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ITERATIVE APPROXIMATION OF FIXED POINTS OF A FINITE FAMILY OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. The purpose of this paper is to use viscosity approximation methods for a finite family of asymptotically nonexpansive mappings to establish the necessary and sufficient conditions for the iterative sequence to converge to a common fixed point of those mappings in uniformly convex Banach spaces.

1. INTRODUCTION

Let X be a Banach space, C a nonempty subset of X and $T : C \to C$ a mapping. Then T is said to be a *contraction* on C with contractive constant $\alpha \in (0,1)$ if $\|T(x) - T(y)\| \leq \alpha\|x - y\|$, for all $x, y \in C$; T is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$ and T is said to be *uniformly* L-*Lipschitzian* $(L > 0)$ if $\|T^n x - T^n y\| \leq L\|x - y\|$, for all $x, y \in C$ and for all $n \in \mathbf{N}$. If there exists a sequence $\{k_n\}$ of positive numbers with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n\|x - y\|$, for all $x, y \in C$ and for all $n \in \mathbf{N}$, then T is said to be *asymptotically nonexpansive*.

It is clear that an asymptotically nonexpansive mapping is uniformly L-Lipschitzian for some constant $L > 0$. The asymptotically nonexpansive mappings introduced by Goebel and Kirk are important generalizations of nonexpansive mappings. Also, Goebel and Kirk [9] proved that if X is a uniformly convex Banach space and C is a nonempty bounded and closed convex subset of X, then any asymptotically nonexpansive mapping $T : C \to C$ has a nonempty fixed point set $\mathrm{Fix}(T)$. The extension of this result was established as well (see, e.g., [10,15,22]). In particular, the iterative methods for approximating fixed points of asymptotically

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nonexpansive mappings have been studied and many nice results were presented (see, e.g., [2,5,7,14,16-21]).

In [14], Lim and Xu established a path-convergence theorem for asymptotically nonexpansive mappings in uniformly smooth Banach spaces. Besides, C. E. Chidume, J. Li and A. Udomene extended this result to real Banach spaces with uniformly Gâteaux differentiable norm possessing uniform normal structure [7].

Theorem 1.1. (Lim and Xu [14]) *Let X be a uniformly smooth Banach space, C a nonempty closed convex and bounded subset of X, $T : C \to C$ an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$. Let $u \in C$ and let $\{t_n\}$ be a sequence in $(0, 1)$ such that $\lim_{n\to\infty} t_n = 1$ and $\lim_{n\to\infty} (k_n-1)/(k_n-t_n) = 0$. Then for each $n \in \mathbf{N}$ there is a unique point $x_n \in C$ such that*

$$x_n = \left(1 - \frac{t_n}{k_n}\right) u + T^n x_n.$$

Suppose, in addition, that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Then the sequence $\{x_n\}$ converges to a fixed point of T.

It is remarkable that, by using viscosity approximation methods for a finite family of nonexpansive mappings in Banach spaces, the necessary and sufficient conditions for the iterative sequence to converge to a common fixed point of them are recently obtained by Chang [6].

Theorem. (Chang [6]) *Let X be a uniformly smooth Banach space, C a nonempty closed convex subset of X, $f : C \to C$ a contraction, $\{T_i\}_{i=1}^m$ a finite family of nonexpansive self-mappings of C with $\bigcap_{i=1}^m \mathrm{Fix}(T_i) \neq \emptyset$ satisfying the following conditions:*

$$\begin{aligned}\bigcap_{i=1}^m \mathrm{Fix}(T_i) &= \mathrm{Fix}(T_1T_m \cdots T_3T_2) = \cdots = \mathrm{Fix}(T_{m-1}T_{m-2} \cdots T_1T_m)\\ &= \mathrm{Fix}(T_mT_{m-1} \cdots T_1) = \mathrm{Fix}(S),\end{aligned}$$

where $S = T_mT_{m-1} \cdots T_1$. Suppose that $f(p) \neq p$, for all $p \in \bigcap_{i=1}^m \mathrm{Fix}(T_i)$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$. Define the iterative sequence $\{x_n\}$ iteratively by $x_0 \in C$,

$$x_{n+1} = \alpha_{n+1} f(x_n) + (1 - \alpha_{n+1})T_{n+1}x_n, \quad n \in \mathbf{N},$$

where $T_n = T_{n \bmod m}$. The sequence $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, \ldots, T_m$ if and only if $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$.

It is of great interest to know that a new implicit iteration scheme with perturbed mapping for a finite family of nonexpansive self-mappings of a Hilbert space is proposed by Zeng and Yao, and the necessary and sufficient conditions for the strong convergence of this implicit iteration scheme are also established as well [24].

Theorem. (Zeng and Yao[24]) *Let H be a real Hilbert space, $F : H \to H$ a κ-Lipschitzian and η-strongly monotone mapping, $\{T_i\}_{i=1}^m$ a finite family of nonexpansive self-mappings of H such that $C = \bigcap_{i=1}^m \mathrm{Fix}(T_i) \neq \emptyset$. Let $\mu \in (0, 2\eta/\kappa^2)$ and let $\{\lambda_n\} \subset [0,1)$ and $\{\alpha_n\} \subset (0,1)$ be the sequences such that $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\alpha \le \alpha_n \le \beta$, $n \in \mathbf{N}$, for some $\alpha, \beta \in (0,1)$. Let $x_0 \in H$ and let $\{x_n\}$ be defined by*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)[T_n x_n - \lambda_n \mu F(T_n x_n)], \quad n \in \mathbf{N},$$

where $T_n = T_{n \bmod m}$. Then $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, \ldots, T_m$ if and only if $\lim_{n\to\infty} d(x_n, C) = 0$.

