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NONLINEAR (A, η)-MONOTONE OPERATOR INCLUSION SYSTEMS INVOLVING NON-MONOTONE SET-VALUED MAPPINGS

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Abstract. In this paper, we introduce a new concept of (A, η)-monotone operators, which generalizes the (H, η)-monotonicity and A-monotonicity in Hilbert spaces and other existing monotone operators as special cases. We study some properties of (A, η)-monotone operators and define the resolvent operators associated with (A, η)-monotone operators. Further, by using the new resolvent operator technique, we construct some new iterative algorithms for solving a new class of nonlinear (A, η)-monotone operator inclusion systems involving non-monotone set-valued mappings in Hilbert spaces. We also prove the existence of solutions for the nonlinear operator inclusion systems and the convergence of iterative sequences generated by the algorithm. Our results improve and generalize the corresponding results of recent works.

1. INTRODUCTION

It is well known that some systems of variational inequalities, variational inclusions, complementarity problems and equilibrium problems have been studied by some authors in recent years because of their close relations to Nash equilibrium problems. Huang and Fang [1] introduced a system of order complementarity problems and established some existence results for these using fixed point theory. Verma [2] introduced and studied some systems of variational inequalities and developed some iterative algorithms for approximating the solutions of systems of variational inequalities. Cho et al. [3] introduced and studied a new system of nonlinear variational inequalities in Hilbert spaces. They proved some existence and uniqueness

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theorems of solutions for the system of nonlinear variational inequalities. They also constructed an iterative algorithm for approximating the solution of the system of nonlinear variational inequalities. For some more related works, we refer to [1, 4-7].

Furthermore, Fang Huang [8], Yan et al. [9] and Fang et al. [10] introduced and studied some new systems of variational inclusions involving H-monotone operators and (H, η)-monotone operators in Hilbert space. Using the resolvent operator associated with H-monotone operators or (H, η)- monotone operators, the authors prove the existence and uniqueness of solutions for the new systems of variational inclusions and constructed some new algorithms for approximating the solutions of the systems and discuss the convergence of the iterative sequences generated by the algorithms.

On the other hand, Verma [11] announced the notion of the A-monotone mappings and its applications to the solvability of systems of nonlinear variational inclusions. As Verma pointed out, "the class of the A-monotone mappings generalizes the H-monotonicity. On the top of that, A-monotonicity originates from hemivariational inequalities and emerges as a major contributor to the solvability of nonlinear variational problems on nonconvex settings". As a matter of fact, some nice examples on A-monotone (or generalized maximal monotone) mappings can be found in Naniewicz and Panagiotopoulos [12] and Verma [13]. Hemivariational inequalities, which initiated and developed by Panagiotopoulos [14], are connected with nonconvex energy functions and turned out to be useful tools proving the existence of solutions of nonconvex constrained problems. We note that the A-monotonicity is defined in terms of relaxed monotone mappings - a more general notion than the monotonicity or strong monotonicity - which gives a significant edge over the H-monotonicity.

Very recently, Lan et al. [15] introduced and studied a new system of nonlinear A-monotone multivalued variational inclusions in Hilbert spaces. By using the concept and properties of A-monotone mappings and the resolvent operator technique associated with A-monotone mappings due to Verma, the authors constructed a new iterative algorithm for solving the system of nonlinear multivalued variational inclusions associated with A-monotone mappings in Hilbert spaces and proved the existence of solutions for the nonlinear multivalued variational inclusions and the convergence of iterative sequences generated by the algorithms.

Inspired and motivated by the above works, the purpose of this paper is to introduce a new concept of (A, η)-monotone operators, which generalizes the (H, η)-monotonicity and A-monotonicity in Hilbert spaces and other some monotone operators as special cases. We study some properties of (A, η)-monotone operators and define resolvent operators associated with (A, η)-monotone operators. Then, by using the new resolvent operator technique, we construct some new iterative algorithms for solving a new class of systems of nonlinear (A, η)-monotone operator inclusions involving non-monotone set-valued mappings in Hilbert spaces. We also prove the existence of solutions for the nonlinear operator inclusion systems and the convergence of iterative sequences generated by the algorithms. Our results improve and generalize the corresponding results of [2-4, 8-10, 13, 15] and other recent works.

2. Preliminaries

Let $\mathcal{X}$ be a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$, $2^{\mathcal{X}}$ denote the family of all the nonempty subsets of $\mathcal{X}$ and $C(\mathcal{X})$ the collection of all closed subsets of $\mathcal{X}$. If $M : \mathcal{X} \to 2^{\mathcal{X}}$ is a set-valued mapping, then we denote the effective domain $Dom(M)$ of M as $Dom(M) = \{x \in \mathcal{X} : M(x) \neq \emptyset\}$.

In the sequel, let us recall some concepts and lemmas.

Definition 2.1. Let $T, A : \mathcal{X} \to \mathcal{X}$ be two single-valued operators. Then T is said to be

(i) monotone if
$$\langle T(x) - T(y), x - y\rangle \geq 0, \quad \forall x, y \in \mathcal{X};$$

(ii) strictly monotone if, T is monotone and
$$\langle T(x) - T(y), x - y\rangle = 0$$
if and only if $x = y$;

(iii) r-strongly monotone if there exists a constant $r > 0$ such that
$$\langle T(x) - T(y), x - y\rangle \geq r\|x - y\|^2, \quad \forall x, y \in \mathcal{X};$$

(iv) γ-strongly monotone with respect to A if there exists a constant $\gamma > 0$ such that
$$\langle T(x) - T(y), A(x) - A(y)\rangle \geq \gamma\|x - y\|^2, \quad \forall x, y \in \mathcal{X};$$

(v) σ-cocoercive with respect to A if there exists a constant $\sigma > 0$ such that
$$\langle T(x) - T(y), A(x) - A(y)\rangle \geq \sigma\|T(x) - T(y)\|^2, \quad \forall x, y \in \mathcal{X};$$

(vi) m-relaxed cocoercive with respect to A if there exists a constant $m > 0$ such that
$$\langle T(x) - T(y), A(x) - A(y)\rangle \geq -m\|T(x) - T(y)\|^2, \quad \forall x, y \in \mathcal{X};$$

(vii) (α, ϵ)-relaxed cocoercive with respect to A if there exist constants $\alpha, \epsilon > 0$ such that

$$\langle T(x)-T(y), A(x)-A(y)\rangle \geq -\alpha\|T(x)-T(y)\|^2+\epsilon\|x-y\|^2, \quad \forall x, y \in \mathcal{X};$$

(viii) s-Lipschitz continuous if there exists a constant $s > 0$ such that

$$\|T(x) - T(y)\| \leq s\|x - y\|, \quad \forall x, y \in \mathcal{X}.$$

Example 2.1. Let $T : \mathcal{X} \to \mathcal{X}$ be a nonexpansive mapping. Then $I - T$ is $\frac{1}{2}$-cocoercive and γ-relaxed cocoercive with respect to I for $\frac{1}{2} > -\gamma$, where $\gamma > 0$.

