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# ALGORITHM OF SOLUTIONS FOR A SYSTEM OF GENERALIZED MIXED IMPLICIT QUASI-VARIATIONAL INCLUSIONS INVOLVING $h$-$\eta$-MAXIMAL MONOTONE MAPPINGS

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**Abstract.** In this paper, we introduce a new class of $h$-$\eta$-maximal monotone mappings and a new system of generalized mixed implicit quasi-variational inclusions involving set-valued mappings and $h$-$\eta$-maximal monotone mappings. By using resolvent operator technique of $h$-$\eta$-maximal monotone mappings, A new iterative algorithm to compute approximate solutions of the system is suggested and analyzed. The convergence of the iterative sequence generated by the new algorithm is also proved. These results generalize many known results in literature.

## 1. INTRODUCTION

Variational inequality theory has become very effective and powerful tool for studying a wide range of problems arising in differential equations, mechanics, contact problems in elasticity, optimization and control problems, management science, operations research, general equilibrium problems in economics and transportation, unilateral, obstacle, moving, etc. Hassouni and Moudafi [12] introduced and studied a class of mixed type variational inequalities with single-valued mappings which was called variational inclusions. Since then, many authors have obtained important extensions and generalizations of the results in [1] from various different directions, see [1-11,13-18,20-24]. Verma [26,27] introduced and studied some system of variational inequalities and some iterative algorithms to compute approximate solutions

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was also suggested and analyzed in Hilbert spaces. Fang and Huang [11] introduced a class of $H$-monotone operators and studied a new class of variational inclusions involving $H$-monotone operators.

Inspired and motivated by recent research works in this field, in this paper, we introduce a new class of $h$-$\eta$-maximal monotone mappings and a new system of generalized mixed implicit quasi-variational inclusions involving set-valued mappings and $h$-$\eta$-maximal monotone mappings. By using resolvent operator technique of $h$-$\eta$ maximal monotone mappings, A new iterative algorithm to compute approximate solutions is suggested and analyzed. The convergence of the sequences of approximate solutions generated by the new algorithm to exact solution is also proved. These results are new and improve and generalize many known results in the fields.

## 2. Preliminaries

Let $H$ be a real Hilbert space with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$. Let $2^H$ and $CB(H)$ denote the family of all nonempty subsets of $H$ and the family of all nonempty bounded closed subsets of $H$ respectively. Let $\tilde{H}(\cdot,\cdot)$ denote the Hausdorff metric on $CB(H)$ defined by

$$\tilde{H}(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\},\ \forall\ A,B\in CB(H),$$

where $d(a,B) = \inf_{b\in B} \| a-b \|: b\in B\}$ and $d(A,b) = \inf_{a\in A} \| a-b \|$ .

**Definition 2.1.** [20] A mapping $\eta : H\times H \to H$ is said to be

(i) monotone if

$$\langle x-y, \eta(x,y)\rangle \geq 0,\ \forall x,y\in H;$$

(ii) strictly monotone if $\eta$ is monotone and the equality holds if and only if $x=y$;

(iii) $\alpha_\eta$-strongly monotone if there exists a constant $\alpha_\eta > 0$ such that

$$\langle x-y, \eta(x,y)\rangle \geq \alpha_\eta \|x-y\|^2,\ \forall\ x,y\in H;$$

(iv) $L_\eta$-Lipschitz continuous if there exists a constant $L_\eta > 0$ such that

$$\|\eta(x,y)\| \leq L_\eta \|x-y\|,\ \forall\ x,y\in H.$$

**Definition 2.2.** Let $h : H \to H$ and $\eta : H\times H \to H$ be single-valued mappings. $h$ is said to be

(i) $\eta$-monotone if

$$\langle h(x)-h(y), \eta(x,y)\rangle \geq 0,\ \forall\ x,y\in H;$$

(ii) $\eta$-strictly monotone if $h$ is $\eta$-monotone and

$$\langle h(x) - h(y), \eta(x, y)\rangle = 0 \text{ if and only if } x = y;$$

(iii) $\alpha_h$-$\eta$-strongly monotone if there exists a constant $\alpha_h > 0$ such that

$$\langle h(x) - h(y), \eta(x, y)\rangle \geq \alpha_h \|x - y\|^2, \ \forall\ x, y \in H;$$

(iv) Lipschitz continuous if there exists a constant $L_h > 0$ such that

$$\|h(x) - h(y)\| \leq L_h \|x - y\|, \ \forall\ x, y \in H.$$

It is clear that if $h = I$, the identity mapping on $H$, then the notions (i), (ii) and (iii) in definition 2.2 reduce to (i), (ii) and (iii) in Definition 2.1, respectively.

**Definition 2.3.** [17] Let $\eta : H \times H \to H$ be a single-valued mapping and $M : H \to 2^H$ be a set-valued mapping. $M$ is said to be

(i) $\eta$-monotone if

$$\langle u - v, \eta(x, y)\rangle \geq 0, \ \forall\ x, y \in H, \ u \in M(x), \ v \in M(y);$$

(ii) $\eta$-maximal monotone if $M$ is $\eta$-monotone and $(I + \rho M)(H) = H$ for all $\rho > 0$, where $I$ is the identity mapping on $H$.

**Definition 2.4.** Let $h : H \to H$ and $\eta : H \times H \to H$ be single-valued mappings and $M : H \to 2^H$ be a set-valued mapping. $M$ is said to be $h$-$\eta$-maximal monotone if $M$ is $\eta$-monotone and $(h + \rho M)(H) = H$ for all $\rho > 0$.

**Remark 2.1.** It is clear that if $h = I$, the identity mapping, the concept of $I$-$\eta$-maximal monotone mapping coincides with that of $\eta$-maximal monotone mapping defined in [17]. If $\eta(x, y) = x - y$ for all $x, y \in H$, the concept of $h$-$\eta$-maximal monotone mapping reduces to that of $h$-maximal monotone mapping (which is called $h$-monotone operator in [11]).

