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APPROXIMATION FOR BASKAKOV-KANTOROVICH-BÉZIER OPERATORS IN THE SPACE $L_p[0,\infty)$

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Abstract. In this paper we give the direct, inverse and equivalence theorem for Baskakov-Kantorovich-Bézier operators in the space $L_p[0,\infty)$ $(1 \le p \le \infty)$ with Ditzian-Totik modulus.

1. Introduction

In computer aided geometric design, the Bézier method is often used to contruct curve and surface. Bézier[2] introduced the Bézier basis functions. Some properties of convergence and approximation for some Bézier-type operators has been studied (cf.[1, 3, 6-10]), but the studies of this kind is insufficient. In this paper we will consider the direct, inverse and equivalence theorems for Baskakov-Kantorovich-Bézier operator(BKB operator). The BKB operator is defined by (cf. [1, 6]): For $f \in L_p[0,\infty)$,

$$V_{n\alpha}(f,x) = n\sum_{k=0}^{\infty}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt\left(J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)\right), \tag{1.1}$$

where $\alpha \ge 1$, $J_{n,k}(x) = \sum_{j=k}^{\infty} v_{n,j}(x)$, $v_{n,j}(x) = \binom{n+j-1}{j}x^j(1+x)^{-n-j}$. When $\alpha = 1$, $V_{n1}(f,x)$ is the well-known Baskakov-Kantorovich operator. For $\alpha \ge 1$, $a^\alpha - b^\alpha \le \alpha(a-b)$ $(1 \ge a \ge b \ge 0)$, we know

$$|V_{n\alpha}(f,x)| \le \alpha\sum_{k=0}^{\infty}\left|n\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt\right| v_{n,k}(x). \tag{1.2}$$

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Noticing that $\int_0^\infty v_{n,k}(x)dx = \frac{1}{n-1}$, one sees that $V_{n\alpha}(f,x)$ is bounded and positive in the space $L_p[0,\infty)$.

To describe our results, we will give the definition of the modulus of smoothness and the K-functional first(cf. [4]).

For $f \in L_p[0,\infty)$ $(1 \le p \le \infty)$, $\varphi(x) = \sqrt{x(1+x)}$,

$$\omega_\varphi(f,t)_p = \sup_{0<h\le t}\left\{\left\| f\left(x+\frac{h\varphi(x)}{2}\right) - f\left(x-\frac{h\varphi(x)}{2}\right)\right\|_p,\ x-\frac{h\varphi(x)}{2} \ge 0\right\},$$

$$K_\varphi(f,t)_p = \inf_{g\in W_p}\left\{\|f-g\|_p + t\|\varphi g'\|_p\right\},$$

$$\overline{K}_\varphi(f,t)_p = \inf_{g\in W_p}\left\{\|f-g\|_p + t\|\varphi g'\|_p + t^2\|g'\|_p\right\},$$

where $W_p = \{f|\ f \in A.C._{loc},\ \|\varphi f'\|_p < \infty,\ \|f'\|_p < \infty\}$.
It is known that [4].

$$\omega_\varphi(f,t)_p \sim K_\varphi(f,t)_p \sim \overline{K}_\varphi(f,t)_p, \tag{1.3}$$

here $a \sim b$ means that there exists $C > 0$ such that $C^{-1}a \le b \le Ca$.

Now we state our equivalence theorem as follows:

Theorem. *For* $f \in L_p[0,\infty)$ $(1 \le p \le \infty)$, $\varphi(x) = \sqrt{x(1+x)}$, $0 < \beta < 1$,

$$\|V_{n\alpha}f - f\|_p = O\left(\left(\frac{1}{\sqrt{n}}\right)^\beta\right) \tag{1.4}$$

$$\Leftrightarrow\ \ \omega_\varphi(f,t)_p = O\left(t^\beta\right). \tag{1.5}$$

Throughout this paper, C denotes a positive constant independent of n and x, but it is not necessarily the same in different cases.

