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ON ϵ-APPROXIMATE SOLUTIONS FOR CONVEX SEMIDEFINITE OPTIMIZATION PROBLEMS

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Abstract. In this paper, we discuss ϵ-optimality conditions and ϵ-saddle point theorems for ϵ-approximate solutions for convex semidefinite optimization problem which hold under a weakened constraint qualification or which hold without any constraint qualification.

Moreover, we formulate a Wolfe type dual problem for the convex semidefinite optimization problem, and prove ϵ-weak duality and ϵ-strong duality between the primal problem and the dual problem, which hold under a weakened constraint qualification.

1. INTRODUCTION

Convex semidefinite optimization problem is to optimize an objective convex function over a linear matrix inequality. When the objective function is linear and the corresponding matrices are diagonal, this problem become a linear optimization problem. So, this problem is an extension of a linear optimization problem. In particular, convex semidefinite optimization problem includes many important applications in systems and control theory [4], approximate theory ([3,9,11]) and combinatorial optimization ([1,23,26]). Hence convex semidefinite optimization has been intensively studied ([20,25,28] and [30]). Polynomial time interior point algorithms are now available for solving semidefinite programming problem. Many authors ([2,18,19,24,32,33]) have developed interior point algorithms for linear, semidefinite and convex optimization problems.

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In particular, for convex semidefinite optimization problem, strong duality without constraint qualification [27], complete dual characterization conditions of solutions ([13,16]), saddle point theorems [5] and characterizations of optimal solution sets [14] have been investigated.

From computational viewpoint, algorithms which have been proposed in literature to solve optimization problems compute only ϵ-approximate solutions for the problems. To get the ϵ-approximate solution, many authors have established ϵ-optimality conditions, ϵ-saddle point theorems and ϵ-duality theorems for optimization problems ([6-8,21,22,29,31]).

Recently, Jeyakumar et al. [13] established sequential optimality conditions for exact solutions of convex optimization problems which holds without any constraint qualification. Jeyakumar et al. [12] gave ϵ-optimality conditions for convex optimization problems, which hold without any constraint qualification. Yokoyama et al. [31] gave a special case of convex optimization problem which satisfies ϵ-optimality conditions. Kim et al. [17] proved sequential ϵ-saddle point theorems and ϵ-saddle point theorems for convex optimization problems.

In this paper, we consider ϵ-approximate solutions for a convex semidefinite optimization problem and prove ϵ-optimality theorems and ϵ-saddle point theorems for such solutions which hold under a weakened constraint qualification or which hold without any constraint qualification. Moreover, we formulate a Wolfe type dual problem for the convex semidefinite optimization problem and its dual problem, which hold under a weakened constraint qualification.

2. Preliminaries

Consider the convex semidefinite optimization problem:

$$\textbf{(SDP)} \qquad \text{Minimize} \quad f(x)$$
$$\text{subject to} \quad F_0 + \sum_{i=1}^{m} x_i F_i \succeq 0,$$

where $f : \mathbb{R}^m \to \mathbb{R}$ is a convex function, and for $i = 0, 1, \cdots, m$, $F_i \in S_n$, the space of $n \times n$ real symmetric matrices. The space S_n is partially ordered by the Löwner order; that is, for $M, N \in S_n$, $M \succeq N$ if and only if $M - N$ is positive semidefinite. The inner product in S_n is defined by $(M, N) = \mathrm{Tr}[MN]$, where $\mathrm{Tr}[\cdot]$ is the trace operation.

Let $S := \{M \in S_n \mid M \succeq 0\}$. Then S is self-dual, that is,

$$S^+ = \{\theta \in S_n \mid (\theta, Z) \geqq 0 \; \forall Z \in S\} = S.$$

Let $F(x) := F_0 + \sum_{i=1}^m x_i F_i$, $\hat{F}(x) = \sum_{i=1}^m x_i F_i$, $x = (x_1 \cdots, x_m) \in \mathbb{R}^m$. Then $\hat{F}$ is a linear operator from $\mathbb{R}^m$ $to\, S_n$ and its dual is defined by

$$\hat{F}^*(Z) = (\mathrm{Tr}[F_1 Z], \cdots, \mathrm{Tr}[F_m Z])$$

for any $Z \in S_n$. Clearly, $A := \{x \in \mathbb{R}^m \mid F(x) \in S\}$ is the feasible set of **(SDP)**. Let

$$D := \bigcup_{(Z,\delta)\in S\times\mathbb{R}_+} \{(\hat{F}^*(Z), -\mathrm{Tr}[ZF_0] - \delta)\}.$$

Clearly, D is a convex cone, but is not necessarily closed.

If the interior point condition holds for **(SDP)**, that is, there exists $\hat{x} \in \mathbb{R}^m$ such that $F_0 + \sum_{i=1}^m \hat{x}_i F_i$ is positive definite, then D is closed, but the converse may not hold. For instance, let

$$F_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, F_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} and\ F_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then it is clear that there does not exists $\hat{x} \in \mathbb{R}^m$ such that $F_0 + \sum_{i=1}^2 \hat{x}_i F_i$ is positive definite. However,

$$\begin{aligned} D: &= \bigcup_{(Z,\delta)\in S\times\mathbb{R}_+} \{(\mathrm{Tr}[ZF_1], \mathrm{Tr}[ZF_2], -\mathrm{Tr}[ZF_0] - \delta)\} \\ &= \{(x, 0, z) \in \mathbb{R}^3 \mid x \in \mathbb{R}, z \leqq 0\}. \end{aligned}$$

So, D is closed in $\mathbb{R}^3$.

We will use ths closedness of the set D as a constraint qualification for **(SDP)**.

We give the definition of ϵ-approximate solutions.

Definition 2.1. Let $\epsilon \geqq 0$. Then $\bar{x} \in A$ is called an ϵ-approximate solution of **(SDP)** if for any $x \in A$,

$$f(x) \geqq f(\bar{x}) - \epsilon.$$

Definition 2.2. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a convex function.

(1) The subdifferential of g at a is given by

$$\partial g(a) := \{v \in \mathbb{R}^n \mid g(x) \geqq g(a) + \langle v, x - a\rangle, \quad \forall x \in \mathbb{R}^n\},$$

where $\langle \cdot, \cdot \rangle$ is the scalar product on $\mathbb{R}^n$.

(2) The ϵ-subdifferential of g at a is given by

$$\partial g_\epsilon(a) := \{v \in \mathbb{R}^n \mid g(x) \geqq g(a) + \langle v, x - a\rangle - \epsilon, \quad \forall x \in \mathbb{R}^n\}.$$

From Theorem 3.1.1 in [10], we can obtain the following proposition.

Proposition 2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and δ_C is the indicator function with respect to a closed convex subset C of $\mathbb{R}^n$. Let $\epsilon \geqq 0$. Then there exist $\epsilon_0, \epsilon_1 \geqq 0$ such that $\epsilon_0 + \epsilon_1 = \epsilon$ and*

$$\partial_\epsilon (f + \delta_c)(\bar{x}) = \partial_{\epsilon_0} f(\bar{x}) + \partial_{\epsilon_1} f(\bar{x}).$$

From Theorem 1 (Feasibility Theorem) in [4], we can obtain the following Lemma.