**Definition 2.5.** [7,8,17] Let $\eta : H \times H \to H$ and $\varphi : H \to \mathbf{R} \bigcup \{+\infty\}$ be a proper functional. $\varphi$ is said to be $\eta$-subdifferentiable at a point $x \in H$ if there exists a point $f^* \in H$ such that

$$\varphi(y) - \varphi(x) \geq \langle f^*, \eta(y, x)\rangle, \ \forall\ y \in H,$$

where $f^*$ is called a $\eta$-subdifferential of $\varphi$ at $x$. The set of all $\eta$-subdifferential of $\varphi$ at $x$ is denoted by $\Delta\varphi(x)$. The mapping $\Delta\varphi : H \to 2^H$ defined by

$$\Delta\varphi(x) = \{f^* \in H : \varphi(y) - \varphi(x) \geq \langle f^*, \eta(y, x)\rangle, \ \forall\ y \in H\}$$

is said to be $\eta$-subdifferential of $\varphi$ at $x$.

**Proposition 2.1.** [7,8,17] *Let $\eta : H \times H \to H$ be continuous and strongly monotone with constant $r > 0$ such that $\eta(x,y) = -\eta(y,x)$ for all $x, y \in H$, and for any given $x \in H$, the function $h(y,u) = \langle x - u, \eta(y,u)\rangle$ is 0-diagonally quasi-concave in $y$ (see [7]) . Let $\varphi : H \to \mathbf{R} \bigcup \{+\infty\}$ be a lower semicontinuous $\eta$-subdifferentiable proper functional. Then $\Delta\varphi : H \to 2^{H}$ is $\eta$-maximal monotone, i.e., for any $\rho > 0$, $(I + \rho\Delta\varphi)(H) = H$.*

**Proposition 2.2.** *Let $h : H \to H$ be a $\eta$-strictly monotone mapping and $M : H \to 2^H$ be $h$-$\eta$-maximal monotone. Then $M$ is $\eta$-maximal monotone.*

*Proof.* Since $M$ is $\eta$-monotone, by Proposition 1 of Lee et al. [17], it is sufficient to prove

$$\langle u - v, \eta(x,y)\rangle \geq 0,\ \forall\ (y,v) \in Gr(M) \text{ implies } u \in M(x),$$

where $Gr(M) = \{(x,u) \in H \times H : u \in M(x)\}$ denotes the graph of $M$. Suppose that $M$ is not maximal $\eta$-monotone, then there exists $(x_0, u_0) \notin Gr(M)$ such that

$$\langle u_0 - v, \eta(x_0, y)\rangle \geq 0,\ \forall\ (y,v) \in Gr(M).$$

By assumption that for any $\rho > 0$, $(h + \rho M)(H) = H$, there exists $(x_1, u_1) \in Gr(M)$ such that

$$h(x_1) + \rho u_1 = h(x_0) + \rho u_0.$$

It follows that

$$\rho\langle u_0 - u_1, \eta(x_0, x_1)\rangle = -\langle h(x_0) - h(x_1), \eta(x_0, x_1)\rangle \geq 0.$$

Since $h$ is strict $\eta$-monotone, we must have $x_0 = x_1$ and so $u_0 = u_1$. Hence $(x_0, u_0) \in Gr(M)$ which is a contradiction. Therefore $M$ is $\eta$-maximal monotone.

**Remark 2.2.** Proposition 2.2 unifies and generalizes Proposition 2.1 of Fang and Huang [11].

**Example 1.** Let $H = \mathbf{R}$, and $h$, $\eta$ and $M$ be defined as follows: $h(x) = x^3$, $\eta(x,y) = x^2 - y^2$, $M(x) = \{x^2\}$, $\forall\ x, y \in \mathbf{R}$. Then $M$ is $h$-$\eta$-maximal monotone, but not $\eta$-maximal monotone.

*Proof.* By the definitions of $h$,$\eta$ and $M$, we have

$$\langle M(x) - M(y), \eta(x,y)\rangle = (x^2 - y^2)^2 \geq 0,\ \forall\ x, y \in \mathbf{R}.$$

Hence $M$ is $\eta$-monotone. Since for any $\rho > 0$, we have

$$(h + \rho M)(x) = x^3 + \rho x^2, \ \forall \ x \in \mathbf{R}.$$

It is easy to see that $(h + \rho M)(\mathbf{R}) = \mathbf{R}$. So $M$ is $h$-$\eta$-maximal monotone. Since $(I + M)(x) = x + x^2 \in [-\frac{1}{4}, +\infty)$ for all $x \in \mathbf{R}$, therefore $(I + M)(\mathbf{R}) \neq \mathbf{R}$ and hence $M$ is not $\eta$-maximal monotone.

**Example 2.** Let $H = \mathbf{R}$, and $h$, $\eta$ and $M$ be defined as follows: $h(x) = x^2, \ \eta(x, y) = |xy|(x - y), \ M(x) = \{x\}, \ \forall \ x, y \in \mathbf{R}$. Then $M$ is $\eta$-maximal monotone, but not $h$-$\eta$-maximal monotone.

*Proof.* By the definitions of $h$, $\eta$ and $M$, we have

$$\langle M(x) - M(y), \eta(x, y)\rangle = |xy|(x - y)^2 \geq 0, \ \forall \ x, y \in \mathbf{R}.$$

Hence $M$ is $\eta$-monotone. For all $\rho > 0$, we have

$$(I + \rho M)(x) = (1 + \rho)x, \ \forall \ x \in \mathbf{R}.$$

It is easy to see that $(I + \rho M)(\mathbf{R}) = \mathbf{R}$ and hence $M$ is $\eta$-maximal monotone. Since

$$(h + M)(x) = x^2 + x \in [-\frac{1}{4}, +\infty), \ \forall \ x \in \mathbf{R},$$

we have $(h + M)(\mathbf{R}) \neq \mathbf{R}$ and so $M$ is not $h$-$\eta$-maximal monotone.

**Theorem 2.1.** *Let $\eta : H \times H \to H$. Let $h : H \to H$ be $\eta$-strictly monotone and $M : H \to 2^H$ is $h$-$\eta$-maximal monotone. Then for any $\rho > 0$, the inverse operator $(h + \rho M)^{-1} : H \to H$ is single-valued.*

*Proof.* For any given $u \in H$, let $x, y \in (h + \rho M)^{-1}(u)$. Then we have

$$u - h(x) \in \rho M(x) \text{ and } u - h(y) \in \rho M(y).$$

Since $M$ is $\eta$-monotone, we have

$$0 \leq \langle u - h(x) - (u - h(y)), \eta(x, y)\rangle = -\langle h(x) - h(y), \eta(x, y)\rangle.$$

It follows from the strict $\eta$-monotonicity of $h$ that $x = y$. Therefore $(h + \rho M)^{-1}$ a single-valued mapping.