2. Direct Theorem

For convenience, we list some basic properties which will be used later and can be found in [9] or obtained by simple computation.

$$1 = J_{n,0}(x) > J_{n,1}(x) > \cdots > J_{n,k}(x) > J_{n,k+1}(x) > \cdots > 0; \tag{2.1}$$

$$v'_{n,k}(x) = n(v_{n+1,k-1}(x) - v_{n+1,k}(x)),\ v'_{n,0}(x) = -nv_{n+1,0}(x),\ k = 1,2,\cdots; \tag{2.2}$$

$$J'_{n,0}(x) = 0, \;\; J'_{n,k}(x) = nv_{n+1,k-1}(x) > 0, \; k = 1, 2, \cdots ; \tag{2.3}$$

$$v'_{n,k}(x) = \frac{n}{\varphi^2(x)} \left(\frac{k}{n} - x \right) v_{n,k}(x), \;\; x \in (0, \infty); \tag{2.4}$$

$$V_{n1}((t-x)^2, x) \leq \frac{\delta_n^2(x)}{n}, \quad V_{n1}((t-x)^{2r}, x) \leq Cn^{-r} \left(\varphi(x) + \frac{1}{\sqrt{n}} \right)^{2r}, \tag{2.5}$$

where $\delta_n(x) = \varphi(x) + \frac{1}{\sqrt{n}}$.

Now we give the direct theorem.

Theorem 2.1. *For $f \in L_p[0, \infty)$ $(1 \leq p \leq \infty)$, $\varphi(x) = \sqrt{x(1+x)}$, one has*

$$\|V_{n\alpha} f - f\|_p \leq C\omega_\varphi \left(f, \frac{1}{\sqrt{n}} \right)_p . \tag{2.6}$$

Proof. By the definition of $\overline{K}_\varphi(f,t)_p$ and the relation (1.3), for the fixed n, we can choose $g = g_n$ such that

$$\|f - g\|_p + \frac{1}{\sqrt{n}} \|\varphi g'\|_p + \frac{1}{n} \|g'\|_p \leq C\omega_\varphi \left(f, \frac{1}{\sqrt{n}} \right)_p . \tag{2.7}$$

Since

$$\begin{aligned} \|V_{n\alpha} f - f\|_p &\leq \|V_{n\alpha}(f-g)\|_p + \|f-g\|_p + \|V_{n\alpha} g - g\|_p \\ &\leq C \|f-g\|_p + \|V_{n\alpha} g - g\|_p . \end{aligned}$$

All we have to do is to estimate the second term in the above relation. By the Riesz-Thorin theorem, we separate the proof of the assertions for $p = \infty$ and $p = 1$.

I. $p = \infty$**.** We will have to split the estimate into two domains, that is $x \in E_n^c = [0, \frac{1}{n}]$ and $x \in E_n = (\frac{1}{n}, \infty)$.

Note that $g(t) = g(x) + \int_x^t g'(u)du$, we write

$$|V_{n\alpha}(g,x) - g(x)| \leq \left| V_{n\alpha} \left(\int_x^t g'(u)du, x \right) \right| .$$

1°. $x \in E_n = (\frac{1}{n}, \infty)$, $\delta_n(x) \sim \varphi(x)$: By simple computation we have

$$\left| \int_x^t g'(u)du \right| \leq \|\varphi g'\|_\infty \left| \int_x^t \varphi^{-1}(u)du \right| ,$$

$$\left|\int_x^t \varphi^{-1}(u)du\right| \le 2\left(\varphi^{-1}(x)+\frac{1}{\sqrt{x(1+t)}}\right)|t-x|.$$

One has

$$\begin{aligned}|V_{n\alpha}(g,x)-g(x)| &\le 2\|\varphi g'\|_\infty\left[\varphi^{-1}(x)V_{n\alpha}(|t-x|,x)+\frac{1}{\sqrt{x}}V_{n\alpha}\left(\frac{|t-x|}{\sqrt{1+t}},x\right)\right]\\ &=: 2\|\varphi g'\|_\infty\left(M_1+M_2\right).\end{aligned}$$