Lemma 2.1. *Let $F_i \in S_n,\ i = 0, 1, \cdots, m$. Suppose that $A \neq \emptyset$. Let $u \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$. Then the following are equivalent:*

$$(i) \qquad \{x \in \mathbb{R}^m \mid F_0 + \sum_{i=1}^{m} F_i x_i \succeq 0\} \subset \{x \in \mathbb{R}^m \mid \langle u, x \rangle \geqq \alpha\}.$$

$$(ii) \qquad \begin{pmatrix} u \\ \alpha \end{pmatrix} \in cl \left(\bigcup_{(Z,\delta) \in S \times \mathbb{R}_+} \left\{ \begin{pmatrix} \hat{F}^*(Z) \\ -\mathrm{Tr}[ZF_0] - \delta \end{pmatrix} \right\} \right).$$

Lemma 2.2. *Let $\bar{x} \in A$ and $\epsilon \geqq 0$. Then $\bar{x}$ is an ϵ-approximate solution of* **(SDP)** *if and only if there exist $\epsilon_0, \epsilon_1 \geqq 0$, $v \in \partial_{\epsilon_0} f(\bar{x})$ such that $\epsilon_0 + \epsilon_1 = \epsilon$ and*

$$\begin{pmatrix} v \\ \langle v, \bar{x} \rangle - \epsilon_1 \end{pmatrix} \in cl \left(\bigcup_{(Z,\delta) \in S \times \mathbb{R}_+} \left\{ \begin{pmatrix} \hat{F}^*(Z) \\ -\mathrm{Tr}[ZF_0] - \delta \end{pmatrix} \right\} \right).$$

Proof. $\bar{x} \in A$ is an ϵ-approximate solution of **(SDP)**
$\iff$ $0 \in \partial_\epsilon (f + \delta_A)(\bar{x})$,where δ_A is the indicator function with respect to A.
$\iff$ (by Proposition 2.1) there exist $\epsilon_0, \epsilon_1 \geqq 0$, $v \in \partial_{\epsilon_0} f(\bar{x})$ such that $\epsilon_0 + \epsilon_1 = \epsilon$ and

$$\langle v, x \rangle \geqq \langle v, \bar{x} \rangle - \epsilon_1, \text{ for any } x \in A$$

Thus it follows from Lemma 2.1 that Lemma 2.2 holds. ■

Lemma 2.3. [15] *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a continuous convex function and $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then*

$$\mathrm{epi}(h + u)^* = \mathrm{epi}h^* + \mathrm{epi}u^*.$$

3. ϵ-Optimality Theorems

Now we give ϵ-optimality theorems for **(SDP)**.

Theorem 3.1. *Let* $\bar{x} \in A$ *and* $\bigcup_{(Z,\delta)\in S\times\mathbb{R}_+}\{\begin{pmatrix} \hat{F}^*(Z) \\ -\mathrm{Tr}[ZF_0]-\delta \end{pmatrix}\}$ *is closed in* $\mathbb{R}^m \times \mathbb{R}$*. Then* $\bar{x} \in A$ *is an* ϵ*-approximate solution of* **(SDP)** *if and only if there exist* $\epsilon_0, \epsilon_1 \geqq 0$, $v \in \partial_{\epsilon_0} f(\bar{x}),\ Z \in S$ *such that* $\epsilon_0 + \epsilon_1 = \epsilon$*,*

$$v = \hat{F}^*(Z)$$

and

$$0 \leqq \mathrm{Tr}[Z \cdot F(\bar{x})] \leqq \epsilon_1.$$

Proof. $\bar{x} \in A$ is an ϵ-approximate solution of **(SDP)**.

$\Longleftrightarrow$ (by Lemma 2.2 and assumption) there exist $\epsilon_0, \epsilon_1 \geqq 0,\ v \in \partial_{\epsilon_0} f(\bar{x})$ such that $\epsilon_0 + \epsilon_1 = \epsilon$ and

$$\begin{pmatrix} v \\ \langle v, \bar{x}\rangle - \epsilon_1 \end{pmatrix} \in \bigcup_{(Z,\delta)\in S\times\mathbb{R}_+} \begin{pmatrix} \hat{F}^*(Z) \\ -\mathrm{Tr}[ZF_0]-\delta \end{pmatrix}.$$

$\Longleftrightarrow$ there exist $\epsilon_0, \epsilon_1 \geqq 0,\ v \in \partial_{\epsilon_0} f(\bar{x}),\ Z \in S,\ \delta \geqq 0$ such that $\epsilon_0 + \epsilon_1 = \epsilon$,

$$v = \hat{F}^*(Z) \ \text{ and}$$

$$-\mathrm{Tr}[ZF_0] - \delta = \langle v, \bar{x}\rangle - \epsilon_1.$$

$\Longleftrightarrow$ there exist $\epsilon_0, \epsilon_1 \geqq 0,\ v \in \partial_{\epsilon_0} f(\bar{x}),\ Z \in S,\ \delta \geqq 0$ such that $\epsilon_0 + \epsilon_1 = \epsilon$,

$$v = \hat{F}^*(Z) \ \text{ and}$$

$$\epsilon_1 - \delta = \mathrm{Tr}[Z \cdot F(\bar{x})].$$

$\Longleftrightarrow$ there exist $\epsilon_0, \epsilon_1 \geqq 0$, $v \in \partial_{\epsilon_0} f(\bar{x}),\ Z \in S$ such that $\epsilon_0 + \epsilon_1 = \epsilon$

$$v = \hat{F}^*(Z) \ \text{ and}$$

$$0 \leqq \mathrm{Tr}[Z \cdot F(\bar{x})] \leqq \epsilon_1.$$

■

Example 3.1. Consider the following semidefinite program.

$$\textbf{(SDP)} \quad \text{minimize} \quad x_1 + x_2^2$$
$$\text{subject to} \quad \begin{pmatrix} 0 & x_1 \\ x_1 & 0 \end{pmatrix} \succeq 0.$$

Let $f(x_1, x_2) = x_1 + x_2^2$,

$$F_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ F_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } F_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then **(SDP)** becomes

$$\begin{aligned} &\text{minimize} \quad f(x_1, x_2) \\ &\text{subject to} \quad F_0 + \sum_{i=1}^{2} x_i F_i \succeq 0. \end{aligned}$$

