. Applying Lemma 2.2 we obtain $\Lambda'_{\hat{u},\hat{y}}(\zeta) = \{\eta \in \mathbb{R}^n \mid \eta = \nabla_u f(\hat{u}, \hat{x})(\zeta) + \nabla_x f(\hat{u}, \hat{x})(\xi)$ and $\xi \in \Gamma'_{\hat{u},\hat{x}}(\zeta)\}$. The proof is complete. ∎

4. Approximation for the Proto-derivative of the Efficient Frontier Map

It is well known that the computation of the proto-derivative of the efficient frontier map $E(\cdot)$ is not easy in general. A number of interesting results concerning the computation of the proto-derivative of a multifunction are found in [6-8, 11-13]. In this section, we discuss inner and outer approximation of the proto-derivative of the efficient frontier map.

Theorem 4.1. *Let $\hat{u} \in \mathbb{R}^d$, $\hat{x} \in S(\hat{u})$ and $\hat{y} = f(\hat{u}, \hat{x})$. Suppose that the following properties hold:*

(i) $\Gamma(\cdot)$ *is proto-differentiable at $\hat{u}$ relative to $\hat{x}$;*

(ii) f *is continuously Fréchet differentiable at $(\hat{u}, \hat{x})$;*

(iii) $D^p E(\hat{u}, \hat{y})(0) \cap (-C) = \{0\}$;

(iv) $\Lambda(\cdot)$ *has $C-$dominated property in a neighborhood of $\hat{u}$, i.e.,there is a neighborhood$U$ of $\hat{u}$ such that $\Lambda(u) \subset E(u) + C \quad \forall u \in U$.*

Then $E(\Lambda'_{\hat{u},\hat{y}}(\zeta)) \subset E'_{\hat{u},\hat{y}}(\zeta) \quad \forall \zeta \in \mathbb{R}^d$.

Proof. By $\hat{x} \in S(\hat{u})$, we have $\hat{y} \in E(\hat{u}) \subset \Lambda(\hat{u})$. Let $\zeta \in \mathbb{R}^d$ and $\eta \in E(\Lambda'_{\hat{u},\hat{y}}(\zeta))$. Then $\eta \in \Lambda'_{\hat{u},\hat{y}}(\zeta)$. From (i)-(iv) and Lemma 2.4, it follows that

$$\Lambda'_{\hat{u},\hat{y}}(\zeta) = E'_{\hat{u},\hat{y}}(\zeta) + C.$$

Hence there exist $\eta' \in E'_{\hat{u},\hat{y}}(\zeta)$ and $c \in C$ such that $\eta = \eta' + c$. We claim that $c = 0$. Indeed, if $c \in C \setminus \{0\}$ then $\eta - \eta' \in C \setminus \{0\}$. Obviously, $\eta' \in \Lambda'_{\hat{u},\hat{y}}(\zeta)$. It follows that $\eta \notin E(\Lambda'_{\hat{u},\hat{y}}(\zeta))$, which is impossible. Thus $\eta \in E'_{\hat{u},\hat{y}}(\zeta)$. The proof is complete. ■

Theorem 4.2. *Let $\hat{u} \in \mathbb{R}^d$, $\hat{x} \in S(\hat{u})$ and $\hat{y} = f(\hat{u}, \hat{x})$. Suppose that the following properties hold:*

(i) $\Gamma(\cdot)$ *is semi-differentiable at $\hat{u}$ relative to $\hat{x}$;*

(ii) f *is continuously Fréchet differentiable at $(\hat{u}, \hat{x})$;*

(iii) $D^p\widetilde{\Gamma}(\hat{u}, \hat{y})(0, 0) = \{0\}$.

Then $E'_{\hat{u},\hat{y}}(\zeta) \subset E^w(\Lambda'_{\hat{u},\hat{y}}(\zeta)) \quad \forall \zeta \in \mathbb{R}^d$.

Proof. Let $\zeta \in \mathbb{R}^d$ and $\eta \in E'_{\hat{u},\hat{y}}(\zeta)$. Then $\eta \in \Lambda'_{\hat{u},\hat{y}}(\zeta)$. Suppose the assertion of the theorem is false. Then we can find $\tilde{\eta} \in \Lambda'_{\hat{u},\hat{y}}(\zeta)$ such that