**Remark 2.3.** If $\eta(x, y) = x - y$ for all $x, y \in H$, then Theorem 2.1 reduces to Theorem 2.1 of Fang and Huang [11].

**Definition 2.6.** Let $\eta : H \times H \to H$ and let $h : H \to H$ be $\eta$-strictly monotone and $M : H \to 2^H$ be $h$-$\eta$-maximal monotone. The resolvent operator $R^h_{M,\rho} : H \to H$ of $M$ is defined by

$$R^h_{M,\rho}(x) = (h + \rho M)^{-1}(x),\ \forall\, x \in H.$$

**Theorem 2.2.** *Let $\eta : H \times H \to H$ be $L_\eta$-Lipschitz continuous, $h : H \to H$ be $\alpha_h$- $\eta$-strongly monotone and $M : H \to 2^H$ be $h$-$\eta$-maximal monotone. Then the resolvent operator $R^h_{M,\rho}$ of $M$ is $\frac{L_\eta}{\alpha_h}$-Lipschitz continuous, i.e.,*

$$\|R^h_{M,\rho}(x) - R^h_{M,\rho}(y)\| \leq \frac{L_\eta}{\alpha_h}\|x - y\|,\ \forall\, x, y \in H.$$

*Proof.* By the definition of the resolvent operator $R^h_{M,\rho}$ of $M$, for any $x, y \in H$, we have

$$R^h_{M,\rho}(x) = (h + \rho M)^{-1}(x) \text{ and } R^h_{M,\rho}(y) = (h + \rho M)^{-1}(y).$$

It follows that

$$\frac{1}{\rho}(x - h(R^h_{M,\rho}(x))) \in M(R^h_{M,\rho}(x)) \text{ and } \frac{1}{\rho}(y - h(R^h_{M,\rho}(y))) \in M(R^h_{M,\rho}(y)).$$

Since $M$ is $\eta$-monotone, we have

$$\langle x - h(R^h_{M,\rho}(x)) - (y - h(R^h_{M,\rho}(y))), \eta(R^h_{M,\rho}(x), R^h_{M,\rho}(y))\rangle \geq 0.$$

It follows that

$$\begin{aligned} L_\eta\|x - y\|\|R^h_{M,\rho}(x) - R^h_{M,\rho}(y)\| &\geq \|x - y\|\|\eta(R^h_{M,\rho}(x), R^h_{M,\rho}(y))\| \\ &\geq\ \langle x - y, \eta(R^h_{M,\rho}(x), R^h_{M,\rho}(y))\rangle \\ &\geq\ \langle h(R^h_{M,\rho}(x)) - h(R^h_{M,\rho}(y)), \eta(R^h_{M,\rho}(x), R^h_{M,\rho}(y))\rangle \\ &\geq\ \alpha_h\|R^h_{M,\rho}(x) - R^h_{M,\rho}(y)\|^2. \end{aligned}$$

It follows that

$$\|R^h_{M,\rho}(x) - R^h_{M,\rho}(y)\| \leq \frac{L_\eta}{\alpha_h}\|x - y\|,\ \forall\, x, y \in H.$$

**Remark 2.4.** If $\eta(x, y) = x - y$ for all $x, y \in H$, then Theorem 2.2 reduces to Theorem 2.2 of Fang and Huang [11].

**Definition 2.7.** Let $N : H \times H \to H$ be a single-valued mapping and $A : H \to CB(H)$ be set-valued mappings.

(i) $N(\cdot,\cdot)$ is said to be $\alpha_N$-strongly monotone with respect to $A$ in first argument if there exists $\alpha_N > 0$ such that

$$\langle N(u_1,v) - N(u_2,v), x_1 - x_2\rangle \geq \alpha_N\|x_1 - x_2\|^2,\ \forall\ x_1, x_2,$$
$$v \in H,\ u_1 \in A(x_1),\ u_2 \in A(x_2),$$

(ii) $N(\cdot,\cdot)$ is said to be $L_N$-Lipschitz continuous in first argument if there exists $L_N > 0$ such that

$$\|N(u_1,v) - N(u_2,v)\| \leq L_N\|u_1 - u_2\|,\ \forall\ u_1, u_2, v \in H,$$

(iii) (Ref. 24) $N(\cdot,\cdot)$ is said to be mixed Lipschitz continuous with respect to first and second arguments if there exist $r_1, r_2 > 0$ such that

$$\|N(u_1,v_1)-N(u_2,v_2)\| \leq r_1\|u_1-u_2\|+r_2\|v_1-v_2\|,\ \forall\ u_i, v_i \in H,\ i = 1,2,$$

(iv) $A$ is said to be $L_A$-Lipschitz continuous if there exists $L_A > 0$ such that

$$\tilde{H}(A(x), A(y)) \leq \mathrm{L}_A\|x - y\|,\ \forall x, y \in H.$$

Let $h_1, h_2, : H \to H$ and $N_1, N_2, \eta_1, \eta_2, Q : H\times H \to H$ be single-valued mappings, and $M_1 : H\times H \to 2^H$ and $M_2 : H\times H \to 2^H$ be $h_1$-$\eta_1$-maximal monotone and $h_2$-$\eta_2$-maximal monotone in first argument respectively. Let $A, B, C, D, E : H \to CB(H)$ be set-valued mappings.

Throughout this paper, we will consider the following system of generalized mixed implicit quasi-variational-like inclusion problems ($SGMIQVLIP$): find $(\hat{x},\hat{y}) \in H \times H$, $\hat{u} \in A(\hat{x})$, $\hat{v} \in B(\hat{y})$, $\hat{w} \in C(\hat{y})$, $\hat{d} \in D(\hat{x})$ and $\hat{e} \in E(\hat{x})$ such that

$$\begin{cases} 0 \in h_1(\hat{x}) - \hat{x} + \rho N_1(\hat{u},\hat{v}) + \rho Q(\hat{x},\hat{y}) + \rho M_1(\hat{x},\hat{w}), \\ 0 \in h_2(\hat{y}) - \gamma N_2(\hat{d},\hat{x}) + \gamma M_2(\hat{y},\hat{e}), \end{cases} \tag{2.1}$$

where $\rho > 0$ and $\gamma > 0$ are two constants.