The purpose of this paper is to use viscosity approximation methods for a finite family $\{T_i\}_{i=1}^m$ of asymptotically nonexpansive self-mappings of a closed convex subset C of a uniformly convex Banach space with sequences $\{k_{i+m(j-1)}\}_{j=1}^{\infty} \subset [1, \infty)$ $(1 \le i \le m)$ to establish the necessary and sufficient conditions for the following iterative sequence to converge to a common fixed point $\hat{z}$ of $\{T_i\}_{i=1}^m$:

$$z_{n+1} = \left(1 - \frac{t_n}{k_n}\right) f(z_n) + \frac{t_n}{k_n} T_{\bar{n}}^{j_n} z_n, \quad n \in \mathbf{N},$$

where $\bar{n} \equiv n \bmod m$, $n = \bar{n} + m(j_n - 1)$, z_1 is a given point of C, $f : C \to C$ is a contraction and $\{t_n\}$ is a sequence in $(0,1)$. Under some restrictions, this common fixed point $\hat{z}$ is, in fact, the unique solution of the variational inequality

$$\langle f(\hat{z}) - \hat{z}, J(y - \hat{z}) \rangle \le 0, \quad \text{for all } y \in \bigcap_{i=1}^m \mathrm{Fix}(T_i).$$

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2. Preliminaries

Let X be a Banach space and let X^* be the dual space of X. Let $J : X \to 2^{X^*}$ be the *normalized duality mapping* defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\| \cdot \|x^*\|,\ \|x\| = \|x^*\|\}, \qquad x \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing. If J is single-valued, J is odd, that is, $J(-x) = -J(x)$, $x \in X$. A Banach space X is said to admit a *weakly sequentially continuous* normalized duality mapping $J : X \to 2^{X^*}$ if J is single-valued and weak-to-weak* continuous. A Banach space X is said to be *uniformly convex* [1] if for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for $x, y \in X$ with $\|x\| \leq 1$ and $\|y\| \leq 1$,

$$\left\| \frac{x+y}{2} \right\| \leq 1 - \delta(\epsilon), \qquad \text{whenever } \|x - y\| \geq \epsilon.$$

A mapping $f : E \to X$, where $E \subset X$, is said to be *demiclosed* at $y \in X$ if, for any sequence $\{x_n\}$ in E, the conditions $x_n \to x \in E$ weakly and $f(x_n) \to y$ strongly together imply $f(x) = y$. Recall also that a Banach space X is said to satisfy *Opial's condition* [11] if whenever a sequence $\{x_n\}$ in X converges weakly to x, then $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$, for $y \neq x$. Gossez and Lami Dozo have shown that a Banach space with weakly sequentially continuous normalized duality mapping must satisfy Opial's condition [13, Theorem 1]. Specially, if X is a reflexive Banach space which satisfies Opial's condition, C is a closed convex subset of X and $T : C \to C$ is a nonexpansive mapping, then $I - T$ is demiclosed, where I denotes the identity mapping of X [11, Theorem 10.3]. Most importantly, Gornicki's demiclosedness principle states that if X is a uniformly convex Banach space satisfying Opial's condition, C is a nonempty closed convex subset of X and $T : C \to C$ is an asymptotically nonexpansive mapping, then $I - T$ is demiclosed at zero [12].

Let $S = \{x \in X : \|x\| = 1\}$ be the unit sphere of X. We say that X is *smooth* (or X has a *Gâteaux differentiable norm*) if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1}$$

exists for all $x, y \in S$; X is said to have a *uniformly Gâteaux differentiable norm* if for each $y \in X$ the limit (1) is attained uniformly in $x \in S$. If the limit (1) exists and is attained uniformly in $x, y \in S$, X is said to be *uniformly smooth*. It is well known that X is smooth if and only if the duality mapping J is single-valued [8]; if X has a uniformly Gâteaux differentiable norm, then J is single-valued and norm-to-weak*, uniformly continuous on bounded subsets of X [8]. Let C be a nonempty bounded subset of X and let $\operatorname{diam} C = \sup\{\|x - y\| : x, y \in C\}$ be the *diameter* of C. For each $x \in C$, let $r(x, C) = \sup\{\|x - y\| : y \in C\}$ so that the *Chebyshev radius* of C relative to itself is defined by $r(C) = \inf\{r(x, C) : x \in C\}$.

The *normal structure coefficient* $N(X)$ of X, cf. [1,3], is defined by

$$N(X) = \inf\left\{\frac{\operatorname{diam} C}{r(C)} : C \text{ is a bounded and closed convex subset of } X \text{ with } \operatorname{diam} C > 0\right\}.$$

A space X is said to have *uniform normal structure* if $N(X) > 1$. It is worth noting that every space with uniform normal structure is reflexive [11], and that all uniformly convex or uniformly smooth Banach spaces have uniform normal structure [1].

The following lemmas will be needed to prove our results.

Lemma 2.1. ([4]) *Let X be a real Banach space. Then*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle,$$

for all $x, y \in X$ and $j(x+y) \in J(x+y)$.

Lemma 2.1. ([23]) *Let $\{a_n\}$ ba a sequence of nonnegative real numbers satisfying the following condition:*

$$a_{n+1} \le (1-\alpha_n)a_n + \sigma_n, \qquad n \in \mathbf{N},$$

where each $0 < \alpha_n < 1$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If either $\sigma_n = o(\alpha_n)$, or $\limsup_{n\to\infty} \sigma_n \le 0$, then $\lim_{n\to\infty} a_n = 0$.