Proof. For any two elements $x, y \in \mathcal{X}$, we have

$$\begin{aligned}\|(I-T)(x)-(I-T)(y)\|^2 &= \langle (I-T)(x)-(I-T)(y), (I-T)(x)-(I-T)(y)\rangle \\ &\leq 2[\|x-y\|^2-\langle x-y, T(x)-T(y)\rangle] \\ &= 2\langle x-y, (I-T)(x)-(I-T)(y)\rangle,\end{aligned}$$

i.e.

$$\begin{aligned}\langle (I-T)(x) - (I-T)(y), x-y\rangle &\geq \frac{1}{2}\|(I-T)(x) - (I-T)(y)\|^2 \\ &\geq -\gamma\|(I-T)(x) - (I-T)(y)\|^2.\end{aligned}$$

Therefore, $I - T$ is $(\frac{1}{2})$-cocoercive and γ-relaxed cocoercive with respect to I for $\frac{1}{2} > -\gamma$.

Example 2.2. Let $T : \mathcal{X} \to \mathcal{X}$ be an r-strongly monotone (and hence r-expanding) mapping. Then T is $(1, r + r^2)$-relaxed cocoercive with respect to I.

Proof. For any two elements $x, y \in \mathcal{X}$, we have

$$\begin{aligned}&\|T(x) - T(y)\| \geq r\|x - y\|, \\ &\langle T(x) - T(y), x - y\rangle \geq r\|x - y\|^2,\end{aligned}$$

and so

$$\|T(x) - T(y)\|^2 + \langle T(x) - T(y), x - y\rangle \geq (r + r^2)\|x - y\|^2,$$

i.e., for all $x, y \in \mathcal{X}$, we get

$$\langle T(x) - T(y), x - y\rangle \geq (-1)\|T(x) - T(y)\|^2 + (r + r^2)\|x - y\|^2.$$

Therefore, T is $(r+r^2,1)$-relaxed cocoercive with respect to I.

Definition 2.2. A single-valued operator $\eta : \mathcal{X}\times\mathcal{X}\to\mathcal{X}$ is said to be τ-Lipschitz continuous if there exists a constant $\tau>0$ such that

$$\|\eta(x,y)\| \le \tau\|x-y\|, \quad \forall x,y\in\mathcal{X}.$$

Definition 2.3. Let $\eta : \mathcal{X}\times\mathcal{X}\to\mathcal{X}$ and $A,H : \mathcal{X}\to\mathcal{X}$ be single-valued operators. Then set-valued operator $M : \mathcal{X}\to 2^{\mathcal{X}}$ is said to be

(i) ζ-**H**-Lipschitz continuous, if there exists a constant $\zeta>0$ such that

$$\mathbf{H}(M(x),M(y)) \le \zeta\|x-y\|, \quad \forall x,y\in\mathcal{X},$$

where $\mathbf{H} : 2^{\mathcal{X}}\times 2^{\mathcal{X}}\to(-\infty,+\infty)\cup\{+\infty\}$ is the Hausdorff pseudo-metric, i.e.,

$$\mathbf{H}(D,B)=\max\{\sup_{x\in D}\inf_{y\in B}\|x-y\|,\sup_{x\in B}\inf_{y\in D}\|x-y\|\}, \quad \forall D,B\in 2^{\mathcal{X}}.$$

Note that if the domain of $\mathbf{H}$ is restricted to closed subsets $C(\mathcal{X})$, then $\mathbf{H}$ is the Hausdorff metric.

(ii) monotone if

$$\langle u-v,x-y\rangle \ge 0, \quad \forall x,y\in\mathcal{X}, u\in M(x), v\in M(y);$$

(iii) η-monotone if

$$\langle u-v,\eta(x,y)\rangle \ge 0, \quad \forall x,y\in\mathcal{X}, u\in M(x), v\in M(y);$$

(iv) strictly η-monotone if M is η-monotone and equality holds if and only if $x=y$;

(v) r-strongly η-monotone if there exists a constant $r>0$ such that

$$\langle u-v,\eta(x,y)\rangle \ge r\|x-y\|^2, \quad \forall x,y\in\mathcal{X}, u\in M(x), v\in M(y);$$

(vi) α-relaxed η-monotone if there exists a constant $\alpha>0$ such that

$$\langle u-v,\eta(x,y)\rangle \ge -\alpha\|x-y\|^2, \quad \forall x,y\in\mathcal{X}, u\in M(x), v\in M(y);$$

(vii) maximal monotone if and only if M is monotone and for every $x\in Dom(M)$ and $u\in E$ such that

$$\langle u-v,x-y\rangle \ge 0, \quad \forall y\in Dom(M), v\in M(y)$$

implies $u\in M(x)$;

(viii) maximal η-monotone if M is η-monotone and $(I+\rho M)(\mathcal{X}) = \mathcal{X}$ for all $\rho > 0$;

(ix) H-monotone if M is monotone and $(H+\rho M)(\mathcal{X}) = \mathcal{X}$ for all $\rho > 0$;

(x) A-monotone with constant m if M is m-relaxed monotone and $A+\lambda M$ is maximal monotone for all $\lambda > 0$.

(xi) (H,η)-monotone if M is η-monotone and $(H+\rho M)(\mathcal{X}) = \mathcal{X}$ for every $\rho > 0$.

Remark 2.1. (1) If $\eta(x,y) = x-y$ for all $x, y \in \mathcal{X}$, then the (H,η)-monotone operator reduces to an H-monotone operator, which was first introduced by Fang and Huang [16].

(2) Obviously, if $m = 0$, that is, M is 0-relaxed monotone, then the A-monotone operator reduces to an H-monotone operator. Therefore, the class of A-monotone mappings provides a unifying framework for classes of maximal monotone operators and H-monotone operators. For details about these operators, we refer the reader to [8, 9, 11, 15, 17, 18] and the references therein.

Example 2.3. [17, Theorem 5.1] Let $\mathcal{X}$ be a reflexive Banach space with $\mathcal{X}^*$ its dual, and $A : \mathcal{X} \to \mathcal{X}^*$ be $m - strongly$ monotone and $B : \mathcal{X} \to \mathcal{X}^*$ be c-Lipschitz continuous. Let $f : \mathcal{X} \to \mathbb{R}$ be locally Lipschitz such that ∂f is α-relaxed monotone. Then ∂f is $(A-B)$-monotone.

3. (A,η)-Monotone Operators and Resolvent Operators

Definition 3.1. Let $A : \mathcal{X} \to \mathcal{X}, \eta : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be two single-valued operators. Then a set-valued operator $M : \mathcal{X} \to 2^{\mathcal{X}}$ is called (A,η)-monotone with m if M is m-relaxed η-monotone and $(A+\rho M)(\mathcal{X}) = \mathcal{X}$ for every $\rho > 0$.

Remark 3.1.

(1) If $m = 0$, then Definition 3.1 reduces to the definition of (H,η)-monotone operators [10].

(2) When $m = 0$ and $\eta(x,y) = x-y$ for all $x, y \in \mathcal{X}$, Definition 3.1 reduces to the definition of H-monotone operators [16], which includes many known monotone operators as special cases (see, for example, [8, 9, 16, 18, 19] and the references therein).