$$\eta - \tilde{\eta} \in \operatorname{int} C. \tag{4.1}$$

By $\eta \in E'_{\hat{u},\hat{y}}(\zeta)$, there exist $\{t^k\} \subset \mathbb{R}_{++}$, $t^k \to 0^+$, $\{(\zeta^k, \eta^k)\} \subset \mathbb{R}^d \times \mathbb{R}^n$, $(\zeta^k, \eta^k) \to (\zeta, \eta)$ such that $\hat{y} + t^k\eta^k \in E(\hat{u} + t^k\zeta^k)$ for all k. From (i)-(iii) and Lemma 2.3, it follows that $\Lambda(\cdot)$ is semi-differentiable at $\hat{u}$ relative to $\hat{y}$. Consequently, for the preceeding sequences $\{t^k\}$ and $\{\zeta^k\}$, there exists $\{\tilde{\eta}^k\} \subset \mathbb{R}^n$, $\lim\limits_{k\to\infty} \tilde{\eta}^k = \tilde{\eta}$ such that $\hat{y} + t^k\tilde{\eta}^k \in \Lambda(\hat{u} + t^k\zeta^k)$ for all k. Hence, by (4.1), we have $(\hat{y} + t^k\eta^k) - (\hat{y} + t^k\tilde{\eta}^k) \in \operatorname{int} C$ for all k large enough. The result is $\hat{y} + t^k\eta^k \notin E(\hat{u} + t^k\zeta^k)$ for all k large enough, which is impossible. Thus $\eta \in E^w(\Lambda'_{\hat{u},\hat{y}}(\zeta))$. The proof is complete. ∎

Acknowledgment

This research was supported by the Brain Korea 21 Project in 2003. Main Results of this work were obtained while the second author had stayed in Korea by the financial support of KOSEF (2004 KOSEF Postdoctoral Fellowship).

References

1. J. P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser Boston, Inc., Boston, MA, 1990.
2. A. L. Dontchev and R. T. Rockafellar, Ample parameterization of variational inclusions, *SIAM Journal on Optimization*, **12** (2001), 170-187.
3. F. Flores-Bazán, Radial epiderivatives and asymptotic functions in nonconvex vector optimization. *SIAM Journal on Optimization*, 14 (2003), 284-305.
4. H. Kuk, T. Tanino and M. Tanaka, Sensitivity analysis in vector optimization, *Journal of Optimization Theory and Applications*, **89** (1996), 713-730.
5. H. Kuk, T. Tanino and M. Tanaka, Sensitivity analysis in parametrized convex vector optimization, *Journal of Mathematical Analysis and Applications*, **202** (1996), 511-522.
6. G. M. Lee and N. Q. Huy, On proto-differentiability of generalize perturbation maps, *Journal of Mathematical Analysis and Applications*, **324** (2006), 1297-1309.
7. A. B. Levy and R. T. Rockafellar, Sensitivity analysis of solutions to generalized equations, *Transactions of American Mathematical Society*, **345** (1994), 661-671.
8. A. B. Levy and R. T. Rockafellar, Variational conditions and the proto-differentiation of partial subgradient mappings, *Nonlinear Analysis*, **26** (1996), 1951-1964.
9. A. B. Levy, Sensitivity of solutions to variational inequalities on Banach spaces, *SIAM Journal on Control and Optimization*, **38** (1999), 50-60.
10. D. T. Luc, Contingent derivatives of set-valued maps and applications to vector optimization, *Mathematical Programming*, **50** (1991), 99-111.

11. J. P. Penot, Differentiability of relations and differential stability of perturbed optimization problems, *SIAM Journal on Control and Optimization* **22(4)** (1984), 529-551.

12. R. A. Poliquin and R. T. Rockafellar, Proto-derivatives of partial subgradient mappings, *Journal of Convex Analysis*, **4(2)** (1997), 221-234.

13. R. T. Rockafellar, Proto-differentiability of set-valued mappings and its applications in optimization, *Annales de l'Institut Henri Poincaré* **-** *Analyse non linéaire*, **6** (1989), 449-482.

14. R. T. Rockafellar and R. J. Wets, *Variational Analysis*, Springer-Verlag, Berlin, 1998.

15. D. S. Shi, Contingent derivative of the perturbation map in multiobjective optimization, *Journal of Optimization Theory and Applications*, **70(2)** (1991), 385-396.

16. D. S. Shi, Sensitivity analysis in convex vector optimization, *Journal of Optimization Theory and Applications*, **77(1)** (1993), 145-159.

17. T. Tanino, Sensitivity analysis in multiobjective optimization, *Journal of Optimization Theory and Applications,* **56(3)** (1988), 479-499.

18. Tanino, T. Stability and sensitivity analysis in convex vector optimization, *SIAM Journal on Control and Optimization*, **26(3)** (1988), 521-536.

G. M. Lee
Department of Applied Mathematics,
Pukyong National University,
Pusan 608-737,
Korea

N. Q. Huy
Department of Mathematics,
Hanoi Pedagogical University, No. 2,
Xuan Hoa, Phuc Yen, Vinh Phuc,
Vietnam