Then $A := \{(0, x_2) \in \mathbb{R}^2 \mid x_2 \in \mathbb{R}\}$ is the set of all feasible solutions of **(SDP)**. and the set of all ϵ-approximate solution is $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0, \ x_2 \geqq 0\}$. Let $\epsilon_0 \in [0, \epsilon]$ and $\epsilon_1 \geqq 0$ be such that $\epsilon_0 + \epsilon_1 = \epsilon$. Then for any $\bar{x}_2 \in \mathbb{R}$, $\partial_{\epsilon_0} f(0, \bar{x}_2) = \{1\} \times [2\bar{x}_2 - 2\sqrt{\epsilon_0}, 2\bar{x}_2 + 2\sqrt{\epsilon_0}]$. Let

$$E := \bigcup_{Z \in S, \epsilon_0 \geqq 0, \epsilon_1 \geqq 0. \epsilon_0 + \epsilon_1 = \epsilon} \{(\hat{F}^*(Z) \in \partial_{\epsilon_0} f(\bar{x}_1, \bar{x}_2), 0 \leqq \mathrm{Tr}[ZF(\bar{x}_1, \bar{x}_2)] \leqq \epsilon_1\}.$$

Then by Theorem 3.1, E is the set of all ϵ-approximate solution of **(SDP)**. Now we caculate the set of E.

$$\begin{aligned} E &= \bigcup_{Z \in S, \epsilon_0 \in [0, \epsilon]} \{(0, \bar{x}_2) \in \mathbb{R}^2 \ |\hat{F}^*(Z) \in \partial_{\epsilon_0} f(0, \bar{x}_2)\}. \\ &= \bigcup_{\epsilon_0 \in [0, \epsilon]} \{(0, \bar{x}_2) \in \mathbb{R}^2 \ |\hat{F}^* \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \in \partial_{\epsilon_0} f(0, \bar{x}_2), a \geqq 0, c \geqq 0, b^2 \leqq ac\}. \\ &= \bigcup_{0 \leqq \epsilon_0 \leqq \epsilon} \{(0, \bar{x}_2) \in \mathbb{R}^2 \ |0 \in [2\bar{x}_2 - 2\sqrt{\epsilon_0}, 2\bar{x}_2 + 2\sqrt{\epsilon_0}]\}. \\ &= \{(0, \bar{x}_2) \in \mathbb{R}^2 \ |0 \in [2\bar{x}_2 - 2\sqrt{\epsilon}, 2\bar{x}_2 + 2\sqrt{\epsilon}]\}. \\ &= \{(0, \bar{x}_2) \in \mathbb{R}^2 \ | - \sqrt{\epsilon} \leqq \bar{x}_2 \leqq \sqrt{\epsilon}]\}. \end{aligned}$$

and

$$\begin{aligned} D : &= \bigcup_{(Z, \delta) \in S \times \mathbb{R}_+} \{(\hat{F}^*(Z), -\mathrm{Tr}[ZF_0] - \delta)\}. \\ &= \bigcup_{(Z, \delta) \in S \times \mathbb{R}_+} \{(\mathrm{Tr}[ZF_1], \mathrm{Tr}[ZF_2], -\mathrm{Tr}[ZF_0] - \delta)\} \\ &= \{(x, 0, z) \in \mathbb{R}^3 \ |x \in \mathbb{R}, z \leqq 0\} \end{aligned}$$

Thus D is closed in $\mathbb{R}^3$. Hence Theorem 3.1 holds for this example.

Theorem 3.2. *Let $\bar{x} \in A$. Then $\bar{x}$ is an ϵ-approximate solution of* **(SDP)** *if and only if there exist* $\epsilon_0,\ \epsilon_1 \geqq 0,\ v \in \partial_{\epsilon_0} f(\bar{x}), Z_n \in S,\ \delta_n \geqq 0$ *such that* $\epsilon_0 + \epsilon_1 = \epsilon$,

$$v = \lim_{n\to\infty} \hat{F}^*(Z_n)$$

and

$$\epsilon_1 = \lim_{n\to\infty} (\mathrm{Tr}[Z_n F(\bar{x})] + \delta_n).$$

Proof. $\bar{x}$ is an ϵ-approximate solution of **(SDP)**
$\Longleftrightarrow$ (by Lemma 2.2) there exist $\epsilon_0, \epsilon_1 \geqq 0,\ v \in \partial_{\epsilon_0} f(\bar{x}),\ Z_n \in S$ such that $\epsilon_0 + \epsilon_1 = \epsilon$ and

$$\begin{pmatrix} v \\ \langle v, \bar{x}\rangle - \epsilon_1 \end{pmatrix} \in cl \left(\bigcup_{(Z,\delta)\in S\times\mathbb{R}_+} \begin{pmatrix} \hat{F}^*(Z) \\ -\mathrm{Tr}[ZF_0] - \delta \end{pmatrix} \right)$$

$\Longleftrightarrow$ there exist $\epsilon_0, \epsilon_1 \geqq 0,\ v \in \partial_{\epsilon_0} f(\bar{x}),\ Z_n \in S,\ \delta_n \geqq 0$ such that $\epsilon_0 + \epsilon_1 = \epsilon$,

$$v = \lim_{n\to\infty} \hat{F}^*(Z_n) \text{ and}$$

$$\epsilon_1 = \lim_{n\to\infty} (\mathrm{Tr}[Z_n F_0] + \delta_n).$$

$\Longleftrightarrow$ there exist $\epsilon_0, \epsilon_1 \geqq 0,\ v \in \partial_{\epsilon_0} f(\bar{x}),\ Z_n \in S,\ \delta_n \geqq 0$ such that $\epsilon_0 + \epsilon_1 = \epsilon$,

$$v = \lim_{n\to\infty} \hat{F}^*(Z_n) \text{ and}$$

$$\epsilon_1 = \lim_{n\to\infty} (\mathrm{Tr}[Z_n F(\bar{x})] + \delta_n).$$

■

Example 3.2. Let $\epsilon \in [0, \frac{1}{2})$. Consider the following semidefinite program.

$$\textbf{(SDP)} \quad \text{minimize} \quad x_1 + \frac{1}{2}\, x_1^2$$
$$\text{subject to} \quad \begin{pmatrix} 0 & \frac{1}{2}\, x_1 \\ \frac{1}{2}\, x_1 & x_2 \end{pmatrix} \succeq 0.$$

Let $f(x_1, x_2) = x_1 + \frac{1}{2}\, x_1^2$,

$$F_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ F_1 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \text{ and } F_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then **(SDP)** becomes

$$\begin{aligned} &\text{minimize} \quad f(x_1,x_2)\\ &\text{subject to} \quad F_0+\sum_{i=1}^{2} x_iF_i \succeq 0, \end{aligned}$$

and the feasible set of **(SDP)** is $\{(x_1,x_2)\in\mathbb{R}^2 \mid x_1=0,\ x_2\geqq 0\}$ and the set of all ϵ-approximate solution is $\{(x_1,x_2)\in\mathbb{R}^2 \mid x_1=0,\ x_2\geqq 0\}$. Let $\epsilon_0\geqq 0$ and $\epsilon_1\geqq 0$ be such that $\epsilon_0+\epsilon_1=\epsilon$. Let $(\bar{x}_1,\bar{x}_2)=(0,0)$. Then $(\bar{x}_1,\bar{x}_2)$ is an ϵ-approximate solution of **(SDP)** and $\partial_{\epsilon_0}f(0,0)=[1-\sqrt{2\epsilon_0},1+\sqrt{2\epsilon_0}]\times\{0\}$.