3. The Main Results

Let X be a real Banach space, C a nonempty closed convex subset of X, $\{T_i\}_{i=1}^m$ a finite family of asymptotically nonexpansive self-mappings of C with sequences $\{k_{i+m(j-1)}\}_{j=1}^{\infty}$ $(1 \le i \le m)$, $f : C \to C$ a contraction with contractive constant $\beta \in (0,1)$. We may always assume that $k_n \ge 1$, for all $n \in \mathbf{N}$. Let $\{t_n\}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty} t_n = 1$. Throughout the rest of this paper, for each $n \in \mathbf{N}$, we denote $\bar{n} \equiv n \bmod m$ and $j_n = (n-\bar{n})/m + 1$. Then $n = \bar{n} + m(j_n - 1)$. Thus $j_n = 1$ if $1 \le n \le m$, $j_n = 2$ if $m+1 \le n \le 2m$, and so on. Define a mapping $S_n : C \to C$ by

$$S_n(x) = \left(1 - \frac{t_n}{k_n}\right) f(x) + \frac{t_n}{k_n} T_{\bar{n}}^{j_n} x.$$

Then for $x, y \in C$ we have

$$\begin{aligned}\|S_n(x) - S_n(y)\| &\leq \left(1 - \frac{t_n}{k_n}\right)\|f(x) - f(y)\| + \frac{t_n}{k_n}\|T_{\bar{n}}^{j_n}x - T_{\bar{n}}^{j_n}y\| \\ &\leq \left[\beta\left(1 - \frac{t_n}{k_n}\right) + t_n\right]\|x - y\|.\end{aligned}$$

In addition, suppose that $\beta(1-t_n/k_n)+t_n < 1$. For example, if $\lim_{n\to\infty}(k_n-1)/(k_n-t_n) = 0$, then

$$\lim_{n\to\infty}\frac{1-t_n}{1-t_n/k_n} = \lim_{n\to\infty}k_n\left(1 - \frac{k_n-1}{k_n-t_n}\right) = 1 \tag{2}$$

and thus $\beta(1 - t_n/k_n) + t_n < 1$, for all sufficiently large n. Then by Banach Contraction Principle, S_n has a unique fixed point $x_n \in C$, that is,

$$x_n = \left(1 - \frac{t_n}{k_n}\right)f(x_n) + \frac{t_n}{k_n}T_{\bar{n}}^{j_n}x_n. \tag{3}$$

Theorem 3.1. *Let X be a uniformly convex Banach space with weakly sequentially continuous normalized duality mapping $J : X \to X^*$, C a nonempty closed convex subset of X, $f : C \to C$ a contraction with contractive constant $\beta \in (0, 1)$, $\{T_i\}_{i=1}^m$ a finite family of asymptotically nonexpansive self-mappings of C with sequences $\{k_{i+m(j-1)}\}_{j=1}^\infty \subset [1,\infty)$ ($1 \leq i \leq m$) such that $\bigcap_{i=1}^m \mathrm{Fix}(T_i) \neq \emptyset$. Let $\{t_n\}$ be a sequence in $(0, 1)$ such that $\lim_{n\to\infty} t_n = 1$ and $\lim_{n\to\infty}(k_n-1)/(k_n-t_n) = 0$. The sequence $\{x_n\}$ defined by* (3) *converges strongly to a point $\hat{x} \in \bigcap_{i=1}^m \mathrm{Fix}(T_i)$ which is the unique solution of the variational inequality in $\bigcap_{i=1}^m \mathrm{Fix}(T_i)$:*

$$\langle f(\hat{x}) - \hat{x}, J(y - \hat{x})\rangle \leq 0, \quad \textit{for all } y \in \bigcap_{i=1}^m \mathrm{Fix}(T_i) \tag{4}$$

if and only if $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$, for $i = 1, \ldots, m$.

Proof. By (2), discarding a few terms if necessary, we may assume that there exists a positive number $\delta < 1$ such that

$$\frac{1-t_n}{1-(t_n/k_n)} > \beta + \delta, \quad \text{for all } n \in \mathbf{N}. \tag{5}$$

Notice that the variational inequality (4) has at most one solution in $\bigcap_{i=1}^{m} \mathrm{Fix}(T_i)$. For, if u and v are two solutions of (4) in $\bigcap_{i=1}^{m} \mathrm{Fix}(T_i)$, we have

$$\langle f(u) - u, J(v-u)\rangle \leq 0,$$
$$\langle f(v) - v, J(u-v)\rangle \leq 0.$$

Adding these two inequalities, we obtain

$$\langle [u - f(u)] - [v - f(v)], J(u-v)\rangle \leq 0$$

which implies

$$(1-\beta)\|u-v\|^2 \leq \langle [u - f(u)] - [v - f(v)], J(u-v)\rangle \leq 0.$$

Therefore $u = v$.

If the sequence $\{x_n\}$ converges strongly to $\hat{x} \in \bigcap_{i=1}^{m} \mathrm{Fix}(T_i)$ which is the unique solution of the variational inequality (4) in $\bigcap_{i=1}^{m} \mathrm{Fix}(T_i)$, then for $i = 1, \ldots, m$,

$$\|x_n - T_i x_n\| \leq \|x_n - \hat{x}\| + \|T_i\hat{x} - T_i x_n\| \leq (1+k_i)\|x_n - \hat{x}\|$$

and hence $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$, for $i = 1, \ldots, m$.