**Special Cases**

(I) If $h_2$ is the identity mapping, $N_2(d,x) = \frac{x}{\gamma}$ for all $d, x \in H$, $M_2 \equiv 0$, and $Q(x,y) = \frac{x}{\rho}$, then the $SGMIQVLIP$ (2.1) reduces to the following generalized mixed implicit quasi-variational-like inclusion problem ( $GMIQVLIP$ ): find $\hat{x} \in H$, $\hat{u} \in A(\hat{x})$, $\hat{v} \in B(\hat{x})$ and $\hat{w} \in C(\hat{x})$ such that

$$0 \in h_1(\hat{x}) + \rho N_1(\hat{u},\hat{v}) + \rho M_1(\hat{x},\hat{w}), \tag{2.2}$$

where $\rho > 0$ is a constant.

Clearly, the variational inclusion (3.1) studied by Fang and Huang (Ref. 11) is very special case of the $GMIQVLIP$ (2.2). The problem (2.2) includes many generalized mixed implicit quasi-variational inclusions, generalized mixed implicit quasi-variational inequalities as special cases.

(II) If $h_1, h_2$ are both the identity mapping, $N_2(d,x) = \frac{x}{\gamma}$ for all $d, x \in H$, $M_2 \equiv 0$, $\eta_1(x,y) = x - y$ for all $x, y \in H$, and $M_1$ is maximal monotone, then the $GMIQVLIP$ (2.1) reduces to the following multivalued quasi-variational inclusion problem ( $MQVIP$ ): find $\hat{x} \in H$, $\hat{u} \in A(\hat{x})$, $\hat{v} \in B(\hat{x})$ and $\hat{w} \in C(\hat{x})$ such that

$$0 \in N_1(\hat{u}, \hat{v}) + Q(\hat{x}, \hat{x}) + M_1(\hat{x}, \hat{w}). \tag{2.3}$$

The $MQVIP$ (2.3) with $C$ being single-valued mapping and $Q \equiv 0$ was introduced and studied by Moudafi and Noor [18] which has many important applications in pure and applied sciences. Indeed, a number of problems arising in structural analysis, mechanics, composite problems, and economics, see, for example [1-18,20-25].

(III) If $h_1$ and $h_2$ are both the identity mapping, and $M_1$ and $M_2$ are $\eta_1$-maximal monotone and $\eta_2$-maximal momotone in first argument respectively, then the $SGMIQVLIP$ (2.1) reduces to the following $SGMIQVLIP$: find $(\hat{x}, \hat{y}) \in H \times H$, $\hat{u} \in A(\hat{x})$, $\hat{v} \in B(\hat{y})$, $\hat{w} \in C(\hat{y})$, $\hat{d} \in D(\hat{x})$ and $\hat{e} \in E(\hat{x})$ such that

$$\begin{cases} 0 \in N_1(\hat{u}, \hat{v}) + Q(\hat{x}, \hat{y}) + M_1(\hat{x}, \hat{w}), \\ 0 \in \hat{y} - \gamma N_2(\hat{d}, \hat{x}) + \gamma M_2(\hat{y}, \hat{e}), \end{cases} \tag{2.4}$$

where $\gamma > 0$ is a constant.

(IV) If $C$ and $E$ are both single-valued mappings, then the $SGMIQVLIP$ (2.4) is equivalent to the following system of generalized mixed quasi-variational-like inclusion problems ( $SGMQVLIP$ ) : find $(\hat{x}, \hat{y}) \in H \times H$, $\hat{u} \in A(\hat{x})$, $\hat{v} \in B(\hat{y})$ and $\hat{d} \in D(\hat{x})$ such that

$$\begin{cases} 0 \in N_1(\hat{u}, \hat{v}) + Q(\hat{x}, \hat{y}) + M_1(\hat{x}, C(\hat{y})), \\ 0 \in \hat{y} - \gamma N_2(\hat{d}, \hat{x}) + \gamma M_2(\hat{y}, E(\hat{x})), \end{cases} \tag{2.5}$$

where $\gamma > 0$ is a constant.

(V) Let $\varphi : H \times H \to \mathbf{R} \bigcup \{+\infty\}$ and $\psi : H \times H \to \mathbf{R} \bigcup \{+\infty\}$ be such that for each fixed $w \in H$, $\varphi(\cdot, w)$ and $\psi(\cdot, w)$ are both lower semicontinuous, and are

$\eta_1$-subdiffrentiable and $\eta_2$-sundifferentiable proper functionals respectively where $\eta_1$ and $\eta_2$ satisfy the conditions in Proposition 2.1. Then, for each fixed $w \in H$, $\Delta\varphi(\cdot, w)$ and $\Delta\psi(\cdot, w)$ are $\eta_1$-maximal monotone and $\eta_2$-maximal monotone respectively by Proposition 2.1. Let $M_1(\cdot, w) = \Delta\varphi(\cdot, w)$ and $M_2(\cdot, e) = \Delta\psi(\cdot, e)$ for all $w, e \in H$ respectively. By Definition 2.5, it is easy to see that the $SGMIQVLIP$ (2.4) reduces to the following system of generalized mixed implicit quasi-variational-like inclusion problems: find $(\hat{x}, \hat{y}) \in H \times H$, $\hat{u} \in A(\hat{x})$, $\hat{v} \in B(\hat{y})$, $\hat{w} \in C(\hat{y})$, $\hat{d} \in D(\hat{x})$ and $\hat{e} \in E(\hat{x})$ such that

$$(2.6) \qquad \begin{cases} \langle N_1(\hat{u}, \hat{v}) + Q(\hat{x}, \hat{y}), \eta_1(x, \hat{x})\rangle \geq \varphi(\hat{x}, \hat{w}) - \varphi(x, \hat{w}), \ \forall\ x \in H, \\ \langle \hat{y} - \gamma N_2(\hat{d}, \hat{x}), \eta_2(y, \hat{y})\rangle \geq \gamma\psi(\hat{y}, \hat{e}) - \gamma\psi(y, \hat{e}), \ \forall\ y \in H, \end{cases}$$

where $\gamma > 0$ is a constant.