(3) When $\eta(x,y) = x-y$ for all $x, y \in \mathcal{X}$, Definition 3.1 reduces to the definition of A-monotone operators [11, 15, 17].

Theorem 3.1. *Let $A:\mathcal{X}\to\mathcal{X}$ be an r-strongly η-monotone operator, $M:\mathcal{X}\to 2^{\mathcal{X}}$ be an (A,η)-monotone operator with m, and $x,u\in\mathcal{X}$ be given points. If $\langle u-v,\eta(x,y)\rangle\geq 0$ holds for all $(y,v)\in Graph\,(M)$, where $Graph(M)=\{(a,b)\in\mathcal{X}\times\mathcal{X}:\ b\in M(a)\}$, then $(x,u)\in Graph\,(M)$.*

Proof. Since M is (A,η)-monotone with m, $(A+\rho M)(\mathcal{X})=\mathcal{X}$ holds for every $\rho>0$. Then there exists $(x_0,u_0)\in Graph(M)$ such that

$$A(x_0)+\rho u_0=A(x)+\rho u. \tag{3.1}$$

Since M is m-relaxed η-monotone and A is r-strongly η-monotone, we have

$$-m\|x-x_0\|^2\leq\rho\langle u-u_0,\eta(x,x_0)\rangle=-\langle A(x)-A(x_0),\eta(x,x_0)\rangle\leq -r\|x-x_0\|^2.$$

This implies that $x=x_0$. From (3.1), we know that $u=u_0$. Thus $(x,u)\in Graph(M)$. ∎

Remark 3.2. Theorem 3.1 generalizes and improves Proposition 2.1 of [16].

Theorem 3.2. *Let $A:\mathcal{X}\to\mathcal{X}$ be an r-strongly η-monotone operator, $M:\mathcal{X}\to 2^{\mathcal{X}}$ be an (A,η)-monotone operator with m. Then the operator $(A+\rho M)^{-1}$ is single-valued.*

Proof. For any given $z\in\mathcal{X}$, and $x,y\in(A+\rho M)^{-1}(z)$, it follows that $-A(x)+z\in\rho M(x)$ and $-A(y)+z\in\rho M(y)$. Since M is m-relaxed η-monotone and A is r-strongly η-monotone,

$$\begin{aligned}-m\|x-y\|^2&\leq\langle(-A(x)+z)-(-A(y)+z),\eta(x,y)\rangle\\&=-\langle A(x)-A(y),\eta(x,y)\rangle\leq -r\|x-y\|^2.\end{aligned}$$

This implies that $x=y$. Thus $(A+\rho M)^{-1}$ is single-valued. ∎

Remark 3.3. Theorem 3.2 generalizes and improves Theorem 2.1 of [16] and Lemma 2.1 of [10], respectively.

Based on Theorem 3.2, we can define the resolvent operator $J_{\eta,M}^{\rho,A}$ associated with an (A,η)-monotone operator M as follows:

Definition 3.2. Let $A:\mathcal{X}\to\mathcal{X}$ be a strictly η-monotone operator and $M:\mathcal{X}\to 2^{\mathcal{X}}$ be an (A,η)-monotone operator with constant m. The resolvent operator $J_{\eta,M}^{\rho,A}:\mathcal{X}\to\mathcal{X}$ is defined by

$$J_{\eta,M}^{\rho,A}(x)=(A+\rho M)^{-1}(x),\quad\forall x\in\mathcal{X}. \tag{3.2}$$

Remark 3.4. The resolvent operators associated with (A, η)-monotone operators include as special cases the corresponding resolvent operators associated with (H, η)-monotone operators [10], H-monotone operators [16] and A-monotone operators [11].

Theorem 3.3. *Let $\eta : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be τ-Lipschitz continuous, $A : \mathcal{X} \to \mathcal{X}$ be a r-strongly η-monotone operator and $M : \mathcal{X} \to 2^{\mathcal{X}}$ be an (A, η)-monotone operator with m. Then the resolvent operator $J_{\eta,M}^{\rho,A} : \mathcal{X} \to \mathcal{X}$ is $\frac{\tau}{r-\rho m}$-Lipschitz continuous, i.e.,*

$$\|J_{\eta,M}^{\rho,A}(x) - J_{\eta,M}^{\rho,A}(y)\| \leq \frac{\tau}{r - \rho m}\|x - y\|, \quad \forall x, y \in \mathcal{X},$$

where $\rho \in (0, \frac{r}{m})$ is a constant.

Proof. For given $x, y \in \mathcal{X}$, from (3.2), we have

$$J_{\eta,M}^{\rho,A}(x) = (A + \rho M)^{-1}(x)$$

and

$$J_{\eta,M}^{\rho,A}(y) = (A + \rho M)^{-1}(y).$$

It follows that

$$\frac{1}{\rho}(x - A(J_{\eta,M}^{\rho,A}(x))) \in M(J_{\eta,M}^{\rho,A}(x))$$

and

$$\frac{1}{\rho}(y - A(J_{\eta,M}^{\rho,A}(y))) \in M(J_{\eta,M}^{\rho,A}(y)).$$

Since M is m-relaxed η-monotone, we get

$$\begin{aligned}
&-m\|J_{\eta,M}^{\rho,A}(x) - J_{\eta,M}^{\rho,A}(y)\|^2 \\
\leq\ &\frac{1}{\rho}\langle x - A(J_{\eta,M}^{\rho,A}(x)) - (y - A(J_{\eta,M}^{\rho,A}(y))), \eta(J_{\eta,M}^{\rho,A}(y), J_{\eta,M}^{\rho,A}(y))\rangle \\
=\ &\frac{1}{\rho}\langle x - y - (A(J_{\eta,M}^{\rho,A}(x)) - A(J_{\eta,M}^{\rho,A}(y))), \eta(J_{\eta,M}^{\rho,A}(x), J_{\eta,M}^{\rho,A}(y))\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\tau\|x - y\| \cdot \|J_{\eta,M}^{\rho,A}(x) - J_{\eta,M}^{\rho,A}(y)\| \\
\geq\ &\|x - y\| \cdot \|\eta(J_{\eta,M}^{\rho,A}(x), J_{\eta,M}^{\rho,A}(y))\|
\end{aligned}$$

$$\geq \langle x - y, \eta(J^{\rho,A}_{\eta,M}(x), J^{\rho,A}_{\eta,M}(y))\rangle$$
$$\geq \langle A(J^{\rho,A}_{\eta,M}(x)) - A(J^{\rho,A}_{\eta,M}(y)), \eta(J^{\rho,A}_{\eta,M}(x), J^{\rho,A}_{\eta,M}(y))\rangle$$
$$-\rho m\|J^{\rho,A}_{\eta,M}(x) - J^{\rho,A}_{\eta,M}(y)\|^2$$
$$\geq (r - \rho m)\|J^{\rho,A}_{\eta,M}(x) - J^{\rho,A}_{\eta,M}(y)\|^2.$$

Therefore,

$$\|J^{\rho,A}_{\eta,M}(x)-J^{\rho,A}_{\eta,M}(y)\| \leq \frac{\tau}{r - \rho m}\|x-y\|, \quad \forall x, y \in \mathcal{X}. \qquad \blacksquare$$

Remark 3.5. Theorem 3.3 extends Lemma 2 of [17], Theorem 2.2 of [16] and Lemma 2.2 of [10].