Conversely, suppose that $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$, for $i = 1, \ldots, m$. We first prove that the sequence $\{x_n\}$ is bounded. Given $u \in \bigcap_{i=1}^{m} \mathrm{Fix}(T_i)$ it follows from (3) that

$$\begin{aligned}
\|x_n - u\|^2 &= \left\langle \left(1 - \frac{t_n}{k_n}\right)[f(x_n) - u] + \frac{t_n}{k_n}(T_{\bar{n}}^{j_n} x_n - T_{\bar{n}}^{j_n} u), J(x_n - u)\right\rangle \\
&\leq \left(1 - \frac{t_n}{k_n}\right)\|f(x_n) - f(u)\| \cdot \|x_n - u\| + \frac{t_n}{k_n} \cdot k_n \|x_n - u\|^2 \\
&\quad + \left(1 - \frac{t_n}{k_n}\right)\langle f(u) - u, J(x_n - u)\rangle \\
&\leq \left[\beta\left(1 - \frac{t_n}{k_n}\right) + t_n\right]\|x_n - u\|^2 + \left(1 - \frac{t_n}{k_n}\right)\langle f(u) - u, J(x_n - u)\rangle,
\end{aligned}$$

and so

$$\left[\frac{1 - t_n}{1 - (t_n/k_n)} - \beta\right]\|x_n - u\|^2 \leq \langle f(u) - u, J(x_n - u)\rangle. \tag{6}$$

By (5), the inequality (6) implies that

$$\tag{7} \begin{aligned} \|x_n - u\|^2 &\leq \frac{1}{\delta}\langle f(u) - u, J(x_n - u)\rangle \\ &\leq \frac{1}{\delta}\|f(u) - u\| \cdot \|x_n - u\|, \quad \text{for } n \in \mathbf{N} \end{aligned}$$

and thus

$$\|x_n - u\| \leq \frac{1}{\delta}\|f(u) - u\|, \quad \text{for } n \in \mathbf{N}.$$

This shows that $\{x_n\}$ is bounded.

On the other hand, given any subsequence $\{x_{n_l}\}$ of $\{x_n\}$, since X is reflexive, there is a subsequence of $\{x_{n_l}\}$, still denoted by $\{x_{n_l}\}$, converging weakly to a point $\hat{x} \in C$. Since each $I - T_i$ is demiclosed at zero and $\lim\limits_{n_l \to \infty} \|x_{n_l} - T_i x_{n_l}\| = 0$, we have $\hat{x} - T_i\hat{x} = 0$ and so $\hat{x} \in \bigcap\limits_{i=1}^{m} \mathrm{Fix}(T_i)$. Choosing $u = \hat{x}$ in (7) and letting $n_l \to \infty$, we obtain

$$\limsup_{n_l \to \infty} \|x_{n_l} - \hat{x}\|^2 \leq \lim_{n_l \to \infty} \frac{1}{\delta}\langle f(\hat{x}) - \hat{x}, J(x_{n_l} - \hat{x})\rangle = 0.$$

Therefore $\{x_{n_l}\}$ converges strongly to $\hat{x}$.

Next, we shall prove that $\hat{x}$ is a solution of the variational inequality (4) in $\bigcap\limits_{i=1}^{m} \mathrm{Fix}(T_i)$. Let $y \in \bigcap\limits_{i=1}^{m} \mathrm{Fix}(T_i)$. It follows from (3) that

$$\begin{aligned} \|x_n - y\|^2 &= \left(1 - \frac{t_n}{k_n}\right)\langle f(x_n) - x_n, J(x_n - y)\rangle + \left(1 - \frac{t_n}{k_n}\right)\langle x_n - y, J(x_n - y)\rangle \\ &\quad + \frac{t_n}{k_n}\langle T_{\tilde{n}}^{j_n} x_n - T_{\tilde{n}}^{j_n} y, J(x_n - y)\rangle \\ &\leq \left(1 - \frac{t_n}{k_n}\right)\langle f(x_n) - x_n, J(x_n - y)\rangle + \left(1 + t_n - \frac{t_n}{k_n}\right)\|x_n - y\|^2 \end{aligned}$$

which implies that

$$\tag{8} \langle f(x_n) - x_n, J(y - x_n)\rangle \leq t_n \cdot \frac{k_n - 1}{k_n - t_n}\|x_n - y\|^2.$$

Observe that

$$\tag{9} \begin{aligned} &|\langle x_n - f(x_n), J(x_n - y)\rangle - \langle \hat{x} - f(\hat{x}), J(\hat{x} - y)\rangle| \\ &= |\langle [x_n - f(x_n)] - [\hat{x} - f(\hat{x})], J(x_n - y)\rangle \\ &\qquad + \langle \hat{x} - f(\hat{x}), J(x_n - y) - J(\hat{x} - y)\rangle| \\ &\leq \|[x_n - f(x_n)] - [\hat{x} - f(\hat{x})]\| \cdot \|x_n - y\| \\ &\qquad + |\langle \hat{x} - f(\hat{x}), J(x_n - y) - J(\hat{x} - y)\rangle|. \end{aligned}$$

Since $\{x_{n_l}\}$ converges strongly to $\hat{x}$ and J is weak-to-weak* continuous, it follows from (8) and (9) that

$$\langle f(\hat{x}) - \hat{x}, J(y - \hat{x})\rangle = \lim_{n_l \to \infty} \langle f(x_{n_l}) - x_{n_l}, J(y - x_{n_l})\rangle \leq 0.$$