(VI) If $C(y) = \{y\}$ and $E(x) = \{x\}$ for all $y, x \in H$, then the $SGMIQVIP$ (2.6) reduces to the following system of generalized mixed quasi-variational-like inclusion problem: find $(\hat{x}, \hat{y}) \in H \times H$, $\hat{u} \in A(\hat{x})$, $\hat{v} \in B(\hat{y})$ and $\hat{d} \in D(\hat{x})$ such that

$$(2.7) \qquad \begin{cases} \langle N_1(\hat{u}, \hat{v}) + Q(\hat{x}, \hat{y}), \eta_1(x, \hat{x})\rangle \geq \varphi(\hat{x}, \hat{y}) - \varphi(x, \hat{y}), \ \forall\ x \in H, \\ \langle \hat{y} - \gamma N_2(\hat{d}, \hat{x}), \eta_2(y, \hat{y})\rangle \geq \gamma\psi(\hat{y}, \hat{x}) - \gamma\psi(y, \hat{x}), \ \forall\ y \in H, \end{cases}$$

where $\gamma > 0$ is a constant.

(VII) If $K_1 : H \to 2^H$ and $K_2 : H \to 2^H$ are two set-valued mappings such that for each $x, y \in H$, $K_1(y)$ and $K_2(x)$ are both closed convex subsets of $H$, $\eta_1(x, y) = \eta_2(x, y) = x - y$ for all $x, y \in H$, and $\varphi(\cdot, y) = I_{K_1(y)}(\cdot)$ and $\psi(\cdot, x) = I_{K_2(x)}(\cdot)$ are the indicator function of $K_1(y)$ and $K_2(x)$ respectively, i.e.,

$$I_{K_1(y)}(u) = \begin{cases} 0, & \text{if } u \in K_1(y), \\ +\infty, & \text{otherwise}, \end{cases}$$

$$I_{K_2(x)}(u) = \begin{cases} 0, & \text{if } u \in K_2(x), \\ +\infty, & \text{otherwise}, \end{cases}$$

then the $SGMQVLIP$ (2.7) reduces to the following system of generalized nonlinear mixed quasi-variational inequality problems: find $(\hat{x}, \hat{y}) \in K_1(\hat{y}) \times K_2(\hat{x})$, $\hat{u} \in A(\hat{x})$, $\hat{v} \in B(\hat{y})$ and $\hat{d} \in D(\hat{x})$ such that

$$(2.8) \qquad \begin{cases} \langle N_1(\hat{u}, \hat{v}) + Q(\hat{x}, \hat{y}), x - \hat{x}\rangle \geq 0, \ \forall\ x \in K_1(\hat{y}), \\ \langle \hat{y} - \gamma N_2(\hat{d}, \hat{x}), y - \hat{y}\rangle \geq 0, \ \forall\ y \in K_2(\hat{x}), \end{cases}$$

where $\gamma > 0$ is a constant.

In brief, for appropriate and suitable choices of $h_1, h_2$, $\eta_1, \eta_2$, $A, B, C, D, E$, $M_1$ and $M_2$, it is easy to see that the $SGMIQVLIP$ (2.1) includes a number of systems of generalized mixed implicit quasi-variational-like inclusions, systems of generalized mixed implicit quasi-variational-like inequalities, generalized mixed implicit quasi-variational-like inclusions and generalized mixed quasi-variational-like inequalities studied by many authors as special cases, for example, see [1-18,20-28].

From the definition of resolvent operator of $h$-$\eta$-maximal monotone mapping, we have the following result.

**Theorem 2.3** $(\hat{x}, \hat{y}) \in H \times H$, $\hat{u} \in A(\hat{x})$, $\hat{v} \in B(\hat{y})$, $\bar{w} \in C(\hat{y})$, $\hat{d} \in D(\hat{x})$ and $\hat{e} \in E(\hat{x})$ is a solution of the $SGMIQVLIP$ (2.1) if and only if

$$\hat{x} = R^{h_1}_{M_1(\cdot,\hat{w}),\rho}[\hat{x} - \rho N_1(\hat{u}, \hat{v}) - \rho Q(\hat{x}, \hat{y})] \text{ and } \hat{y} = R^{h_2}_{M_2(\cdot,\hat{e}),\gamma}[\gamma N_2(\hat{d}, \hat{x})],$$

where $R^{h_1}_{M_1(\cdot,\hat{w}),\rho}(u) = (h_1(\cdot) + \rho M(\cdot, \hat{w}))^{-1}(u)$ and $R^{h_2}_{M_2(\cdot,\hat{e}),\gamma}(u) = (h_2(\cdot) + \gamma(M_2(\cdot, \hat{e}))^{-1}(u)$.

## 3. EXISTENCE AND ALGORITHM

In order to compute approximate solutions of the $SGMIQVLIP$ (2.1), we suggest the following iterative algorithm.

**Algorithm 3.1** For any given $x_0 \in H$ and $\lambda \in (0, 1]$, take any $d_0 \in D(x_0)$ and $e_0 \in E(x_0)$. Let

$$y_0 = R^{h_2}_{M_2(\cdot,e_0),\gamma}[\gamma N_2(d_0, x_0)]. \tag{3.1}$$

Take any $u_0 \in A(x_0)$, $v_0 \in B(y_0)$ and $w_0 \in C(y_0)$. Let

$$x_1 = (1 - \lambda)x_0 + \lambda R^{h_1}_{M_1(\cdot,w_0),\rho}[x_0 - \rho N_1(u_0, v_0) - \rho Q(x_0, y_0)]. \tag{3.2}$$

By Nadler [19], there $u_1 \in A(x_1)$, $d_1 \in D(x_1)$ and $e_1 \in E(x_1)$ such that

$$\|u_1 - u_0\| \leq (1 + 1)\tilde{H}(A(x_1), A(x_0)),$$
$$\|d_1 - d_0\| \leq (1 + 1)\tilde{H}(D(x_1), D(x_0)),$$
$$\|e_1 - e_0\| \leq (1 + 1)\tilde{H}(E(x_1), E(x_0)).$$