Then since $\epsilon\in[0,\frac{1}{2})$, $1-\sqrt{2\epsilon_0}\leqq 1$ and hence $(1,0)\in\partial_{\epsilon_0}f(0,0)$. Let

$$Z_n=\begin{pmatrix} n & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{n}\end{pmatrix}.$$

Then $Z_n\succeq 0$ for all n, $\hat{F}^*(Z_n)=(\mathrm{Tr}[Z_nF_1],\ \mathrm{Tr}[Z_nF_2])=(1,\frac{1}{n})$, $\mathrm{Tr}[Z_nF_0]=0$ *and* $\lim\limits_{n\to\infty}\hat{F}^*(Z_n)=\lim\limits_{n\to\infty}(1,\frac{1}{n})=(1,0)$. Let $\delta_n=\epsilon_1$. Then we have, $(1,0)\in\partial_{\epsilon_0}f(\bar{x}_1,\bar{x}_2)$, $(1,0)=\lim\limits_{n\to\infty}\hat{F}^*(Z_n)$, and $\epsilon_1=\lim\limits_{n\to\infty}(\mathrm{Tr}[Z_nF_0]+\delta_n)$. Thus Theorem 3.2 holds for this example. But we can not apply Theorem 3.1 to this example. Indeed, let $D=\bigcup_{(Z,\delta)\in S\times\mathbb{R}_+}\{(\mathrm{Tr}[ZF_1],\mathrm{Tr}[ZF_2],-\mathrm{Tr}[ZF_0]-\delta)\}$ Then

$$\begin{aligned} D: &= \{(b,c,-\delta)\in\mathbb{R}^3\,|a\geqq 0,c\geqq 0,b^2\leqq ac,\delta\geqq 0\}\\ &= \{(b,c)\in\mathbb{R}^2\,|a\geqq 0,c\geqq 0,b^2\leqq ac\}\times(-\mathbb{R}_+)\\ &= \{(0,0,z)\in\mathbb{R}^3\,|z\leqq 0\}\cup\{(x,y,z)\in\mathbb{R}^3\,|x\in\mathbb{R},y>0,z\leqq 0\} \end{aligned}$$

This means that D is not closed, and hence we can not apply Theorem 3.1 to this example.

4. ϵ-Saddle Point Theorems and ϵ-Duality Theorem

Now we give ϵ-saddle point theorems and ϵ-duality theorems for **(SDP)**. Using Lemma 2.1, we can obtain the following lemmas which are useful in proving our ϵ-saddle point theorems for **(SDP)**.

Lemma 4.1. *Let $\bar{x}\in A$. Then $\bar{x}$ is an ϵ-approximate solution of* **(SDP)** *if and only if there exists a sequence $\{Z_n\}$ in S such that*

$$f(x)-\liminf_{n\to\infty}\mathrm{Tr}[Z_nF(x)]\geqq f(\bar{x})-\epsilon,\quad \text{for any } x\in\mathbb{R}^m.$$

Proof. $(\Rightarrow)$ Let $\bar{x} \in A$ be an ϵ-approximate solution of **(SDP)**. Then $f(x) \geqq f(\bar{x}) - \epsilon$, for any $x \in A$. Let $h(x) = f(x) - f(\bar{x}) + \epsilon$. Then $h(x) + \delta_A(x) \geqq 0$, for any $x \in \mathbb{R}^m$. Thus we have, from Lemma 2.3,

$$\begin{aligned} 0 \in \mathrm{epi}(h + \delta_A)^* &= \mathrm{epi}h^* + \mathrm{epi}{\delta_A}^* \\ &= \mathrm{epi}f^* + (0, f(\bar{x}) - \epsilon) + \mathrm{epi}{\delta_A}^* \end{aligned}$$

and hence $(0, \epsilon - f(\bar{x})) \in \mathrm{epi}f^* + \mathrm{epi}{\delta_A}^*$. So there exists $(u, r) \in \mathrm{epi}f^*$ such that $(-u, \epsilon - f(\bar{x}) - r) \in \mathrm{epi}{\delta_A}^*$ and hence there exists $(u, r) \in \mathrm{epi}f^*$ such that $\langle -u, x\rangle \leqq \epsilon - f(\bar{x}) - r$ for any $x \in A$. Since $f^*(u) \leqq r$, $\langle -u, x\rangle \leqq \epsilon - f(\bar{x}) - f^*(u)$ for any $x \in A$, and hence it follows from Lemma 2.1 that

$$\begin{pmatrix} u \\ -\epsilon + f(\bar{x}) + f^*(u) \end{pmatrix} \in cl \left(\bigcup_{(Z,\delta)\in S\times\mathbb{R}_+} \begin{pmatrix} \hat{F}^*(Z) \\ -\mathrm{Tr}[ZF_0] - \delta \end{pmatrix} \right).$$

So, there exists $(Z_n, \delta_n) \in S \times \mathbb{R}_+$ such that

$$\begin{aligned} u &= \lim_{n\to\infty} \hat{F}^*(Z_n), \\ -\epsilon + f(\bar{x}) + f^*(u) &= -\lim_{n\to\infty}\Big[\mathrm{Tr}[Z_nF_0] + \delta_n\Big]. \end{aligned}$$

This gives

$$\langle u, x\rangle - f(x) \leqq f^*(u) = -\lim_{n\to\infty}\Big[\mathrm{Tr}[Z_nF_0] + \delta_n\Big] - f(\bar{x}) + \epsilon, \text{ for any } x \in \mathbb{R}^m.$$

Thus we have, for any $x \in \mathbb{R}^n$,

$$\begin{aligned} f(\bar{x}) - \epsilon &\leqq \langle -u, x\rangle + f(x) - \lim_{n\to\infty}\Big[\mathrm{Tr}[Z_nF_0] + \delta_n\Big] \\ &= f(x) - \lim_{n\to\infty}\Big\langle \hat{F}^*(Z_n), x\Big\rangle - \lim_{n\to\infty}\Big[\mathrm{Tr}[Z_nF_0] + \delta_n\Big] \\ &= f(x) - \lim_{n\to\infty}\Big[\Big\langle \hat{F}^*(Z_n), x\Big\rangle + \mathrm{Tr}[Z_nF_0] + \delta_n\Big] \\ &= f(x) - \lim_{n\to\infty}\Big[\mathrm{Tr}[Z_nF(x)] + \delta_n\Big] \\ &\leqq f(x) - \liminf_{n\to\infty}\mathrm{Tr}[Z_nF(x)] - \liminf_{n\to\infty}\delta_n \\ &\leqq f(x) - \liminf_{n\to\infty}\mathrm{Tr}[Z_nF(x)]. \end{aligned}$$

$(\Leftarrow)$ Suppose that there exists a sequence $\{Z_n\}$ in S such that $f(x) - \liminf_{n\to\infty}\mathrm{Tr}[Z_nF(x)] \geqq f(\bar{x}) - \epsilon$, for any $x \in \mathbb{R}^m$. Then we have,

$$f(x) \geqq f(x) - \liminf_{n\to\infty}\mathrm{Tr}[Z_nF(x)] \geqq f(\bar{x}) - \epsilon,$$

for any $x \in A$ and hence $f(x) \geqq f(\bar{x}) - \epsilon$, for any $x \in A$. So $\bar{x}$ is an ϵ-approximate solution of **(SDP)**. ■