This shows that $\hat{x}$ is a solution of the variational inequality (4) in $\bigcap_{i=1}^{m} \mathrm{Fix}(T_i)$. Consequently, any subsequence of $\{x_n\}$ has a strongly convergent subsequence with limit $\hat{x}$, and hence $\{x_n\}$ converges strongly to $\hat{x}$. ∎

Theorem 3.2. *Let X be a uniformly convex Banach space with weakly sequentially continuous normalized duality mapping $J : X \to X^*$, C a nonempty closed convex subset of X, $f : C \to C$ a contraction with contractive constant $\beta \in (0, 1)$, $\{T_i\}_{i=1}^{m}$ a finite family of asymptotically nonexpansive self-mappings of C with sequences $\{k_{i+m(j-1)}\}_{j=1}^{\infty} \subset [1, \infty)$ ($1 \leq i \leq m$) such that $\bigcap_{i=1}^{m} \mathrm{Fix}(T_i) \neq \emptyset$. Let $\{t_n\}$ be a sequence in $(0, 1)$ such that $\lim_{n\to\infty} (k_n - 1)/(k_n - t_n) = 0$ and let $\{x_n\}$ be a sequence defined by (3). Define the sequence $\{z_n\}$ iteratively by $z_1 \in C$,*

$$z_{n+1} = \left(1 - \frac{t_n}{k_n}\right) f(z_n) + \frac{t_n}{k_n} T_{\bar{n}}^{j_n} z_n, \quad n \in \mathbf{N}. \tag{10}$$

For any C, f and $\{T_i\}_{i=1}^{m}$ as above, the sequences $\{x_n\}$ and $\{z_n\}$ both converge strongly to a point $\hat{x} \in \bigcap_{i=1}^{m} \mathrm{Fix}(T_i)$ which is the unique solution of the variational inequality (4) in $\bigcap_{i=1}^{m} \mathrm{Fix}(T_i)$ if and only if the following conditions are satisfied:

(*i*) $\lim_{n\to\infty} t_n = 1$;

(*ii*) $\sum_{n=1}^{\infty} [\, 1 - (t_n/k_n)\,] = \infty$;

(*iii*) $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ *and* $\lim_{n\to\infty} \|z_n - T_i z_n\| = 0$, *for* $i = 1, \ldots, m$.

Proof. Suppose that the sequences $\{x_n\}$ and $\{z_n\}$ both converge strongly to a point $\hat{x}$ of $\bigcap_{i=1}^{m} \mathrm{Fix}(T_i)$ which is the unique solution of the variational inequality (4) in $\bigcap_{i=1}^{m} \mathrm{Fix}(T_i)$. Then for $i = 1, \ldots, m$,

$$\|x_n - T_i x_n\| \leq \|x_n - \hat{x}\| + \|T_i\hat{x} - T_i x_n\| \leq (1 + k_i)\|x_n - \hat{x}\| \to 0 \quad \text{as } n \to \infty,$$

and similarly

$$\|z_n - T_i z_n\| \le \|z_n - \hat{x}\| + \|T_i\hat{x} - T_i z_n\| \le (1+k_i)\|z_n - \hat{x}\| \to 0 \quad \text{as } n \to \infty.$$

Hence condition (iii) holds.

On the other hand, for any $n \in \mathbf{N}$, since $\|T_{\bar{n}}^{j_n} z_n - \hat{x}\| = \|T_{\bar{n}}^{j_n} z_n - T_{\bar{n}}^{j_n}\hat{x}\| \le k_n\|z_n - \hat{x}\|$, it follows that $\lim_{n\to\infty} T_{\bar{n}}^{j_n} z_n = \hat{x}$. Letting $f \equiv u \in C$ with $u \notin \bigcap_{i=1}^{m} \mathrm{Fix}(T_i)$, by (10) we have

$$\liminf_{n\to\infty} \frac{t_n}{k_n}\|\hat{x} - u\| = \liminf_{n\to\infty} \frac{t_n}{k_n}\|T_{\bar{n}}^{j_n} z_n - u\| = \liminf_{n\to\infty} \|z_{n+1} - u\| = \|\hat{x} - u\|$$

which implies that $\lim_{n\to\infty} t_n/k_n = 1$ and condition (i) holds.

To prove condition (ii) is satisfied, we may take $C = \{x \in X : \|x\| \le 1\}$ to be the closed unit ball and $z_1 \neq 0$. Set $f \equiv 0$ and $T_i = -I : C \to C$, for $i = 1, \ldots, m$, so that it follows from (10) that

$$z_{n+1} = (-1)^{j_n} \cdot \frac{t_n}{k_n} \cdot z_n = \cdots\cdots = (-1)^{m(j_n-1)j_n/2 + \bar{n}j_n} \prod_{l=1}^{n} \frac{t_l}{k_l} \cdot z_1.$$

Since 0 is the only common fixed point of T_i's, we obtain $z_n \to \hat{x} = 0$ and

$$0 = \lim_{n\to\infty} \|z_{n+1} - 0\| = \lim_{n\to\infty} \prod_{l=1}^{n} \frac{t_l}{k_l} \cdot \|z_1 - 0\|;$$

hence $\prod_{n=1}^{\infty} t_n/k_n = 0$, that is, $\sum_{n=1}^{\infty} [\,1 - (t_n/k_n)\,] = \infty$.