4. Variational Inclusion Systems and Iterative Algorithms

In this section, we shall introduce a new system of nonlinear (A, η)-monotone operator inclusions involving non-monotone set-valued mappings and construct a new iterative algorithm for solving this kind of system of nonlinear operator inclusions in Hilbert spaces.

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two real Hilbert spaces, $S : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{X}_1$, $T : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{X}_2$, $p : \mathcal{X}_1 \to \mathcal{X}_1$, $q : \mathcal{X}_2 \to \mathcal{X}_2$, $\eta_1 : \mathcal{X}_1 \times \mathcal{X}_1 \to \mathcal{X}_1$ and $\eta_2 : \mathcal{X}_2 \times \mathcal{X}_2 \to \mathcal{X}_2$ be single-valued operators, $E : \mathcal{X}_1 \to 2^{\mathcal{X}_1}$, $F : \mathcal{X}_2 \to 2^{\mathcal{X}_2}$ be any two non-monotone set-valued mappings. Let $A_1 : \mathcal{X}_1 \to \mathcal{X}_1$, $A_2 : \mathcal{X}_2 \to \mathcal{X}_2$, $M : \mathcal{X}_1 \times \mathcal{X}_1 \to 2^{\mathcal{X}_1}$ and $N : \mathcal{X}_2 \times \mathcal{X}_2 \to 2^{\mathcal{X}_2}$ be any nonlinear mappings such that $M(\cdot, a) : \mathcal{X}_1 \to 2^{\mathcal{X}_1}$ is an (A_1, η_1)-monotone operator for all $a \in \mathcal{X}_1$ and $N(\cdot, b) : \mathcal{X}_2 \to 2^{\mathcal{X}_2}$ is an (A_2, η_2)-monotone operator for all $b \in \mathcal{X}_2$, $f : \mathcal{X}_1 \to \mathcal{X}_1$, $g : \mathcal{X}_2 \to \mathcal{X}_2$ be nonlinear mappings with $(f(\mathcal{X}_1), \mathcal{X}_1) \cap Dom(M) \neq \emptyset$ and $(g(\mathcal{X}_2), \mathcal{X}_2) \cap Dom(N) \neq \emptyset$, respectively. Then the problem of finding $(a, b) \in \mathcal{X}_1 \times \mathcal{X}_2$, $u \in E(a), v \in F(b)$ such that

$$\text{(4.1)} \qquad \begin{cases} 0 \in S(p(a), v) + M(f(a), a), \\ 0 \in T(u, q(b)) + N(g(b), b) \end{cases}$$

is called a nonlinear (A, η)-monotone operator inclusion system involving non-monotone set-valued mappings.

Some special cases of the problem (4.1) are as follows:

(1) If $M(a,b) = M(a)$ for all $(a,b) \in \mathcal{X}_1 \times \mathcal{X}_1$ and $N(x,y) = N(x)$ for all $(x,y) \in \mathcal{X}_2 \times \mathcal{X}_2$, then the problem (4.1) reduces to the following generalized system of set-valued variational inclusion problem:

$$\begin{cases} 0 \in S(p(a), v) + M(f(a)), \\ 0 \in T(u, q(b)) + N(g(b)), \end{cases} \tag{4.2}$$

which is studied by Lan et al. [15] when M, N are A-monotone mappings.

(2) If $p = q = f = g \equiv I$, the identity mapping, then the problem (4.2) is equivalent to the following system of set-valued variational inclusion problem:

Find $(a,b) \in \mathcal{X}_1 \times \mathcal{X}_2$, $u \in E(a), v \in F(b)$ such that

$$\begin{cases} 0 \in S(a, v) + M(a), \\ 0 \in T(u, b) + N(b), \end{cases} \tag{4.3}$$

which is considered by Huang and Fang [1].

(3) If $E : \mathcal{X}_1 \to \mathcal{X}_1$ and $F : \mathcal{X}_2 \to \mathcal{X}_2$ are two single-valued mappings, then the problem (4.3) collapses to the following system of nonlinear variational inclusion problem: find $(a,b) \in \mathcal{X}_1 \times \mathcal{X}_2$ such that

$$\begin{cases} 0 \in S(a, F(b)) + M(a), \\ 0 \in T(E(a), b) + N(b), \end{cases} \tag{4.4}$$

which is studied by Verma [11] and Fang et al. [10] with $E = F = I$.

(4) If $M(x) = \partial\phi(x)$ and $N(y) = \partial\varphi(y)$ for all $x \in \mathcal{X}_1$ and $y \in \mathcal{X}_2$, where $\phi : \mathcal{X}_1 \to \mathbb{R} \cup \{+\infty\}$ and $\varphi : \mathcal{X}_2 \to \mathbb{R} \cup \{+\infty\}$ are two proper, convex and lower semi-continuous functionals, and $\partial\phi$ and $\partial\varphi$ denote the subdifferential operators of ϕ and φ, respectively, then the problem (4.4) reduces to the following problem: find $(x,y) \in \mathcal{X}_1 \times \mathcal{X}_2$ such that

$$\begin{cases} \langle S(a, F(b)), x - a \rangle + \phi(x) - \phi(a) \geq 0, \quad \forall x \in \mathcal{X}_1, \\ \langle T(E(x), y), y - b \rangle + \phi(y) - \phi(b) \geq 0, \quad \forall y \in \mathcal{X}_2, \end{cases} \tag{4.5}$$

which is called a system of nonlinear mixed variational inequalities. Some special cases of the problem (4.5) can be found in [7].

(5) Further, if $E = F \equiv I$, then the problem (4.5) reduces to the system of nonlinear variational inequalities problem considered by Cho et al. [3].

(6) If $M(x) = \partial\delta_{K_1}(x)$ and $N(y) = \partial\delta_{K_2}(y)$ for all $x \in K_1$ and $y \in K_2$, where K_1 and K_2, respectively, are nonempty closed convex subsets of $\mathcal{X}_1$ and $\mathcal{X}_2$,

and $\partial\delta_{K_1}$ and $\partial\delta_{K_2}$ denote indicator functions of K_1 and K_2, respectively, then the problem (4.5) becomes to determining an element $(a, b) \in K_1 \times K_2$ such that

$$\begin{cases} \langle S(a, F(b)), x - a\rangle \geq 0, \quad \forall x \in K_1, \\ \langle T(E(a), b), y - b\rangle \geq 0, \quad \forall y \in K_2, \end{cases} \tag{4.6}$$

which is just the problem in [5] when E and F are single-valued and $E = F \equiv I$.