Let

$$y_1 = R^{h_2}_{M_2(\cdot,e_1),\gamma'}[\gamma N_2(d_1, x_1)], \tag{3.3}$$

By Nadler [19], there exist $v_1 \in B(y_1)$ and $w_1 \in C(y_1)$ such that

$$\|v_1 - v_0\| \le (1+1)\tilde{H}(B(y_1), B(y_0)),$$

$$\|w_1 - w_0\| \le (1+1)\tilde{H}(C(y_1), C(y_0)).$$

Following this way, we can define sequences $\{x_n\},\ \{y_n\},\ \{u_n\},\ \{v_n\},\ \{w_n\},\ \{d_n\}$ and $\{e_n\}$ as follows:

$$(3.4)\quad \begin{cases} u_n \in A(x_n),\ v_n \in B(y_n),\ w_n \in C(y_n),\ d_n \in D(x_n),\ e_n \in E(x_n), \\ \|u_{n+1} - u_n\| \le (1+\frac{1}{n+1})\tilde{H}(A(x_{n+1}), A(x_n)), \\ \|v_{n+1} - v_n\| \le (1+\frac{1}{n+1})\tilde{H}(B(y_{n+1}), B(y_n)), \\ \|w_{n+1} - w_n\| \le (1+\frac{1}{n+1})\tilde{H}(C(y_{n+1}), C(y_n)), \\ \|d_{n+1} - d_n\| \le (1+\frac{1}{n+1})\tilde{H}(D(x_{n+1}), D(x_n)), \\ \|e_{n+1} - e_n\| \le (1+\frac{1}{n+1})\tilde{H}(E(x_{n+1}), E(x_n)), \\ y_n = R^{h_2}_{M_2(\cdot,e_n),\gamma}[\gamma N_2(d_n, x_n)], \\ x_{n+1} = (1-\lambda)x_n + \lambda R^{h_1}_{M_1(\cdot,w_n),\rho}[x_n - \rho N_1(u_n, v_n) - \rho Q(x_n, y_n)], \\ n = 0, 1, 2, \cdots . \end{cases}$$

**Theorem 3.1** *Let $\eta_1, \eta_2 : H \times H \to H$ be $L_{\eta_1}$-Lipshitz continuous and $L_{\eta_2}$-Lipschitz continuous respectively. Let $h_1, h_2 : H \to H$ be $\alpha_{h_1}$-$\eta_1$-strongly monotone and $\alpha_{h_2}$-$\eta_2$-strongly monotone respectively. Let $M_1, M_2 : H \times H \to 2^H$ be $h_1$-$\eta_1$-maximal monotone and $h_2$-$\eta_2$-maximal monotone in first argument respectively. Let $A, B, C, D, E : H \to CB(H)$ be Lipachitz continuous with Lipschitz constants $L_A$, $L_B$, $L_C$, $L_D$ and $L_E$. Let $N_1, N_2, Q : H \times H \to H$ be such that*

(i) $N_1(\cdot,\cdot)$ is $\alpha_{N_1}$-strongly monotone in first argument with respect to $A$, $r_1$-Lipschitz continuous in first argument, and $r_2$-Lipschitz continuous in second argument,

(ii) $N_2(\cdot,\cdot)$ is mixed Lipschith continuous with constants $k_1, k_2 > 0$,

(iii) $Q(\cdot,\cdot)$ is mixed Lipschitz continuous with constants $s_1, s_2 > 0$.

Assume that there exist $\rho, \gamma, \delta, \mu > 0$ satisfying

$$(3.5)\qquad \|R^{h_1}_{M_1(\cdot,w_1),\rho}(u) - R^{h_1}_{M(\cdot,w_2),\rho}(u)\| \le \delta\|w_1 - w_2\|,\ \forall\ w_1, w_2, u \in H,$$

$$(3.6)\qquad \|R^{h_2}_{M_2(\cdot,e_1),\gamma}(u) - R^{h_2}_{M_2(\cdot,e_2),\gamma}(u)\| \le \mu\|e_1 - e_2\|,\ \forall\ e_1, e_2, u \in H.$$

If

$$
\text{(3.7)}\quad \begin{cases} \delta L_C < 1,\ r_2L_B + s_1 + s_2 < r_1L_A,\ \alpha_{h_1}(1-\delta L_C) < L_{\eta_1}, \\ \alpha_{N_1} > L_{\eta_1}^{-1}\alpha_{h_1}(1-\delta L_C)(r_2L_B + s_1 + s_2) \\ \quad +\sqrt{[r_1^2L_A^2 - (r_2L_B + s_1 + s_2)^2](1 - L_{\eta_1}^{-2}\alpha_{h_1}^2(1-\delta L_C)^2)}, \\ |\rho - \dfrac{\alpha_{N_1} - L_{\eta_1}^{-1}\alpha_{h_1}(1-\delta L_C)(r_2L_B + s_1 + s_2)}{r_1^2L_A^2 - (r_2L_B + s_2 + s_2)^2}| \\ < \frac{\sqrt{[\alpha_{N_1} - L_{\eta_1}^{-1}(1-\delta L_C)(r_2L_B+s_1+s_2)]^2 - (r_1^2L_A^2 - (r_2L_B+s_1+s_2)^2)(1-L_{\eta_1}^{-2}\alpha_{h_1}^2(1-\delta L_C)^2)}}{r_1^2L_A^2 - (r_2L_B+s_1+s_2)^2}, \\ \mu L_E < 1,\ 0 < \gamma < \dfrac{\alpha_{h_2}(1-\mu L_E)}{L_{\eta_2}(k_1L_D + k_2)}, \end{cases}
$$

Then there exist $(\hat{x}, \hat{y}) \in H \times H$, $\hat{u} \in A(\hat{x})$, $\hat{v} \in B(\hat{y})$, $\hat{w} \in C(\hat{y})$, $\hat{d} \in D(\hat{x})$ and $\hat{e} \in E(\hat{x})$ such that

$$
\text{(3.8)}\qquad \begin{cases} 0 \in h_1(\hat{x}) - \hat{x} + \rho N_1(\hat{u}, \hat{v}) + \rho Q(\hat{x}, \hat{y}) + \rho M_1(\hat{x}, \hat{w}), \\ 0 \in h_2(\hat{y}) - \gamma N_2(\hat{d}, \hat{x}) + \gamma M_2(\hat{y}, \hat{e}), \end{cases}
$$

and the iterative sequences generated by the Algorithm 3.1 satisfy: $x_n \to \hat{x}$, $y_n \to \hat{y}$, $u_n \to \hat{u}$, $v_n \to \hat{v}$, $w_n \to \hat{w}$, $d_n \to \hat{d}$ and $e_n \to \hat{e}$.