From the above discussion, conditions (i), (ii) and (iii) are seen to be necessary. So we proceed to show their sufficiency. It follows from Theorem 3.1 that the sequence $\{x_n\}$ converges strongly to a point $\hat{x} \in \bigcap_{i=1}^{m} \mathrm{Fix}(T_i)$ which is the unique solution of the variational inequality (4) in $\bigcap_{i=1}^{m} \mathrm{Fix}(T_i)$.

We will prove that the sequence $\{z_n\}$ is bounded. By (2), there exists a positive number $\delta < 1$ and $n_0 \in \mathbf{N}$ such that $\dfrac{1 - t_n}{1 - (t_n/k_n)} > \beta + \delta$, for all $n \ge n_0$. Given

$u \in \bigcap_{i=1}^{m} \mathrm{Fix}(T_i)$ it follows from (10) that for $n \geq n_0$,

$$\begin{aligned}\|z_{n+1}-u\| &\leq \left(1-\frac{t_n}{k_n}\right)\|f(z_n)-f(u)+f(u)-u\| + \frac{t_n}{k_n}\|T_{\bar{n}}^{j_n} z_n - T_{\bar{n}}^{j_n} u\| \\ &\leq \left[\beta\left(1-\frac{t_n}{k_n}\right)+t_n\right]\|z_n-u\| + \left(1-\frac{t_n}{k_n}\right)\|f(u)-u\| \\ &\leq \left[1-\delta\left(1-\frac{t_n}{k_n}\right)\right]\|z_n-u\| + \delta\left(1-\frac{t_n}{k_n}\right)\cdot\frac{1}{\delta}\|f(u)-u\| \\ &\leq \max\{\|z_n-u\|,\ \frac{1}{\delta}\|f(u)-u\|\}.\end{aligned}$$

By induction we obtain

$$\|z_{n+1}-u\| \ \leq \max\{\|z_1-u\|,\ \frac{1}{\delta}\|f(u)-u\|\}, \qquad \text{for } n \geq n_0;$$

hence $\{z_n\}$ is bounded. The reflexivity of X asserts that there is a subsequence $\{z_{n_l}\}$ of $\{z_n\}$ which converges weakly to a point $\hat{z} \in C$. Since each $I - T_i$ is demiclosed at zero and $\lim_{n_l\to\infty} \|z_{n_l} - T_i z_{n_l}\| = 0$, we have $\hat{z} - T_i\hat{z} = 0$ and hence $\hat{z} \in \bigcap_{i=1}^{m} \mathrm{Fix}(T_i)$. Since $\hat{x}$ is the unique solution of (4) and J is weakly sequentially continuous, we conclude that

$$\begin{aligned}\limsup_{n\to\infty}\langle f(\hat{x})-\hat{x}, J(z_n-\hat{x})\rangle &= \limsup_{n_l\to\infty}\langle f(\hat{x})-\hat{x}, J(z_{n_l}-\hat{x})\rangle \\ &= \langle f(\hat{x})-\hat{x}, J(\hat{z}-\hat{x})\rangle \leq 0.\end{aligned} \tag{11}$$

To prove $\{z_n\}$ converges strongly to $\hat{x}$, let $\gamma_n = 1 - t_n/k_n$, for $n \in \mathbf{N}$ so that $\gamma_n < 1/2$ for all sufficiently large n. Thus $1 - \beta\gamma_n > 1/2$ for sufficiently large n. It follows from (10) and Lemma 2.1 that

$$\begin{aligned}\|z_{n+1}-\hat{x}\|^2 &\leq (1-\gamma_n)^2\,\|T_{\bar{n}}^{j_n} z_n - \hat{x}\|^2 + 2\gamma_n\langle f(z_n)-\hat{x}, J(z_{n+1}-\hat{x})\rangle \\ &\leq k_n^2(1-\gamma_n)^2\,\|z_n-\hat{x}\|^2 + 2\gamma_n\langle\, f(z_n)-f(\hat{x})+f(\hat{x})-\hat{x}, J(z_{n+1}-\hat{x})\,\rangle \\ &\leq k_n^2(1-\gamma_n)^2\,\|z_n-\hat{x}\|^2 + 2\beta\gamma_n\,\|z_n-\hat{x}\|\cdot\|z_{n+1}-\hat{x}\| \\ &\quad + 2\gamma_n\langle f(\hat{x})-\hat{x}, J(z_{n+1}-\hat{x})\rangle \\ &\leq k_n^2(1-\gamma_n)^2\,\|z_n-\hat{x}\|^2 + \beta\gamma_n\left[\,\|z_n-\hat{x}\|^2 + \|z_{n+1}-\hat{x}\|^2\,\right] \\ &\quad + 2\gamma_n\,\langle f(\hat{x})-\hat{x}, J(z_{n+1}-\hat{x})\rangle.\end{aligned}$$

Therefore

$$\|z_{n+1}-\hat{x}\|^2 \leq \frac{1-(2-\beta)\gamma_n}{1-\beta\gamma_n}\|z_n-\hat{x}\|^2+\frac{(k_n^2-1)(1-2\gamma_n)+k_n^2\gamma_n^2}{1-\beta\gamma_n}\|z_n-\hat{x}\|^2 + \frac{2\gamma_n}{1-\beta\gamma_n}\langle f(\hat{x})-\hat{x}, J(z_{n+1}-\hat{x})\rangle. \tag{12}$$