(7) If $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{X}$, $K_1 = K_2 = K$, $S(x, F(y)) = \rho F(y) + x - y$ and $T(E(x), y) = \lambda E(x) + y - x$ for all $x, y \in \mathcal{X}$, where $\rho > 0$ and $\lambda > 0$ are two constants, then the problem (4.6) is equivalent to finding an element $(x, y) \in K \times K$ such that

$$\begin{cases} \langle \rho F(b) + a - b, x - a\rangle \geq 0, \quad \forall x \in K, \\ \langle \lambda E(a) + b - a, y - b\rangle \geq 0, \quad \forall y \in K, \end{cases} \tag{4.7}$$

which is the system of nonlinear variational inequalities considered by Verma [7] with $E = F$.

(8) If $x = y$, $E = F$ and $\rho = \lambda$, then the problem (4.7) reduces to the following classical nonlinear variational inequality problem: find an element $a \in K$ such that

$$\langle F(a), z - a\rangle \geq 0, \quad \forall z \in K.$$

Lemma 4.1. *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two real Hilbert spaces. Suppose that $A_1 : \mathcal{X}_1 \to \mathcal{X}_1$ and $A_2 : \mathcal{X}_2 \to \mathcal{X}_2$ are strictly η-monotone, $M(\cdot, t) : \mathcal{X}_1 \to 2^{\mathcal{X}_1}$ is (A_1, η_1)-monotone for all $t \in \mathcal{X}_1$ and $N(\cdot, \omega) : \mathcal{X}_2 \to 2^{\mathcal{X}_2}$ is (A_2, η_2)-monotone for all $\omega \in \mathcal{X}_2$. Let $S : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{X}_1$ and $T : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{X}_2$ be any two single-valued mappings, $p, f : \mathcal{X}_1 \to \mathcal{X}_1$ and $q, g : \mathcal{X}_2 \to \mathcal{X}_2$ be any nonlinear mappings with $(f(\mathcal{X}_1), \mathcal{X}_1) \cap Dom(M) \neq \emptyset$ and $(g(\mathcal{X}_2), \mathcal{X}_2) \cap Dom(N) \neq \emptyset$, respectively, and $E : \mathcal{X}_1 \to 2^{\mathcal{X}_1}$, $F : \mathcal{X}_2 \to 2^{\mathcal{X}_2}$ be any two set-valued mappings. Then a given element $(a, b, u, v) \in \mathcal{X}_1 \times \mathcal{X}_2$ is a solution to the problem (4.1) if and only if (a, b, u, v) satisfies*

$$\begin{cases} f(a) = J^{\rho, A_1}_{\eta_1, M(\cdot, a)}(A_1(f(a)) - \rho S(p(a), v)), \\ g(b) = J^{\lambda, A_2}_{\eta_2, N(\cdot, b)}(A_2(g(b)) - \lambda T(u, q(b))), \end{cases} \tag{4.8}$$

where $\rho > 0$ and $\lambda > 0$ are two constants.

Remark 4.1. The equality (4.8) can be written as

$$\begin{cases} a = a - f(a) + J^{\rho, A_1}_{\eta_1, M(\cdot, a)}(A_1(f(a)) - \rho S(p(a), v)), \\ b = b - g(b) + J^{\lambda, A_2}_{\eta_2, N(\cdot, b)}(A_2(g(b)) - \lambda T(u, q(b))), \end{cases}$$

where $\rho, \lambda > 0$ are constants. This fixed point formulation enables us to suggest the following iterative algorithm.

Algorithm 4.1. Assume that $\mathcal{X}_1$, $\mathcal{X}_2$, A_1, A_2, η_1, η_2, M, N, S, T, p, f, q, g, E and F are the same as in the problem (4.1). For any given $(a_0, b_0) \in \mathcal{X}_1 \times \mathcal{X}_2$, we choose $u_0 \in E(a_0), v_0 \in F(b_0)$ and let

$$\begin{cases} a_1 = a_0 - f(a_0) + J^{\rho,A_1}_{\eta_1,M(\cdot,a_0)}(A_1(f(a_0)) - \rho S(p(a_0), v_0)) + d_0, \\ b_1 = b_0 - g(b_0) + J^{\lambda,A_2}_{\eta_2,N(\cdot,b_0)}(A_2(g(b_0)) - \lambda T(u_0, q(b_0))) + e_0. \end{cases}$$

Since $u_0 \in E(a_0)$ and $v_0 \in F(b_0)$, for any $a_1 \in \mathcal{X}_1$, $b_1 \in \mathcal{X}_2$, by Nadler [20], there exist $u_1 \in E(a_1), v_1 \in F(b_1)$ such that

$$\begin{cases} \|u_0 - u_1\| \le (1+1)\mathbf{H}_1(E(a_0), E(a_1)), \\ \|v_0 - v_1\| \le (1+1)\mathbf{H}_2(F(b_0), F(b_1)). \end{cases}$$

Let

$$\begin{cases} a_2 = a_1 - f(a_1) + J^{\rho,A_1}_{\eta_1,M(\cdot,a_1)}(A_1(f(a_1)) - \rho S(p(a_1), v_1)) + d_1, \\ b_2 = y_1 - g(b_1) + J^{\lambda,A_2}_{\eta_2,N(\cdot,b_1)}(A_2(g(b_1)) - \lambda T(u_1, q(b_1))) + e_1. \end{cases}$$

Continuing this way, we can obtain sequences $\{a_n\}, \{b_n\}$ satisfying

$$\tag{4.9} \begin{cases} a_{n+1} = a_n - f(a_n) + J^{\rho,A_1}_{\eta_1,M(\cdot,a_n)}(A_1(f(a_n)) - \rho S(p(a_n), v_n)) + d_n, \\ b_{n+1} = b_n - g(b_n) + J^{\lambda,A_2}_{\eta_2,N(\cdot,b_n)}(A_2(g(b_n)) - \lambda T(u_n, q(b_n))) + e_n, \end{cases}$$

and choose $u_{n+1} \in E(a_{n+1})$ and $v_{n+1} \in F(b_{n+1})$ such that

$$\tag{4.10} \begin{cases} \|u_n - u_{n+1}\| \le (1+(n+1)^{-1})\mathbf{H}_1(E(a_n), E(a_{n+1})), \\ \|v_n - v_{n+1}\| \le (1+(n+1)^{-1})\mathbf{H}_2(F(b_n), F(b_{n+1})), \quad \forall n \ge 0, \end{cases}$$

where $\rho, \lambda > 0$ are constants, $d_n \in \mathcal{X}_1, e_n \in \mathcal{X}_2\,(n \ge 0)$ are errors to take into account a possible inexact computation of the resolvent operator point and $\mathbf{H}_i(\cdot,\cdot)$ is the Hausdorff pseudo-metric on $2^{\mathcal{X}_i}$ for $i = 1, 2$.

Remark 4.2. If we choose suitable d_n, e_n, A_1, A_2, η_1, η_2, M, N, S, T, p, f, q, g, E, F and $\mathcal{X}_1$, $\mathcal{X}_2$, then Algorithm 4.1 can be degenerated to a number of algorithms involving many known algorithms which due to classes of variational inequalities, complementarity problems, and variational inclusions (see, for example, [1-4, 7-9, 11, 13-16]).

5. Existence and Convergence Theorems

In this section, we will prove the existence of solutions for problem (4.1) and the convergence of iterative sequences generated by Algorithm 4.1.