*Proof.* For convenience' sake, write

$$
a_n = \|x_n - \rho N_1(u_n, v_n) - \rho Q(x_n, y_n)\|.
$$

By the Algorithm 3.1, (3.5) and Theorem 2.2, we have

$$
\text{(3.9)}\quad \begin{aligned} &\|x_{n+1} - x_n\| \\ &\quad = \|(1-\lambda)x_n + \lambda R^{h_1}_{M_1(\cdot,w_n),\rho}[x_n - \rho N_1(u_n, v_n) - \rho Q(x_n, y_n)] \\ &\quad -(1-\lambda)x_{n-1} - \lambda R^{h_1}_{M_1(\cdot,w_{n-1}),\rho}[x_{n-1} \\ &\quad -\rho N_1(u_{n-1}, v_{n-1}) - \rho Q(x_{n-1}, y_{n-1})]\| \\ &\leq\ (1-\lambda)\|x_n - x_{n-1}\| + \lambda\|R^{h_1}_{M_1(\cdot,w_n),rho}(a_n) - R^{h_1}_{M(\cdot,w_n)\rho}(a_{n-1})\| \\ &\quad +\lambda\|R^{h_1}_{M(\cdot,w_n),\rho}(a_{n-1}) - R^{h_1}_{M(\cdot,w_{n-1}),\rho}(a_{n-1})\| \\ &\leq\ (1-\lambda)\|x_n - x_{n-1}\| + \tfrac{\lambda L_{\eta_1}}{\alpha_{h_1}}\|a_n - a_{n-1}\| + \lambda\delta\|w_n - w_{n-1}\| \end{aligned}
$$

$$
\begin{aligned}
&\|a_n - a_{n-1}\| \\
\leq\ & \|x_n - \rho N_1(u_n, v_n) - \rho Q(x_n, y_n) \\
&-(x_{n-1} - \rho N_1(u_{n-1}, v_{n-1}) - \rho Q(x_{n-1}, y_{n-1}))\| \\
\leq\ & \|x_n - x_{n-1} - \rho(N_1(u_n, v_n) - N_1(u_{n-1}, v_n))\| \\
&+\rho\|N_1(u_{n-1}, v_n) - N_1(u_{n-1}, v_{n-1})\| + \rho\|Q(x_n, y_n) - Q(x_{n-1}, y_{n-1})\|.
\end{aligned} \tag{3.10}
$$

Since $N_1(\cdot,\cdot)$ is $\alpha_{N_1}$-strongly monotone in first argument with respect to $A$ and $r_1$-Lipschitz continuous in first argument, and $A$ is $L_A$-Lipschitz continuous, by (3.4), we have

$$
\begin{aligned}
&\|x_n - x_{n-1} - \rho(N_1(u_n, v_n) - N_1(u_{n-1}, v_n))\|^2 \\
&= \|x_n - x_{n-1}\|^2 - 2\rho\langle N_1(u_n, v_n) - N_1(u_{n-1}, v_n), x_n - x_{n-1}\rangle \\
&\quad +\rho^2\|N_1(u_n, v_n) - N_1(u_{n-1}, v_n)\|^2 \\
&\leq \|x_n - x_{n-1}\|^2 - 2\rho\alpha_{N_1}\|x_n - x_{n-1}\|^2 + \rho^2 r_1^2\|u_n - u_{n-1}\|^2 \\
&\leq (1 - 2\rho\alpha_{N_1})\|x_n - x_{n-1}\|^2 + \rho^2 r_1^2(1 + \tfrac{1}{n})^2[\tilde{H}(A(x_n), A(x_{n-1}))]^2 \\
&\leq [1 - 2\rho\alpha_{N_1} + \rho^2 r_1^2 L_A^2(1 + \tfrac{1}{n})^2]\|x_n - x_{n-1}\|^2.
\end{aligned} \tag{3.11}
$$

Since $N_1(\cdot,\cdot)$ is $r_2$-Lipschitz continuous in second argument and $B$ is $L_B$-Lipschitz continuous, by (3.4), we have

$$
\begin{aligned}
&\|N_1(u_{n-1}, v_n) - N_1(u_{n-1}, v_{n-1})\| \\
&\leq r_2\|v_n - v_{n-1}\| \leq r_2 L_B(1 + \tfrac{1}{n})\|y_n - y_{n-1}\|.
\end{aligned} \tag{3.12}
$$

Since $Q(\cdot,\cdot)$ is mixed Lipschitz continuous with constants $s_1$ and $s_2$, we have

$$
\|Q(x_n, y_n) - Q(x_{n-1}, y_{n-1})\| \leq s_1\|x_n - x_{n-1}\| + s_2\|y_n - y_{n-1})\|. \tag{3.13}
$$

From (3.10)-(3.13) it follows that

$$
\begin{aligned}
\|a_n - a_{n-1}\| \leq\ & [\sqrt{1 - 2\rho\alpha_{N_1} + \rho^2 r_1^2 L_A^2(1 + \tfrac{1}{n})^2} + \rho s_1]\|x_n - x_{n-1}\| \\
&+(\rho r_2 L_B(1 + \tfrac{1}{n}) + \rho s_2)\|y_n - y_{n-1}\|.
\end{aligned} \tag{3.14}
$$

Since $C$ is $L_C$-Lipschitz continuous, by (3.4), we have

$$
\|w_n - w_{n-1}\| \leq (1 + \frac{1}{n})\tilde{H}(C(y_n), C(y_{n-1})) \leq L_C(1 + \frac{1}{n})\|y_n - y_{n-1}\|. \tag{3.15}
$$

By (3.9),(3.14) and (3.15), we obtain

$$
\begin{aligned}
&\|x_{n+1} - x_n\| \leq (1 - \lambda)\|x_n - x_{n-1}\| + \tfrac{\lambda L_{\eta_1}}{\alpha_{h_1}} \\
&\quad (\sqrt{1 - 2\rho\alpha_{N_1} + \rho^2 r_1^2 L_A^2(1 + \tfrac{1}{n})^2} + \rho s_1)\|x_n - x_{n-1}\| \\
&\quad +(\tfrac{\lambda L_{\eta_1}}{\alpha_{h_1}}(\rho r_2 L_B(1 + \tfrac{1}{n}) + \rho s_2) + \lambda\delta L_C(1 + \tfrac{1}{n}))\|y_n - y_{n-1}\|.
\end{aligned} \tag{3.16}
$$