Since

$$\frac{1-(2-\beta)\gamma_n}{1-\beta\gamma_n} = 1 - \frac{2(1-\beta)\gamma_n}{1-\beta\gamma_n} < 1 - 2(1-\beta)\gamma_n,$$

by (12) we have

$$\|z_{n+1}-\hat{x}\|^2 \leq (1-\alpha_n)\|z_n-\hat{x}\|^2 + \sigma_n,$$

where $\alpha_n = 2(1-\beta)\gamma_n$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, and

$$\sigma_n = \frac{(k_n^2-1)(1-2\gamma_n)+k_n^2\gamma_n^2}{1-\beta\gamma_n}\|z_n-\hat{x}\|^2 + \frac{2\gamma_n}{1-\beta\gamma_n}\langle f(\hat{x})-\hat{x}, J(z_{n+1}-\hat{x})\rangle.$$

The boundedness of $\{z_n\}$ and the inequality (11) imply that $\limsup_{n\to\infty}\sigma_n \leq 0$. By Lemma 2.2, $\lim_{n\to\infty}\|z_n-\hat{x}\| = 0$ obtains. ■

Remark. Such a sequence $\{t_n\}$ as in Theorem 3.1 and Theorem 3.2 always exists. We take $t_n = \min\{1-(k_n-1)^{1/2}, 1-(1/n)\}$ for example (see also [7,14]).

4. Applications

When only one asymptotically nonexpansive mapping is considered, the equations (3) and (10) are reduced to the following iterative sequences respectively:

$$x_n = \left(1-\frac{t_n}{k_n}\right)f(x_n) + \frac{t_n}{k_n}T^n x_n, \quad n \in \mathbf{N}, \tag{13}$$

and

$$z_{n+1} = \left(1-\frac{t_n}{k_n}\right)f(z_n) + \frac{t_n}{k_n}T^n z_n, \quad n \in \mathbf{N}. \tag{14}$$

The corresponding results are given as follows.

Theorem 4.1. *Let X be a uniformly convex Banach space with weakly sequentially continuous normalized duality mapping $J : X \to X^*$, C a nonempty closed* *convex subset of* X*,* $f : C \to C$ *a contraction,* $T : C \to C$ *an asymptotically nonexpansive mapping with sequence* $\{k_n\} \subset [1, \infty)$ *such that* $\mathrm{Fix}(T) \neq \emptyset$*. Let* $\{t_n\}$ *be a sequence in* $(0, 1)$ *such that* $\lim_{n\to\infty} t_n = 1$ *and* $\lim_{n\to\infty}(k_n - 1)/(k_n - t_n) = 0$*. The sequence* $\{x_n\}$ *defined by* (13) *converges strongly to a point* $\hat{x} \in \mathrm{Fix}(T)$ *which is the unique solution of the variational inequality* (4) *in* $\mathrm{Fix}(T)$ *if and only if* $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Theorem 4.2. *Let* X *be a uniformly convex Banach space with weakly sequentially continuous normalized duality mapping* $J : X \to X^*$*,* C *a nonempty closed convex subset of* X*,* $f : C \to C$ *a contraction,* $T : C \to C$ *an asymptotically nonexpansive mapping with sequence* $\{k_n\} \subset [1, \infty)$ *such that* $\mathrm{Fix}(T) \neq \emptyset$*. Let* $\{t_n\}$ *be a sequence in* $(0, 1)$ *such that* $\lim_{n\to\infty}(k_n - 1)/(k_n - t_n) = 0$ *and let* $\{x_n\}$ *and* $\{z_n\}$ *be the sequences defined by* (13) *and* (4) *respectively. For any* C*,* f *and* T*, the sequences* $\{x_n\}$ *and* $\{z_n\}$ *both converge strongly to a point* $\hat{x} \in \mathrm{Fix}(T)$ *which is the unique solution of the variational inequality* (4) *in* $\mathrm{Fix}(T)$ *if and only if the following conditions are satisfied:*

(i) $\lim_{n\to\infty} t_n = 1$*;*

(i) $\sum_{n=1}^{\infty} [\, 1 - (t_n/k_n)\,] = \infty$*;*

(iii) $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ *and* $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$.

Let X be a Banach space with uniformly Gâteaux differentiable norm. If X satisfies Opial's condition, then the normalized duality mapping $J : X \to X^*$ is weakly sequentially continuous at zero [13, Theorem 2]. Therefore, with similar approaches to Theorems 3.1 and 3.2, the following two results are established.

Theorem 4.3. *Let* X *be a uniformly convex Banach space with uniformly Gâteaux differentiable norm satisfying Opial's condition,* C *a nonempty closed convex subset of* X*,* $f : C \to C$ *a contraction,* $\{T_i\}_{i=1}^m$ *a finite family of asymptotically nonexpansive self-mappings of* C *with sequences* $\{k_{\,i+m(j-1)}\}_{j=1}^{\infty} \subset [1, \infty)$ *(*$1 \le i \le m$*) such that* $\bigcap_{i=1}^m \mathrm{Fix}(T_i) \neq \emptyset$*. Let* $\{t_n\}$ *be a sequence in* $(0, 1)$ *such that* $\lim_{n\to\infty} t_n = 1$ *and* $\lim_{n\to\infty}(k_n - 1)/(k_n - t_n) = 0$*. The sequence* $\{x_n\}$ *defined by* (3) *converges strongly to a point* $\hat{x} \in \bigcap_{i=1}^m \mathrm{Fix}(T_i)$ *which is the unique solution of the variational inequality* (4) *in* $\bigcap_{i=1}^m \mathrm{Fix}(T_i)$ *if and only if* $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$*, for* $i = 1, \dots, m$*.*