Using the $H\ddot{o}lder$ inequality and (2.5), we have

$$\begin{aligned}M_1 &= \varphi^{-1}(x)n\sum_{k=0}^{\infty}\int_{\frac{k}{n}}^{\frac{k+1}{n}}|t-x|dt[J_{n,k}^{\alpha}(x)-J_{n,k+1}^{\alpha}(x)]\\ &\le \alpha\varphi^{-1}(x)n\sum_{k=0}^{\infty}\int_{\frac{k}{n}}^{\frac{k+1}{n}}|t-x|dtv_{n,k}(x) \le \alpha\varphi^{-1}(x)\frac{\delta_n(x)}{\sqrt{n}} \le \frac{C}{\sqrt{n}}.\end{aligned}$$

Using the relation [4, pp.141 (9.6.3)]: for any integer m, we have

$$V_{n1}((1+t)^{-m}) \le C(m)(1+x)^{-m}.$$

Similarly, we can estimate M_2:

$$\begin{aligned}M_2 &= x^{-\frac{1}{2}}n\sum_{k=0}^{\infty}\int_{\frac{k}{n}}^{\frac{k+1}{n}}\frac{|t-x|}{\sqrt{1+t}}dt[J_{n,k}^{\alpha}(x)-J_{n,k+1}^{\alpha}(x)]\\ &\le \alpha x^{-\frac{1}{2}}n\sum_{k=0}^{\infty}\int_{\frac{k}{n}}^{\frac{k+1}{n}}\frac{|t-x|}{\sqrt{1+t}}dtv_{n,k}(x)\\ &\le \alpha x^{-\frac{1}{2}}\left(n\sum_{k=0}^{\infty}\int_{\frac{k}{n}}^{\frac{k+1}{n}}|t-x|^2dtv_{n,k}(x)\right)^{\frac{1}{2}}\cdot\left(n\sum_{k=0}^{\infty}\int_{\frac{k}{n}}^{\frac{k+1}{n}}\frac{1}{1+t}dtv_{n,k}(x)\right)^{\frac{1}{2}}\\ &\le \alpha\varphi^{-1}(x)\frac{\delta_n(x)}{\sqrt{n}} \le \frac{C}{\sqrt{n}}.\end{aligned}$$

2°. $x\in E_n^c=[0,\frac{1}{n}]$, $\delta_n(x)\sim\frac{1}{\sqrt{n}}$:

$$\left|\int_x^t g'(u)du\right| \le \|g'\|_\infty|t-x|.$$

Therefore

$$|V_{n\alpha}(g,x)-g(x)| \le C\|g'\|_\infty V_{n\alpha}(|t-x|,x) \le C\|g'\|_\infty\frac{\delta_n(x)}{\sqrt{n}} \le \frac{C}{n}\|\, g'\|_\infty.$$

From 1° and 2°, we get for $p=\infty$

$$\|V_{n\alpha}f-f\|_\infty \le C\omega_\varphi(f,\frac{1}{\sqrt{n}})_\infty. \tag{2.8}$$

II. $p=1$**.** We will estimate it into two domains $x\in E_n^c$ and $x\in E_n$ too.

1°. $x\in E_n^c=[0,\frac{1}{n}]$.

$$\begin{aligned}|V_{n\alpha}(g,x)-g(x)| &\le \alpha\sum_{k=0}^{\infty} n\int_{\frac{k}{n}}^{\frac{k+1}{n}} |\int_x^t g'(u)du|dt v_{n,k}(x)\\ &\le \alpha\sum_{k=0}^{\infty} v_{n,k}(x)n\int_{\frac{k}{n}}^{\frac{k+1}{n}}\left(\varphi^{-1}(x)+\frac{1}{\sqrt{x}}\frac{1}{\sqrt{1+t}}\right)dt\int_0^1|\varphi(u)g'(u)|du \qquad (2.9)\\ &\le \alpha\|\varphi g'\|_1\sum_{k=0}^{\infty}\left(\varphi^{-1}(x)+\frac{1}{\sqrt{x}}\sqrt{\frac{n}{k+n}}\right)v_{n,k}(x).\end{aligned}$$