Note that $N_2$ is mixed Lipschitz continuous with constants $k_1, k_2 > 0$, $D$ and $E$ are Lipschitz continuous, by(3.4) and (3.6), we have

$$
\begin{aligned}
\|y_n - y_{n-1}\| &= \|R^{h_2}_{M_2(\cdot,e_n),\gamma}[\gamma N_2(d_n, x_n)] \\
&\quad -R^{h_2}_{M_2(\cdot,e_{n-1}),\gamma}[\gamma N_2(d_{n-1}, x_{n-1})]\| \\
&\leq \|R^{h_2}_{M_2(\cdot,e_n),\gamma}[\gamma N_2(d_n, x_n)] \\
&\quad -R^{h_2}_{M_2(\cdot,e_n),\gamma}[\gamma N_2(d_{n-1}, x_{n-1})]\| \\
&\quad +\|R^{h_2}_{M_2(\cdot,e_n),\gamma}[\gamma N_2(d_{n-1}, x_{n-1})] \\
&\quad -R^{h_2}_{M_2(\cdot,e_{n-1}),\gamma}[\gamma N_2(d_{n-1}, x_{n-1})]\| \\
&\leq \tfrac{\gamma L_{\eta_2}}{\alpha_{h_2}}(\|N_2(d_n, x_n) \\
&\quad -N_1(d_{n-1}, x_{n-1})\|) + \mu\|e_n - e_{n-1}\| \\
&\leq \tfrac{\gamma L_{\eta_2}}{\alpha_{h_2}}(k_1\|d_n - d_{n-1}\| + k_2\|x_n - x_{n-1}\|) \\
&\quad +\mu(1 + \tfrac{1}{n})\tilde{H}(E(x_n), E(x_{n-1})) \\
&\leq [\tfrac{\gamma L_{\eta_2}}{\alpha_{h_2}}(k_1 L_D(1 + \tfrac{1}{n}) + k_2) \\
&\quad +\mu(1 + \tfrac{1}{n})L_E]\|x_n - x_{n-1}\| \\
&= \sigma_n\|x_n - x_{n-1}\|,
\end{aligned}
\tag{3.17}
$$

where $\sigma_n = \frac{\gamma L_{\eta_2}}{\alpha_{h_2}}(k_1 L_D(1+\frac{1}{n})+k_2)+\mu(1+\frac{1}{n})L_E \to \sigma = \frac{\gamma L_{\eta_2}}{\alpha_{h_2}}(k_1 L_D+k_2)+\mu L_E$ whenever $n \to \infty$. The condition(3.7) implies $\sigma < 1$ and so $\sigma_n < 1$ for sufficient large $n$. By (3.17), for sufficient $n$, we have

$$\|y_n - y_{n-1}\| \leq \sigma_n\|x_n - x_{n-1}\| < \|x_n - x_{n-1}\|. \tag{3.18}$$

It follows from (3.17) and (3.18) that for sufficient large $n$,

$$\|x_{n+1} - x_n\| \leq (1 - (1 - \theta_n)\lambda)\|x_n - x_{n-1}\|,$$

where

$$\theta_n = \frac{L_{\eta_1}}{\alpha_{h_1}}(\sqrt{1 - 2\rho\alpha_{N_1} + \rho^2 r_1^2 L_A^2(1 + \frac{1}{n})^2} + \rho(r_2 L_B(1+\frac{1}{n})+s_1+s_2)) + \delta L_c(1+\frac{1}{n}).$$

Hence we have

$$\theta_n \to \theta = \frac{L_{\eta_1}}{\alpha_{h_1}}(\sqrt{1 - 2\rho\alpha_{N_1} + \rho^2 r_1^2 L_A^2} + \rho(r_2 L_B + s_1 + s_2)) + \delta L_C, \text{ as } n \to \infty.$$

By (3.7), we have $\theta < 1$. So there exists $\theta_0 < 1$ such that for sufficiently large $n$, $\theta_n < \theta_0$ and

$$\|x_{n+1} - x_n\| \leq (1 - (1 - \theta_0)\lambda)\|x_n - x_{n-1}\|.$$

It follows that $\{x_n\}$ is a Cauchy sequence. Let $x_n \to \hat{x}$ as $n \to \infty$. By (3.18), $\{y_n\}$ is also a Cauchy sequence. Let $y_n \to \hat{y}$ as $n \to \infty$. The condition (3.4) and Lipschitz continuity of $A, B, C, D, E$ imply that $\{u_n\}, \{v_n\}, \{w_n\}, \{d_n\}$ and $\{e_n\}$ are all Cauchy sequences. Let $u_n \to \hat{u}$, $v_n \to \hat{v}$, $w_n \to \hat{w}$, $d_n \to \hat{d}$ and $e_n \to \hat{e}$ respectively. By (3.4), we have

$$\begin{aligned} d(\hat{u}, A(\hat{x})) &\leq \|\hat{u} - u_n\| + d(u_n, A(\hat{x})) \leq \|\hat{u} - u_n\| + \tilde{H}(A(x_n), A(\hat{x})) \\ &\leq \|\hat{u} - u_n\| + L_A\|x_n - \hat{x}\| \to 0, \end{aligned}$$

and so $\hat{u} \in A(\hat{x})$. Similarly, we can show that $\hat{v} \in B(\hat{y})$, $\hat{w} \in C(\hat{y})$, $\hat{d} \in D(\hat{x})$ and $\hat{e} \in E(\hat{x})$. By (3.4), we have

$$y_n = R^{h_2}_{M_2(\cdot, e_n), \gamma}[\gamma N_2(d_n, x_n)],$$

$$x_{n+1} = (1 - \lambda)x_n + \lambda R^{h_1}_{M_1(\cdot, w_n), \rho}[x_n - \rho N_1(u_n, v_n) - \rho Q(x_n, y_n)].$$

By Theorem 2.2 and the assumptions in Theorem 3.1, letting $n \to \infty$ in the above equalities, we can obtain

$$\hat{y} = R^{h_2}_{M_2(\cdot, \hat{e}), \gamma}[\gamma N_2(\hat{d}, \hat{x})],$$

$$\hat{x} = R^{h_1}_{M_1(\cdot, \hat{w}), \rho}[\hat{x} - \rho N_1(\hat{u}, \hat{v}) - \rho Q(\hat{x}, \hat{y})].$$

By Theorem 2.3, $(\hat{x}, \hat{y}, \hat{u}, \hat{v}, \hat{w}, \hat{d}, \hat{e})$ is a solution of the $SGMIQVIP$ (2.1). This completes the proof.

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