Proof. Since X is a uniformly convex Banach space satisfying Opial's condition, each $I - T_i$ is demiclosed at zero. Due to the weakly sequential continuity of J at zero, the proof is identical to that of Theorem 3.1. ■

Theorem 4.4. *Let X be a uniformly convex Banach space with uniformly Gâteaux differentiable norm satisfying Opial's condition, C a nonempty closed convex subset of X, $f : C \to C$ a contraction with contractive constant $\beta \in (0,1)$, $\{T_i\}_{i=1}^m$ a finite family of asymptotically nonexpansive self-mappings of C with sequences $\{k_{i+m(j-1)}\}_{j=1}^{\infty} \subset [1,\infty)$ ($1 \le i \le m$) such that $\bigcap_{i=1}^m \mathrm{Fix}(T_i) \neq \emptyset$. Let $\{t_n\}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty}(k_n - 1)/(k_n - t_n) = 0$. Let $\{x_n\}$ and $\{z_n\}$ be the sequences defined by* (3) *and* (10) *respectively. For any C, f and $\{T_i\}_{i=1}^m$ as above, the sequences $\{x_n\}$ and $\{z_n\}$ both converge strongly to a point $\hat{x} \in \bigcap_{i=1}^m \mathrm{Fix}(T_i)$ which is the unique solution of the variational inequality* (4) *in $\bigcap_{i=1}^m \mathrm{Fix}(T_i)$ if and only if the conditions (i)-(iii) in Theorem 3.2 are satisfied.*

Proof. Applying the same argument as in Theorem 3.2, we only prove conditions (i)-(iii) are sufficient. First, Theorem 4.3 asserts that the sequence $\{x_n\}$ converges strongly to a point $\hat{x} \in \bigcap_{i=1}^m \mathrm{Fix}(T_i)$ which is the unique solution of the variational inequality (4) in $\bigcap_{i=1}^m \mathrm{Fix}(T_i)$.

On the other hand, the sequence $\{z_n\}$ is bounded as verified in Theorem 3.2, and so has a weakly convergent subsequence $\{z_{n_l}\}$ with limit $\hat{z} \in C$. Since each $I - T_i$ is demiclosed at zero and $\lim_{n_l\to\infty} \|z_{n_l} - T_i z_{n_l}\| = 0$, we have $\hat{z} - T_i\hat{z} = 0$ and hence $\hat{z} \in \bigcap_{i=1}^m \mathrm{Fix}(T_i)$. The weakly sequential continuity of J at zero ensures that

$$\limsup_{n\to\infty}\langle f(\hat{z}) - \hat{z}, J(z_n - \hat{z})\rangle = \limsup_{n_l\to\infty}\langle f(\hat{z}) - \hat{z}, J(z_{n_l} - \hat{z})\rangle = 0. \tag{15}$$

Furthermore, as shown in Theorem 3.2, to prove $\{z_n\}$ converges strongly to $\hat{z}$, let $\gamma_n = 1 - t_n/k_n$, for $n \in \mathbf{N}$ so that $\gamma_n < 1/2$ for all sufficiently large n. Replacing $\hat{x}$ with $\hat{z}$ in (12) yields:

$$\|z_{n+1} - \hat{z}\|^2 \le (1 - \alpha_n)\|z_n - \hat{z}\|^2 + \sigma_n,$$

where $\alpha_n = 2(1-\beta)\gamma_n$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and

$$\sigma_n = \frac{(k_n^2 - 1)(1 - 2\gamma_n) + k_n^2\gamma_n^2}{1 - \beta\gamma_n}\|z_n - \hat{z}\|^2 + \frac{2\gamma_n}{1 - \beta\gamma_n}\langle f(\hat{z}) - \hat{z}, J(z_{n+1} - \hat{z})\rangle.$$

The boundedness of $\{z_n\}$ and (15) imply that $\limsup_{n\to\infty} \sigma_n = 0$. By Lemma 2.2, we have $\lim_{n\to\infty} \|z_n - \hat{z}\| = 0$, as claimed. In fact, $\hat{z} = \hat{x}$. For, if $y \in \bigcap_{i=1}^{m} \mathrm{Fix}(T_i)$, apply (10) and Lemma 2.1 again to obtain the inequality

$$\|z_{n+1} - \hat{x}\|^2 \leq (1 - \alpha_n)\|z_n - \hat{x}\|^2 + \tau_n,$$

where

$$\tau_n = \frac{(k_n^2 - 1)(1 - 2\gamma_n) + k_n^2\gamma_n^2}{1 - \beta\gamma_n}\|z_n - \hat{x}\|^2 + \frac{2\gamma_n}{1 - \beta\gamma_n}\langle f(\hat{x}) - \hat{x}, J(z_{n+1} - \hat{x})\rangle.$$

Note that $\hat{x}$ is the unique solution of (4). Since $\{z_n\}$ converges strongly to $\hat{z}$ and J is norm-to-weak* continuous, we have

$$\limsup_{n\to\infty}\langle f(\hat{x}) - \hat{x}, J(z_n - \hat{x})\rangle = \langle f(\hat{x}) - \hat{x}, J(\hat{z} - \hat{x})\rangle \leq 0,$$

which implies that $\limsup_{n\to\infty} \tau_n \leq 0$. By Lemma 2.2, $\lim_{n\to\infty} \|z_n - \hat{x}\| = 0$ holds, completing the proof. ∎

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