Theorem 5.1. *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two real Hilbert spaces. Suppose that $A_1 : \mathcal{X}_1 \to \mathcal{X}_1$ is r_1-strongly η-monotone and α_1-Lipschitz continuous, and $A_2 : \mathcal{X}_2 \to \mathcal{X}_2$ is r_2-strongly η-monotone and α_2-Lipschitz continuous, $M(\cdot,t) : \mathcal{X}_1 \to 2^{\mathcal{X}_1}$ is (A_1,η_1)-monotone with constant m_1 for all $t \in \mathcal{X}_1$ and $N(\cdot,z) : \mathcal{X}_2 \to 2^{\mathcal{X}_2}$ is (A_2,η_2)-monotone with constant m_2 for all $z \in \mathcal{X}_2$, $S : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{X}_1$ is a single-valued mapping such that $S(\cdot,y)$ is (γ,r)-relaxed cocoercive with respect to f_1 and σ-Lipschitz continuous in the first variable and $S(x,\cdot)$ is ϱ-Lipschitz continuous in the second variable for all $(x,y) \in \mathcal{X}_1 \times \mathcal{X}_2$, $T : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{X}_2$ is a nonlinear mapping such that $T(x,\cdot)$ is (δ,s)-relaxed cocoercive with respect to g_2 and β-Lipschitz continuous in the second variable and $T(\cdot,y)$ is ι-Lipschitz continuous in the first variable for all $(x,y) \in \mathcal{X}_1 \times \mathcal{X}_2$, where $f_1 : \mathcal{X}_1 \to \mathcal{X}_1$ is defined by $f_1(x) = A_1 \circ f(x) = A_1(f(x))$ for all $x \in \mathcal{X}_1$ and $g_2 : \mathcal{X}_2 \to \mathcal{X}_2$ is defined by $g_2(x) = A_2 \circ g(x) = A_2(g(x))$ for all $x \in \mathcal{X}_2$. Let $\eta_1 : \mathcal{X}_1 \times \mathcal{X}_1 \to \mathcal{X}_1$ be τ_1-Lipschitz continuous, $\eta_2 : \mathcal{X}_2 \times \mathcal{X}_2 \to \mathcal{X}_2$ be τ_2-Lipschitz continuous, $p : \mathcal{X}_1 \to \mathcal{X}_1$ be κ-Lipschitz continuous, $q : \mathcal{X}_2 \to \mathcal{X}_2$ be ς-Lipschitz continuous, $f : \mathcal{X}_1 \to \mathcal{X}_1$ be π-strongly monotone and ϵ-Lipschitz continuous, $g : \mathcal{X}_2 \to \mathcal{X}_2$ be ϖ-strongly monotone and ε-Lipschitz continuous, $E : \mathcal{X}_1 \to C(\mathcal{X}_1)$ be ξ-$\mathbf{H}_1$-Lipschitz continuous, $F : \mathcal{X}_2 \to C(\mathcal{X}_2)$ be ζ-$\mathbf{H}_2$-Lipschitz continuous. If, in addition, there exist positive constants ρ and λ such that*

$$\|J^{\rho,A_1}_{\eta_1,M(\cdot,x)}(t) - J^{\rho,A_1}_{\eta_1,M(\cdot,y)}(t)\| \le \mu\|x-y\|, \quad \forall x,y,t \in \mathcal{X}_1, \tag{5.1}$$

$$\|J^{\lambda,A_2}_{\eta_2,N(\cdot,x)}(t) - J^{\lambda,A_2}_{\eta_2,N(\cdot,y)}(t)\| \le \nu\|x-y\|, \quad \forall x,y,t \in \mathcal{X}_2, \tag{5.2}$$

$$\begin{cases} k_1 = \mu + \sqrt{1-2\pi+\epsilon^2} + \dfrac{\tau_1\sqrt{\alpha_1^2\epsilon^2 - 2\rho r + 2\rho\gamma\kappa^2 + \rho^2\sigma^2\kappa^2}}{r_1 - \rho m_1} < 1, \\[2ex] k_2 = \nu + \sqrt{1-2\varpi+\varepsilon^2} + \dfrac{\tau_2\sqrt{\alpha_2^2\varepsilon^2 - 2\lambda s + 2\lambda\delta\varsigma^2 + \lambda^2\beta^2\varsigma^2}}{r_2 - \lambda m_2} < 1, \\[2ex] \rho < \min\{\dfrac{r_1}{m_1}, \dfrac{r_1(1-k_2)}{\tau_1\zeta\varrho + m_1(1-k_2)}\},\ \lambda < \min\{\dfrac{r_2}{m_2}, \dfrac{r_2(1-k_1)}{\tau_2\xi\iota + m_2(1-k_1)}\}, \\[2ex] \displaystyle\sum_{i=1}^{\infty} \|d_i - d_{i-1}\|\, k^{-i} < \infty,\ \sum_{i=1}^{\infty} \|e_i - e_{i-1}\|\, k^{-i} < \infty, \quad \forall k \in (0,1), \\[2ex] \displaystyle\lim_{n\to\infty} d_n = \lim_{n\to\infty} e_n = 0, \end{cases} \tag{5.3}$$

then the iterative sequences $\{a_n\}, \{b_n\}, \{u_n\}$ *and* $\{v_n\}$ *generated by Algorithm 4.1 converge strongly to* a^*, b^*, u^* *and* v^**, respectively, and* (a^*, b^*, u^*, v^*) *is a solution of the system of nonlinear set-valued variational inclusion problem* (4.1).

Proof. It follows from (4.9), (5.1) and Theorem 3.3 that

$$\begin{aligned}
&\|a_{n+1} - a_n\| \\
&= \|a_n - f(a_n) + J^{\rho,A_1}_{\eta_1,M(\cdot,a_n)}(A_1(f(a_n)) - \rho S(p(a_n), v_n)) + d_n \\
&\quad -\{a_{n-1} - f(a_{n-1}) \\
&\qquad +J^{\rho,A_1}_{\eta_1,M(\cdot,a_{n-1})}(A_1(f(a_{n-1})) - \rho S(p(a_{n-1}), v_{n-1})) + d_{n-1}\}\| \\
&\le \|a_n - a_{n-1} - (f(a_n) - f(a_{n-1}))\| \\
&\quad +\|J^{\rho,A_1}_{\eta_1,M(\cdot,a_n)}(A_1(f(a_n)) - \rho S(p(a_n), v_n)) \\
&\qquad -J^{\rho,A_1}_{\eta_1,M(\cdot,a_{n-1})}(A_1(f(a_n)) - \rho S(p(a_n), v_n))\| \\
&\quad +\|J^{\rho,A_1}_{\eta_1,M(\cdot,a_{n-1})}(A_1(f(a_n)) - \rho S(p(a_n), v_n)) \\
&\qquad -J^{\rho,A_1}_{\eta_1,M(\cdot,a_{n-1})}(A_1(f(a_{n-1})) - \rho S(p(a_{n-1}), v_{n-1}))\| + \|d_n - d_{n-1}\| \\
&\le \|a_n - a_{n-1} - (f(a_n) - f(a_{n-1}))\| + \mu\|a_n - a_{n-1}\| \\
&\quad +\tfrac{\tau_1}{r_1-\rho m_1}\|A_1(f(a_n)) - A_1(f(a_{n-1})) - \rho[S(p(a_n), v_n) - S(p(a_{n-1}), v_n)]\| \\
&\quad +\tfrac{\rho\tau_1}{r_1-\rho m_1}\|S(p(a_{n-1}), v_n) - S(p(a_{n-1}), v_{n-1})\| + \|d_n - d_{n-1}\|.
\end{aligned}$$