We obtain

$$\int_{E_n^c}|V_{n\alpha}(g,x)-g(x)|\,dx \le \alpha\|\varphi g'\|_1 \left(\int_0^{\frac{1}{n}}\varphi^{-1}(x)dx+\int_0^{\frac{1}{n}}\frac{1}{\sqrt{x}}\sum_{k=0}^{\infty}\sqrt{\frac{n}{k+n}}v_{n,k}(x)dx\right).$$

Since $\int_0^{\frac{1}{n}}\varphi^{-1}(x)dx\le\frac{2}{\sqrt{n}}$, we can get

$$\int_0^{\frac{1}{n}}\frac{1}{\sqrt{x}}\sum_{k=0}^{\infty}\sqrt{\frac{n}{k+n}}v_{n,k}(x)dx\le\int_0^{\frac{1}{n}}\frac{1}{\sqrt{x}}dx\le\frac{2}{\sqrt{n}}.$$

Then

$$\int_{E_n^c}|V_{n\alpha}(g,x)-g(x)|\,dx\le\frac{C}{\sqrt{n}}\|\varphi g'\|_1. \tag{2.10}$$

2°. $x\in E_n$. By the procedure of the relation (2.9), we can write

$$\begin{aligned}&\int_{E_n}|V_{n\alpha}(g,x)-g(x)|\,dx\\ &\le\alpha\int_{E_n}\sum_{k=0}^{\infty}v_{n,k}(x)n\int_{\frac{k}{n}}^{\frac{k+1}{n}}\left(\varphi^{-1}(x)+\frac{1}{\sqrt{x}}\frac{1}{\sqrt{1+t}}\right)dt \qquad (2.11)\\ &\qquad\left|\int_x^{\frac{k^*}{n}}|\varphi(u)g'(u)|du\right|dx\end{aligned}$$

$$\leq C \int_{E_n} \sum_{k=0}^{\infty} v_{n,k}(x) \left(\varphi^{-1}(x) + \frac{1}{\sqrt{x}} \sqrt{\frac{n}{k+n}} \right)$$
$$\left| \int_x^{\frac{k^*}{n}} |\varphi(u) g'(u)| du \right| dx =: C(R_1 + R_2),$$

here $\left| \int_x^{\frac{k^*}{n}} |\varphi(u)g'(u)|du \right| = \max_{j=k,k+1} \left| \int_x^{\frac{j}{n}} |\varphi(u)g'(u)|du \right|$.

Using the method in [4, pp.146-147], we estimate R_1 and R_2. First let

$$D(l,n,x) = \{k : l\varphi(x)n^{-\frac{1}{2}} \leq |\frac{k}{n} - x| < (l+1)\varphi(x)n^{-\frac{1}{2}}\},$$

then

$$R_1 = \int_{E_n} \varphi^{-1}(x) \sum_{l=0}^{\infty} \sum_{k \in D(l,n,x)} v_{n,k}(x) \left| \int_x^{\frac{k^*}{n}} |\varphi(u)g'(u)|du \right| dx.$$

For $x \in E_n$, $l \geq 1$, we have [4, Lemma 9.4.4]

$$\sum_{k \in D(l,n,x)} v_{n,k}(x) \leq \sum_{k \in D(l,n,x)} \left| \frac{k}{n} - x \right|^4 v_{n,k}(x) \frac{n^2}{l^4 \varphi^4(x)} \leq \frac{C}{(l+1)^4}. \tag{2.12}$$

Let

$$F(l,x) = \left\{ v : \ v \in [0,\infty), \ |v - x| \leq (l+1)\varphi(x)n^{-\frac{1}{2}} + \frac{1}{n} \right\},$$
$$G(l,v) = \{x : \ x \in E_n, \ v \in F(l,x)\}.$$

From the procedure in [4, pp. 147], we know

$$\begin{aligned} R_1 &\leq C \sum_{l=0}^{\infty} \frac{1}{(l+1)^4} \int_{E_n} \varphi^{-1}(x) \int_{F(l,x)} |\varphi(v)g'(v)| dv dx \\ &\leq C \sum_{l=0}^{\infty} \frac{1}{(l+1)^4} \int_0^{\infty} |\varphi(v)g'(v)| \int_{G_{(l,v)}} \varphi^{-1}(x) dx dv \leq C \frac{1}{\sqrt{n}} \|\varphi g'\|_1. \end{aligned} \tag{2.13}$$