Lemma 4.2. *Let $\bar{x} \in A$. Suppose that* $\bigcup_{(Z,\delta)\in S\times\mathbb{R}_+} \left\{ \begin{pmatrix} \hat{F}^*(Z) \\ -\mathrm{Tr}[ZF_0] - \delta \end{pmatrix} \right\}$ *is closed. Then $\bar{x}$ is an ϵ-approximate solution of* **(SDP)** *if and only if there exists $Z \in S$ such that for any $x \in \mathbb{R}^m$,*

$$f(x) - \mathrm{Tr}[ZF(x)] \geqq f(\bar{x}) - \epsilon.$$

Proof. $(\Rightarrow)$ Let $\bar{x}$ be an ϵ-approximate solution of **(SDP)**. Then with the same arguments as in proof of Lemma 4.1, we can check there exists $u \in \mathbb{R}^m$ such that

$$\begin{pmatrix} u \\ -\epsilon + f(\bar{x}) + f^*(u) \end{pmatrix} \in \bigcup_{(Z,\delta)\in S\times\mathbb{R}_+} \left\{ \begin{pmatrix} \hat{F}^*(Z) \\ -\mathrm{Tr}[ZF_0] - \delta \end{pmatrix} \right\}.$$

Thus there exist $(Z, \delta) \in S \times \mathbb{R}_+$ such that

$$u = \hat{F}^*(Z),$$

$$-\epsilon + f(\bar{x}) + f^*(u) = -\mathrm{Tr}[ZF_0] - \delta.$$

This gives

$$\begin{aligned} \left\langle \hat{F}^*(Z), x \right\rangle - f(x) &= \langle u, x\rangle - f(x) \leqq f^*(u) \\ &= -\mathrm{Tr}[ZF_0] - \delta - f(\bar{x}) + \epsilon, \end{aligned}$$

for any $x \in \mathbb{R}^m$. Thus we have, for any $x \in \mathbb{R}^m$,

$$\begin{aligned} f(\bar{x}) - \epsilon &\leqq \langle -u, x\rangle + f(x) - \mathrm{Tr}[ZF_0] - \delta \\ &= f(x) - \left\langle \hat{F}^*(Z), x \right\rangle - \mathrm{Tr}[ZF_0] - \delta \\ &= f(x) - \mathrm{Tr}[ZF(x)] - \delta \\ &\leqq f(x) - \mathrm{Tr}[ZF(x)]. \end{aligned}$$

$(\Leftarrow)$ Suppose that there exists $Z \in S$ such that

$$f(x) - \mathrm{Tr}[ZF(x)] \geqq f(\bar{x}) - \epsilon,$$

for any $x \in \mathbb{R}^m$. Then we have,

$$f(x) \geqq f(x) - \mathrm{Tr}[ZF(x)] \geqq f(\bar{x}) - \epsilon,$$

for any $x \in A$. Thus $f(x) \geqq f(\bar{x}) - \epsilon$, for any $x \in A$. Hence $\bar{x}$ is an ϵ-approximate solution of **(SDP)**. ■

Let $\epsilon \geqq 0$. Consider the following sequential ϵ-saddle point problem and ϵ-saddle point problem:

(SSP)$_\epsilon$ Find $\bar{x} \in \mathbb{R}^m$ and a sequence $\{\bar{Z}_n\} \subset S$ such that

$$f(\bar{x}) - \liminf_{n\to\infty} \mathrm{Tr}[Z_n F(\bar{x})] - \epsilon \leqq f(\bar{x}) - \liminf_{n\to\infty} \mathrm{Tr}[\bar{Z}_n F(\bar{x})]$$
$$\leqq f(x) - \liminf_{n\to\infty} \mathrm{Tr}[\bar{Z}_n F(x)] + \epsilon$$

for any $x \in \mathbb{R}^m$ and any sequence $\{Z_n\} \subset S$.

(SP)$_\epsilon$ Find $\bar{x} \in \mathbb{R}^m$ and $\bar{Z} \in S$ such that

$$f(\bar{x}) - \mathrm{Tr}[ZF(\bar{x})] - \epsilon \leqq f(\bar{x}) - \mathrm{Tr}[\bar{Z}F(\bar{x})] \leqq f(x) - \mathrm{Tr}[\bar{Z}F(x)] + \epsilon$$

for any $x \in \mathbb{R}^m$ and any $Z \in S$.

Now we give a useful characterization of solution of **(SSP)$_\epsilon$**.

Lemma 4.3. *Suppose that $A \neq \emptyset$. Let $(\bar{x}, \bar{Z}_n) \in \mathbb{R}^m \times S$, $n = 1, 2, \cdots$. Then $(\bar{x}, \bar{Z}_n)$ is a solution of* **(SSP)$_\epsilon$** *if and only if*

$$f(\bar{x}) - \liminf_{n\to\infty} \mathrm{Tr}[\bar{Z}_n F(\bar{x})] \leqq f(x) - \liminf_{n\to\infty} \mathrm{Tr}[\bar{Z}_n F(x)] + \epsilon$$

for any $x \in \mathbb{R}^m$,

$$\liminf_{n\to\infty} \mathrm{Tr}[\bar{Z}_n F(\bar{x})] \leqq \epsilon$$

and $F(\bar{x}) \in S$.

Proof. $(\Rightarrow)$ Let $(\bar{x}, \bar{Z}_n)$ be a solution of **(SSP)$_\epsilon$**. Then

$$f(\bar{x}) - \liminf_{n\to\infty} \mathrm{Tr}[\bar{Z}_n F(\bar{x})] \leqq f(x) - \liminf_{n\to\infty} \mathrm{Tr}[\bar{Z}_n F(x)] + \epsilon,$$

for any $x \in \mathbb{R}^m$. Letting $Z_n = 0$ in the first inequality of **(SSP)$_\epsilon$** , we have $\liminf_{n\to\infty} \mathrm{Tr}[\bar{Z}_n F(\bar{x})] \leqq \epsilon$. Now we prove that $\liminf_{n\to\infty} \mathrm{Tr}[\bar{Z}_n F(\bar{x})] \geqq 0$. Assume to the contary that $\liminf_{n\to\infty} \mathrm{Tr}[\bar{Z}_n F(\bar{x})] < 0$.

Then from the first inequality of **(SSP)$_\epsilon$** , $-\liminf_{n\to\infty} \mathrm{Tr}[Z_n F(\bar{x})] - \epsilon \leqq -\liminf_{n\to\infty} \mathrm{Tr}[\bar{Z}_n F(\bar{x})]$, for any $Z_n \in S$.
Letting $Z_n = M\bar{Z}_n$ with $M > 0$. We have that $(1-M)\liminf_{n\to\infty} \mathrm{Tr}[\bar{Z}_n F(\bar{x})] \leqq \epsilon$. Setting $M \to \infty$, this is a contradiction.