Since f is π-strongly monotone and ϵ-Lipschitz continuous, F is ζ-$\mathbf{H}_2$-Lipschitz continuous, A_1 is α_1-Lipschitz continuous, p is κ-Lipschitz continuous and $S(\cdot, y)$ is (γ, r)-relaxed cocoercive with respect to f_1 and σ-Lipschitz continuous in the first variable and $S(x, \cdot)$ is ϱ-Lipschitz continuous in the second variable for all $(x, y) \in \mathcal{X}_1 \times \mathcal{X}_2$, from (4.10) we get

$$\begin{aligned}
&\|a_n - a_{n-1} - (f(a_n) - f(a_{n-1}))\|^2 \\
&= \|a_n - a_{n-1}\|^2 - 2\langle a_n - a_{n-1}, f(a_n) - f(a_{n-1})\rangle + \|f(a_n) - f(a_{n-1})\|^2 \\
&\le (1 - 2\pi + \epsilon^2)\|a_n - a_{n-1}\|^2,
\end{aligned} \tag{5.5}$$

$$\begin{aligned}
&\|S(p(a_{n-1}), v_n) - S(p(a_{n-1}), v_{n-1})\| \\
&\le \varrho\|v_n - v_{n-1}\| \le \varrho(1 + n^{-1})\mathbf{H}_2(F(b_n), F(b_{n-1})) \\
&\le \zeta\varrho(1 + n^{-1})\|b_n - b_{n-1}\|
\end{aligned} \tag{5.6}$$

and

$$
\begin{aligned}
&\|A_1(f(a_n)) - A_1(f(a_{n-1})) - \rho[S(p(a_n), v_n) - S(p(a_{n-1}), v_n)]\|^2 \\
&= \|A_1(f(a_n)) - A_1(f(a_{n-1}))\|^2 + \rho^2\|S(p(a_n), v_n) - S(p(a_{n-1}), v_n)\|^2 \\
&\quad -2\rho\langle S(p(a_n), v_n) - S(p(a_{n-1}), v_n), A_1(f(a_n)) - A_1(f(a_{n-1}))\rangle \\
&\le \alpha_1^2\|f(a_n) - f(a_{n-1})\|^2 + \rho^2\sigma^2\|p(a_n) - p(a_{n-1})\|^2 \\
&\quad -2\rho[-\gamma\|p(a_n) - p(a_{n-1})\|^2 + r\|a_n - a_{n-1}\|^2] \\
&\le (\alpha_1^2\epsilon^2 - 2\rho r + 2\rho\gamma\kappa^2 + \rho^2\sigma^2\kappa^2)\|a_n - a_{n-1}\|^2.
\end{aligned} \tag{5.7}
$$

From (5.4)-(5.7), we have

$$
\begin{aligned}
&\|a_{n+1} - a_n\| \\
&\le [\mu + \sqrt{1 - 2\pi + \epsilon^2} \\
&\quad + \frac{\tau_1}{r_1 - \rho m_1}\sqrt{\alpha_1^2\epsilon^2 - 2\rho r + 2\rho\gamma\kappa^2 + \rho^2\sigma^2\kappa^2}]\|a_n - a_{n-1}\| \\
&\quad + \frac{\rho\tau_1\zeta\varrho}{r_1 - \rho m_1}(1 + n^{-1})\|b_n - b_{n-1}\| + \|d_n - d_{n-1}\| \\
&\quad v = k_1\|a_n - a_{n-1}\| + \theta_n\|b_n - b_{n-1}\| + \|d_n - d_{n-1}\|,
\end{aligned} \tag{5.8}
$$

where

$$
k_1 = \mu + \sqrt{1 - 2\pi + \epsilon^2} + \frac{\tau_1\sqrt{\alpha_1^2\epsilon^2 - 2\rho r + 2\rho\gamma\kappa^2 + \rho^2\sigma^2\kappa^2}}{r_1 - \rho m_1},
$$

$$
\theta_n = \frac{\rho\tau_1\zeta\varrho}{r_1 - \rho m_1}(1 + n^{-1}).
$$

Similarly, by the assumptions of $g, E, A_2, q, T(\cdot, \cdot)$, we can obtain

$$
\begin{aligned}
&c\|b_n - b_{n-1} - (g(b_n) - g(b_{n-1}))\|^2 \le (1 - 2\varpi + \varepsilon^2)\|b_n - b_{n-1}\|^2, \\
&\|T(u_n, q(b_n)) - T(u_{n-1}, q(b_n))\| \\
&\le \iota\|u_n - u_{n-1}\| \le \iota(1 + n^{-1})\mathbf{H}_1(E(a_n), E(a_{n-1})) \\
&\le \xi\iota(1 + n^{-1})\|a_n - a_{n-1}\|, \\
&\|A_2(g(b_n)) -! A_2(g(b_{n-1})) - \lambda[T(u_{n-1}, q(b_n)) - T(u_{n-1}, q(b_{n-1}))]\|^2 \\
&\le (\alpha_2^2\varepsilon^2 - 2\lambda s + 2\lambda\delta\varsigma^2 + \lambda^2\beta^2\varsigma^2)\|b_n - b_{n-1}\|^2,
\end{aligned} \tag{5.9}
$$

and

$$
\begin{aligned}
&\|b_{n+1}-b_n\| \\
&\le \|b_n-b_{n-1}-(g(b_n)-g(b_{n-1}))\|+\nu\|b_n-b_{n-1}\| \\
&\quad +\frac{\tau_2}{r_2\lambda m_2}\|A_2(g(b_n))-A_2(g(b_{n-1})) \\
&\quad -\lambda[T(u_{n-1},q(b_n))-T(u_{n-1},q(b_{n-1}))]\| \\
&\quad +\tfrac{\lambda\tau_2}{r_2-\lambda m_2}\|T(u_n,q(b_n))-T(u_{n-1},q(b_n))\|+\|e_n-e_{n-1}\| \\
&\le [\nu+\sqrt{1-2\varpi+\varepsilon^2} \\
&\quad +\tfrac{\tau_2}{r_2-\rho m_2}\sqrt{\alpha_2^2\varepsilon^2-2\lambda s+2\lambda\delta\varsigma^2+\lambda^2\beta^2\varsigma^2}]\|b_n-b_{n-1}\| \\
&\quad +\tfrac{\lambda\tau_2\xi\iota}{r_2-\rho m_2}(1+n^{-1})\|a_n-a_{n-1}\|+\nu\|e_n-e_{n-1}\| \\
&= k_2\|b_n-b_{n-1}\|+\vartheta_n\|a_n-a_{n-1}\|+\nu\|e_n-e_{n-1}\|,
\end{aligned} \tag{5.10}
$$

where

$$
\begin{aligned}
k_2 &= \nu+\sqrt{1-2\varpi+\varepsilon^2}+\frac{\tau_2\sqrt{\alpha_2^2\varepsilon^2-2\lambda s+2\lambda\delta\varsigma^2+\lambda^2\beta^2\varsigma^2}}{r_2-\lambda m_2}, \\
\vartheta_n &= \frac{\lambda\tau_2\xi\iota}{r_2-\lambda m_2}(1+n^{-1}).
\end{aligned}
$$