($\Leftarrow$) Since $0 \leqq \liminf_{n\to\infty} \mathrm{Tr}[\bar{Z}_n F(\bar{x})] \leqq \epsilon$ and $F(\bar{x}) \in S$, we have, for any sequence $Z_n \subset S$,

$$\begin{aligned} f(\bar{x}) - \liminf_{n\to\infty} \mathrm{Tr}[\bar{Z}_n F(\bar{x})] &\geqq f(\bar{x}) - \epsilon \\ &\geqq f(\bar{x}) - \liminf_{n\to\infty} \mathrm{Tr}[Z_n F(\bar{x})] - \epsilon. \end{aligned}$$

Thus $(\bar{x}, \bar{Z}_n)$ is a solution of **(SSP)**$_\epsilon$. ∎

Using Lemmas 4.1 and 4.3, we give a sequential ϵ-saddle point theorem which holds between **(SDP)** and **(SSP)**$_\epsilon$.

Theorem 4.1. (Sequential ϵ-Saddle Point Theorem)

(1) *If $\bar{x} \in A$ is an ϵ-approximate solution of* **(SDP)***, then there exists a sequence $\bar{Z}_n$ such that $(\bar{x}, \bar{Z}_n)$ is a solution of* **(SSP)**$_\epsilon$

(2) *If $A \neq \emptyset$ and $(\bar{x}, \bar{Z}_n)$ is a solution of* **(SSP)**$_\epsilon$*, then $\bar{x}$ is an 2ϵ-approximate solution of* **(SDP)***.*

Proof. (1) Let $\bar{x} \in A$ be an ϵ-approximate solution of **(SDP)**. Then $f(x) \geqq f(\bar{x}) - \epsilon$ for any $x \in A$. It follows from Lemma 4.1 that there exists a sequence $\bar{Z}_n$ in S such that

$$f(x) - \liminf_{n\to\infty} \mathrm{Tr}[\bar{Z}_n F(x)] \geqq f(\bar{x}) - \epsilon \quad \text{for any } x \in \mathbb{R}^m. \tag{4.1}$$

Since $\liminf_{n\to\infty} \mathrm{Tr}[\bar{Z}_n F(\bar{x})] \geqq 0$, we have, for any $x \in \mathbb{R}^m$,

$$f(x) - \liminf_{n\to\infty} \mathrm{Tr}[\bar{Z}_n F(x)] + \epsilon \geqq f(\bar{x}) - \liminf_{n\to\infty} \mathrm{Tr}[\bar{Z}_n F(\bar{x})].$$

Letting $x = \bar{x}$ in (4.1), we have

$$\liminf_{n\to\infty} \mathrm{Tr}[\bar{Z}_n F(\bar{x})] \leqq \epsilon.$$

Hence it follows from Lemma 4.3 that $(\bar{x}, \bar{Z}_n)$ is a solution of **(SSP)**$_\epsilon$.

(2) Since $(\bar{x}, \bar{Z}_n)$ is a solution of $(\mathbf{SSP})_\epsilon$, it follows from Lemma 4.3 that for any $x \in A$,

$$\begin{aligned} f(x) + \epsilon &\geqq f(x) - \liminf_{n\to\infty} \mathrm{Tr}[\bar{Z}_n F(x)] + \epsilon \\ &\geqq f(\bar{x}) - \liminf_{n\to\infty} \mathrm{Tr}[\bar{Z}_n F(\bar{x})] \\ &\geqq f(\bar{x}) - \epsilon. \end{aligned}$$

Hence $f(x)+2\epsilon \geqq f(\bar{x})$ for any $x \in A$. By Lemma 4.3, $F(\bar{x}) \in S$, i.e., $\bar{x} \in A$. Consequently, $\bar{x}$ is an 2ϵ-approximate solution of **(SDP)**. ∎

Example 4.1. Consider the following semidefinite linear program.

$$\textbf{(SDLP)} \quad \text{minimize} \quad x_1$$
$$\text{subject to} \quad \begin{pmatrix} 0 & x_1 & 0 \\ x_1 & x_2 & 0 \\ 0 & 0 & x_1+1 \end{pmatrix} \succeq 0.$$

Let $f(x_1,x_2)=x_1$,

$$F_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ F_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $\epsilon \geqq 0$. Then the feasible set of **(SDLP)** is $\{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 = 0,\ x_2 \geqq 0\}$ and the set of all ϵ-approximate solution is $\{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 = 0,\ x_2 \geqq 0\}$. Let $(\bar{x}_1,\bar{x}_2)=(0,1)$. Then $(\bar{x}_1,\bar{x}_2)$ is an ϵ-approximate solution of **(SDLP)**. Let

$$\bar{Z}_n = \begin{pmatrix} n & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{n} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then for any sequence $\{Z_n\} \subset S$, $f(\bar{x}_1,\bar{x}_2) - \lim_{n\to\infty} \inf \mathrm{Tr}[Z_n F(\bar{x}_1,\bar{x}_2)] - \epsilon \leqq -\epsilon$, $f(\bar{x}_1,\bar{x}_2) - \lim_{n\to\infty} \inf \mathrm{Tr}[\bar{Z}_n F(\bar{x}_1,\bar{x}_2)] = -\lim_{n\to\infty} \inf \frac{1}{n} = 0$, and for any $(x_1,x_2) \in \mathbb{R}^2$,

$$\begin{aligned} & f(x_1,x_2) - \lim_{n\to\infty} \inf \mathrm{Tr}[\bar{Z}_n F(x_1,x_2)] + \epsilon \\ = \ & x_1 - \lim_{n\to\infty} \inf (x_1 + \frac{1}{n} x_2) + \epsilon \\ = \ & \epsilon. \end{aligned}$$

Thus $((\bar{x}_1,\bar{x}_2),\bar{Z}_n)$ is a solution of $\textbf{(SSP)}_\epsilon$. Hence (1) of Theorem 4.1 holds. ∎

Theorem 4.2. (ϵ- Saddle Point Theorem) *Suppose that* $\bigcup_{(Z,\delta)\in S\times\mathbb{R}_+} \left\{ \begin{pmatrix} \hat{F}^*(Z) \\ -\mathrm{Tr}[ZF_0]-\delta \end{pmatrix} \right\}$ *is closed. If* $\bar{x} \in A$ *is an* ϵ*-approximate solution of* **(SDP)**, *then there exists* $\bar{Z} \in S$ *such that* $(\bar{x},\bar{Z})$ *is a solution of* $\textbf{(SP)}_\epsilon$.

Proof. Let $\bar{x} \in A$ be an ϵ-approximate solution of **(SDP)**. Then $f(x) \geqq f(\bar{x}) - \epsilon$, for any $x \in A$. By Lemma 4.2, there exists $\bar{Z} \in S$ such that

$$f(x) - \mathrm{Tr}[\bar{Z}F(x)] \geqq f(\bar{x}) - \epsilon, \tag{4.2}$$

for any $x \in \mathbb{R}^m$. Since $F(\bar{x}) \in S$ and $\bar{Z} \in S$, $\mathrm{Tr}[\bar{Z}F(\bar{x})] \geqq 0$. Thus from (4.2),

$$f(x) - \mathrm{Tr}[\bar{Z}F(x)] + \epsilon \geqq f(\bar{x}) - \mathrm{Tr}[\bar{Z}F(\bar{x})]$$

for any $x \in \mathbb{R}^m$. Letting $x = \bar{x}$ in (4.2), $0 \leqq \mathrm{Tr}[\bar{Z}F(\bar{x})] \leqq \epsilon$. Hence we have, for any $x \in \mathbb{R}^m$ and any $Z \in S$,

$$\begin{aligned} f(\bar{x}) - \mathrm{Tr}[ZF(\bar{x})] - \epsilon &\leqq f(\bar{x}) - \mathrm{Tr}[\bar{Z}F(\bar{x})] \\ &\leqq f(x) - \mathrm{Tr}[\bar{Z}F(x)] + \epsilon. \end{aligned}$$

Consequently, $(\bar{x}, \bar{Z})$ is a solution of **(SP)**$_\epsilon$. ■

Example 4.2. Consider the following semidefinite program.

$$\textbf{(SDP)} \qquad \text{minimize} \qquad x_1 + x_2^2$$
$$\text{subject to} \quad \begin{pmatrix} 0 & x_1 \\ x_1 & 0 \end{pmatrix} \succeq 0.$$

Let $f(x_1, x_2) = x_1 + x_2^2$,

$$F_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ F_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } F_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then as shown in Example 3.1, $\bigcup_{(Z,\delta) \in S \times \mathbb{R}_+} \{ \begin{pmatrix} \hat{F}^*(Z) \\ -\mathrm{Tr}[ZF_0] - \delta \end{pmatrix} \}$ is closed.

Let $\epsilon \geqq 0$ and $(\bar{x}_1, \bar{x}_2) = (0, \sqrt{\epsilon})$. Then $(\bar{x}_1, \bar{x}_2)$ is an ϵ-approximate solution of **(SDP)**. Let $f(x_1, x_2) = x_1 + x_2^2$ and $F(x_1, x_2) = \begin{pmatrix} 0 & x_1 \\ x_1 & 0 \end{pmatrix}$. Then for any $Z \in S, f(\bar{x}_1, \bar{x}_2) - \mathrm{Tr}[\bar{Z}F(\bar{x}_1, \bar{x}_2)] - \epsilon = \epsilon - \epsilon = 0$. Let $\bar{Z} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$. Then $\bar{Z} \in S, f(\bar{x}_1, \bar{x}_2) - \mathrm{Tr}[\bar{Z}F(x_1, x_2)] + \epsilon = x_2^2 + \epsilon$. Thus $((\bar{x}_1, \bar{x}_2), \bar{Z})$ is a solution of **(SP)**$_\epsilon$. Hence Theorem 4.2 holds.

Theorem 4.3. *If $(\bar{x}, \bar{Z})$ is a solution of* **(SP)**$_\epsilon$, *then $\bar{x}$ is an 2ϵ-approximate solution of* **(SDP)**.

Proof. Since $(\bar{x}, \bar{Z})$ is a solution of **(SP)**$_\epsilon$, we can prove by the similar way in the proof of Lemma 4.3, that

$$f(x) - \mathrm{Tr}[\bar{Z}F(x)] + \epsilon \geqq f(\bar{x}) - \mathrm{Tr}[\bar{Z}F(\bar{x})] \text{for any } x \in \mathbb{R}^m,$$

$$\mathrm{Tr}[\bar{Z}F(\bar{x})] \leqq \epsilon \text{ and } \bar{x} \in A.$$

Thus we have, for any $x \in A$,

$$\begin{aligned} f(x) + \epsilon &\geqq f(x) - \mathrm{Tr}[\bar{Z}F(x)] + \epsilon \\ &\geqq f(\bar{x}) - \mathrm{Tr}[\bar{Z}F(\bar{x})] \\ &\geqq f(\bar{x}) - \epsilon. \end{aligned}$$

Hence $f(x) + 2\epsilon \geqq f(\bar{x})$ for any $x \in A$. Consequently, $\bar{x}$ is an 2ϵ-approximate solution of **(SDP)**. ■

Now we formulate the dual problem **(SDD)** of **(SDP)** as follows:

$$\begin{array}{lll} \textbf{(SDD)} & \text{Maximize} & f(x) - \mathrm{Tr}[ZF(x)] \\ & \text{subject to} & 0 \in \partial_{\epsilon_0} f(x) - \hat{F}^*(Z) \\ & & Z \succeq 0 \\ & & \epsilon_0 \in [0, \epsilon]. \end{array}$$

We prove ϵ-weak and ϵ-strong duality theorems which hold between **(SDP)** and **(SDD)**.

Theorem 4.4. (ϵ-Weak Duality) *For any feasible x of* **(SDP)** *and any feasible (y, Z) of* **(SDD)**,

$$f(x) \geqq f(y) - \mathrm{Tr}[ZF(y)] - \epsilon.$$

Proof. Let x and (y, Z) be feasible solutions of **(SDP)** and **(SDD)**, respectively. Then $\mathrm{Tr}[ZF(x)] \geqq 0$ and there exists $v \in \partial_{\epsilon_0} f(y)$ such that $v = \hat{F}^*(Z)$. Thus, we have

$$\begin{aligned} f(x) - \{f(y) - \mathrm{Tr}[ZF(y)]\} &\geqq \langle v, x - y\rangle - \epsilon_0 + \mathrm{Tr}[ZF(y)] \\ &= \langle \hat{F}^*(Z), x - y\rangle - \epsilon_0 + \mathrm{Tr}[ZF(y)] \\ &= \mathrm{Tr}[Z(\sum_{i=1}^{m} x_i F_i)] + \mathrm{Tr}[ZF_0] - \epsilon_0 \\ &= \mathrm{Tr}[ZF(x)] - \epsilon_0 \\ &\geqq -\epsilon_0 \\ &\geqq -\epsilon. \end{aligned}$$

Hence $f(x) \geqq f(y) - \mathrm{Tr}[ZF(y)] - \epsilon$. ■

Theorem 4.5. (ϵ-Strong Duality) *Suppose that* $\displaystyle\bigcup_{(Z,\delta)\in S\times\mathbb{R}_+}\left\{\begin{pmatrix}\hat{F}^*(Z)\\ -\mathrm{Tr}[ZF_0]-\delta\end{pmatrix}\right\}$ *is closed. If* $\bar{x}\in A$ *is an* ϵ*-approximate solution of* **(SDP)**, *then there exists* $\bar{Z}\in S$ *such that* $(\bar{x},\bar{Z})$ *is an* 2ϵ*-approximate solution of* **(SDD)**.