Now (5.8) and (5.10) imply that

$$
\begin{aligned}
&\|a_{n+1}-a_n\|+\|b_{n+1}-b_n\| \\
&\le (k_1+\vartheta_n)\|a_n-a_{n-1}\|+(\theta_n+k_2)\|b_n-b_{n-1}\| \\
&\quad +(\|d_n-d_{n-1}\|+\|e_n-e_{n-1}\|) \\
&\le t_n(\|a_n-a_{n-1}\|+\|b_n-b_{n-1}\|)+(\|d_n-d_{n-1}\|+\|e_n-e_{n-1}\|),
\end{aligned} \tag{5.11}
$$

where

$$
t_n=\max\{k_1+\vartheta_n,\theta_n+k_2\},\quad \forall n\ge 1.
$$

Letting $t=\max\{k_1+\vartheta,\theta+k_2\}$, where

$$
\theta=\frac{\rho\tau_1\zeta\varrho}{r_1-\rho m_1},\quad \vartheta=\frac{\lambda\tau_2\xi\iota}{r_2-\lambda m_2},
$$

then $t_n\to t$, $a_n\to a$ and $b_n\to b$ as $n\to\infty$. From condition (5.3), we know that $0<t<1$ and hence there exist $n_0>0$ and $t_0\in(t,1)$ such that $t_n\le t_0$ for all $n\ge n_0$. Therefore, it follows from (5.11) that

$$
\begin{aligned}
&\|a_{n+1}-a_n\|+\|b_{n+1}-b_n\| \\
&\le t_{n_0}(\|a_n-a_{n-1}\|+\|b_n-b_{n-1}\|)+(\|d_n-d_{n-1}\|+\|e_n-e_{n-1}\|),\quad \forall n\ge n_0.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\|a_{n+1} - a_n\| + \|b_{n+1} - b_n\| \\
&\leq t_0^{n-n_0}(\|a_{n_0+1} - a_{n_0}\| + \|b_{n_0+1} - b_{n_0}\|) \\
&\quad + \sum_{j=1}^{n-n_0} t_0^{j-1} \varpi_{n-(j-1)} + \sum_{j=1}^{n-n_0} t_0^{j-1} l_{n-(j-1)}, \quad \forall n \geq n_0,
\end{aligned}
$$

where $\varpi_n = \|d_n - d_{n-1}\|$, $l_n = \|e_n - e_{n-1}\|$ for all $n > n_0$. Hence, for any $m \geq n > n_0$, we have

$$
\begin{aligned}
&\|a_m - a_n\| + \|b_m - b_n\| \\
&\leq \sum_{i=n}^{m-1} (\|a_{i+1} - a_i\| + \|b_{i+1} - b_i\|) \\
&\leq \sum_{i=n}^{m-1} t_0^{i-n_0}(\|a_{n_0+1} - a_{n_0}\| + \|b_{n_0+1} - b_{n_0}\|) \\
&\quad + \mu \sum_{i=n}^{m-1} [\sum_{j=1}^{i-n_0} t_0^{j-1} \iota_{i-(j-1)}] + \nu \sum_{i=n}^{m-1} [\sum_{j=1}^{i-n_0} t_0^{j-1} l_{i-(j-1)}] \\
&\leq \sum_{i=n}^{m-1} t_0^{i-n_0}(\|a_{n_0+1} - a_{n_0}\| + \|b_{n_0+1} - b_{n_0}\|) \\
&\quad + \sum_{i=n}^{m-1} t_0^{i} [\sum_{j=1}^{i-n_0} \frac{\iota_{i-(j-1)}}{t_0^{i-(j-1)}}] + \sum_{i=n}^{m-1} t_0^{i} [\sum_{j=1}^{i-n_0} \frac{l_{i-(j-1)}}{t_0^{i-(j-1)}}].
\end{aligned}
\tag{5.12}
$$

Since $\sum_{i=1}^{\infty} \varpi_i \, k^{-i} < \infty$ and $\sum_{i=1}^{\infty} l_i \, k^{-i} < \infty$ for all $k \in (0,1)$, and $t_0 < 1$, it follows from (5.12) that $\|a_m - a_n\| \to 0$ and $\|b_m - b_n\| \to 0$, as $n \to \infty$, and so $\{a_n\}$ and $\{b_n\}$ are both Cauchy sequences in $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively. Thus, there exist $a^* \in \mathcal{X}_1$ and $b^* \in \mathcal{X}_2$ such that $a_n \to a^*$ and $b_n \to b^*$ as $n \to \infty$.

Now we prove that $u_n \to u^* \in E(a^*)$ and $v_n \to v^* \in F(b^*)$. In fact, it follows from (5.6) and (5.9) that $\{u_n\}$ and $\{v_n\}$ are also Cauchy sequences. Let $u_n \to u^*$ and $v_n \to v^*$, respectively. In the sequel, we will show that $u^* \in E(a^*)$ and $v^* \in F(b^*)$. Noting $u_n \in E(a_n)$, we have

$$
\begin{aligned}
d(u^*, E(a^*)) &= \inf\{\|u_n - z\| : z \in E(a^*)\} \\
&\leq \|u^* - u_n\| + d(u_n, E(a^*)) \\
&\leq \|u^* - u_n\| + \mathbf{H}(E(a_n), E(a^*)) \\
&\leq \|u^* - u_n\| + \xi \|a_n - a^*\| \to 0.
\end{aligned}
$$

Hence $d(u^*, E(a^*)) = 0$ and therefore $u^* \in E(a^*)$. Similarly, we can prove that $v^* \in F(b^*)$.

By continuity, (4.9) and $\lim_{n\to\infty} d_n = \lim_{n\to\infty} e_n = 0$, it is easy to see that a^*, b^*, u^*, v^* satisfy the following relation

$$\begin{cases} f(a^*) = J^{\rho,A_1}_{\eta_1,M}(A_1(f(a^*)) - \rho S(p(a^*), v^*)), \\ g(b^*) = J^{\lambda,A_2}_{\eta_2,N}(A_2(g(b^*)) - \lambda T(u^*, q(b^*))). \end{cases}$$

It follows from Lemma 4.1 that (a^*, b^*, u^*, v^*) is a solution of the system of generalized set-valued variational inclusion problem (4.1). ■

Remark 5.1. If M, N are (H, η)-monotone operators, H-monotone operators or A-monotone mappings, respectively, then from Theorem 5.1, we can get the existence and convergence results of solutions for the problems (4.2)-(4.7). Our results improve and generalize many known corresponding results of [2-4, 8-10, 13, 15].

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