Proof. Since $\bar{x}$ is an ϵ-approximate solution of **(SDP)**. It follows from Theorem 4.2 that there exists $\bar{Z}\in S$ such that $(\bar{x},\bar{Z})$ is a solution of **(SP)**$_\epsilon$. Thus we have, for any $x\in\mathbb{R}^m$ and any $Z\in S$,

$$\begin{aligned} f(\bar{x}) - Tr[ZF(\bar{x})] - \epsilon &\leqq f(\bar{x}) - Tr[\bar{Z}F(\bar{x})] \\ &\leqq f(x) - Tr[\bar{Z}F(x)] + \epsilon. \end{aligned}$$

Letting $Z=0$ in the first inequality, we have $Tr[\bar{Z}F(\bar{x})]\leqq\epsilon$. The second inequality means that $\bar{x}$ is an ϵ-approximate solution of the following problem:

$$\begin{array}{ll} \text{Minimize} & f(x) - Tr[\bar{Z}F(x)] \\ \text{subjec to} & x\in\mathbb{R}^m \end{array}$$

and hence there exists $\epsilon_0\in[0,\epsilon]$ such that

$$0\in\partial_{\epsilon_0}f(\bar{x}) - \hat{F}^*(\bar{Z}).$$

So, $(\bar{x},\bar{Z})$ is feasible for **(SDD)**. For any feasible (y,Z) of **(SDD)**,

$$\begin{aligned} f(\bar{x}) - Tr[\bar{Z}F(\bar{x})] - \{f(y) - Tr[ZF(y)]\} &\geqq f(\bar{x}) - \{f(y) - Tr[ZF(y)]\} - \epsilon \\ &\geqq -\epsilon-\epsilon \quad \text{(by ϵ-weak duality)} \\ &= -2\epsilon. \end{aligned}$$

Thus $(\bar{x},\bar{Z})$ is an 2ϵ-approximate solution of **(SDD)**. ■

Example 4.3. Consider the following semidefinite program.

$$\begin{array}{lll} \textbf{(SDP)} & \text{minimize} & x_1 + x_2^2 \\ & \text{subject to} & \begin{pmatrix}0 & x_1\\ x_1 & 0\end{pmatrix}\succeq 0. \end{array}$$

Let $f(x_1,x_2) = x_1 + x_2^2$,

$$F_0 = \begin{pmatrix}0&0\\0&0\end{pmatrix},\ F_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix} \text{ and } F_2 = \begin{pmatrix}0&0\\0&0\end{pmatrix},$$

and $\epsilon \geqq 0$. Let $f(x_1, x_2) = x_1 + x_2^2$ and $F(x_1, x_2) = \begin{pmatrix} 0 & x_1 \\ x_1 & 0 \end{pmatrix}$. Then $A := \{(0, x_2) \in \mathbb{R}^2 \mid x_2 \in \mathbb{R}\}$ is the set of all feasible solutions of **(SDP)** and the set of all ϵ-approximate solution is $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0,\ x_2 \geqq 0\}$. Let $F := \{((x_1, x_2), Z) \,|\, \hat{F}^*(Z) \in \partial_{\epsilon_0} f(x_1, x_2), Z \geqq 0, \epsilon_0 \in [0, \epsilon]\}$.

Then F is the set of all feasible solution of **(SDD)**. Now we can calculate the set F.

$$\begin{aligned} F \;=\; & \left\{ \left((x_1, x_2), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid \hat{F}^* \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \{1\} \times [2x_2 - 2\sqrt{\epsilon_0}, 2x_2 + 2\sqrt{\epsilon_0}], \right. \\ & \qquad\qquad \left. a \geqq 0, c \geqq 0, b^2 \leqq ac, \epsilon_0 \in [0, \epsilon] \right\} \\ =& \left\{ \left((x_1, x_2), \begin{pmatrix} a & \frac{1}{2} \\ \frac{1}{2} & c \end{pmatrix} \right) \mid \hat{F}^* \begin{pmatrix} a & \frac{1}{2} \\ \frac{1}{2} & c \end{pmatrix} \in \{1\} \times [2x_2 - 2\sqrt{\epsilon_0}, 2x_2 + 2\sqrt{\epsilon_0}], \right. \\ & \qquad\qquad \left. a \geqq 0, c \geqq 0, b^2 \leqq ac, \epsilon_0 \in [0, \epsilon] \right\} \\ =& \left\{ \left((x_1, x_2), \begin{pmatrix} a & \frac{1}{2} \\ \frac{1}{2} & c \end{pmatrix} \right) \mid (1, 0) \in \{1\} \times [2x_2 - 2\sqrt{\epsilon_0}, 2x_2 + 2\sqrt{\epsilon_0}], \right. \\ & \qquad\qquad \left. a \geqq 0, c \geqq 0, b^2 \leqq ac, \epsilon_0 \in [0, \epsilon] \right\} \\ =& \left\{ \left((x_1, x_2), \begin{pmatrix} a & \frac{1}{2} \\ \frac{1}{2} & c \end{pmatrix} \right) \mid 0 \in [2x_2 - 2\sqrt{\epsilon_0}, 2x_2 + 2\sqrt{\epsilon_0}], \right. \\ & \qquad\qquad \left. a \geqq 0, c \geqq 0, b^2 \leqq ac, \epsilon_0 \in [0, \epsilon] \right\} \\ =& \left\{ \left((x_1, x_2), \begin{pmatrix} a & \frac{1}{2} \\ \frac{1}{2} & c \end{pmatrix} \right) \mid x_1 \in \mathbb{R}, -\sqrt{\epsilon} \leqq x_2 \leqq \sqrt{\epsilon}, \quad a \geqq 0, c \geqq 0, \frac{1}{4} \leqq ac \right\} \end{aligned}$$

For any $(x_1, x_2) \in Ax_1 + x_2^2$. and any $\left((x_1, x_2), \begin{pmatrix} a & \frac{1}{2} \\ \frac{1}{2} & c \end{pmatrix} \right) \in F$,

$$f(y_1, y_2) - \mathrm{Tr}[\begin{pmatrix} a & \frac{1}{2} \\ \frac{1}{2} & c \end{pmatrix} F(y_1, y_2)] - \epsilon = y_1 + y_2^2 - y_1 - \epsilon = y_2^2 - \epsilon \leqq 0 \leqq f(x_1, x_2).$$

Hence Theorem 4.4 (ϵ-weak daulity) for this example holds.

Let $(\bar{x}_1, \bar{x}_2) \in A$ is an ϵ-approximate solution of **(SDP)**. For any $(x_1, x_2) \in A$ and any $Z \in S, \mathrm{Tr}[ZF(x)] = 0$. So from arguments, in Example 3.1, there exists $\bar{Z} \in S$ and $\epsilon_0 \in [0, \epsilon]$ such that $\hat{F}^*(Z) \in \partial_{\epsilon_0} f(\bar{x}_1, \bar{x}_2)$, that is, there exists $\bar{Z} \in S$ such that $((\bar{x}_1, \bar{x}_2), \bar{Z}) \in F$ and hence by weak duality, $((\bar{x}_1, \bar{x}_2), \bar{Z})$ is an ϵ-approximate solution of **(SDD)**.

So Strong duality (Theorem 4.5) holds.

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