By the Hölder inequality, we can get the estimation for R_2

$$\frac{1}{\sqrt{x}} \sum_{k \in D(l,n,x)} v_{n,k}(x) \sqrt{\frac{n}{k+n}} \leq \frac{1}{\sqrt{x}} \left(\sum_{k \in D(l,n,x)} v_{n,k}(x) \frac{n}{k+n} \right)^{\frac{1}{2}}$$
$$\leq \frac{1}{\sqrt{x}} \left(\frac{2}{1+x} \sum_{k \in D(l,n,x)} v_{n-1,k}(x) \right)^{\frac{1}{2}} \leq \frac{C}{(1+l)^4} \varphi^{-1}(x).$$

$$R_2 \le C\frac{1}{\sqrt{n}}\|\varphi g'\|_1. \tag{2.14}$$

So,

$$\int_{E_n} |V_{n\alpha}(g,x)-g(x)|dx \le \frac{C}{\sqrt{n}}\|\varphi g'\|_1. \tag{2.15}$$

Therefore by (2.10) and (2.15), we obtain (2.6) for $p=1$. Following (2.8) we complete the proof of Theorem 2.1. ■

3. Inverse Theorem

To prove the inverse theorem, we shall first prove the following lemmas.

Lemma 3.1. *For* $f \in L_p[0,\infty)$ $(1 \le p \le \infty)$, $\varphi(x) = \sqrt{x(1+x)}$, $\delta_n(x) = \varphi(x) + \frac{1}{\sqrt{n}}$, *one has*

$$\left\|\delta_n V'_{n\alpha}(f)\right\|_p \le C\sqrt{n}\|f\|_p. \tag{3.1}$$

Proof. We will show (3.1) for two cases $p=\infty$ and $p=1$. Since

$$V'_{n\alpha}(f,x) = \alpha\sum_{k=0}^{\infty} n\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt\left[J_{n,k}^{\alpha-1}(x)J'_{n,k}(x) - J_{n,k+1}^{\alpha-1}(x)J'_{n,k+1}(x)\right].$$

Using (2.1) and (2.3) we write

$$\begin{aligned}\left|V'_{n,\alpha}(f,x)\right| &\le \alpha\|f\|_\infty \\ &\quad \left(\sum_{k=0}^{\infty}\left[J_{n,k}^{\alpha-1}(x)-J_{n,k+1}^{\alpha-1}(x)\right]J'_{n,k+1}(x)+\sum_{k=0}^{\infty}J_{n,k}^{\alpha-1}(x)|v'_{n,k}(x)|\right) \\ &=: \alpha\|f\|_\infty\left(J_1+J_2\right).\end{aligned} \tag{3.2}$$

By (2.2) (let $v_{n,-1}(x)=0$), one has for $x \in E_n^c$

$$\delta_n(x)J_2 \le \frac{\sqrt{2}+1}{\sqrt{n}}\sum_{k=0}^{\infty} n\left|v_{n+1,k-1}(x)-v_{n+1,k}(x)\right| \le \frac{6}{\sqrt{n}}\sum_{k=0}^{\infty} nv_{n+1,k}(x) < 6\sqrt{n}.$$

By (2.4), one has for $x \in E_n$

$$\delta_n(x)J_2 \le 2\varphi(x)\sum_{k=0}^{\infty}\frac{n}{\varphi^2(x)}|\frac{k}{n}-x|v_{n,k}(x) \le \frac{2n}{\varphi(x)}\left(\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right|^2 v_{n,k}(x)\right)^{\frac{1}{2}} < 4\sqrt{n}.$$

For $x \in [0, \infty)$, we obtain

$$\delta_n(x) J_2 \le C\sqrt{n}. \tag{3.3}$$

Note that $J'_{n,0}(x) = 0$, then

$$\begin{aligned} J_1 &= \sum_{k=0}^{\infty} \left(J_{n,k}^{\alpha-1}(x) - J_{n,k+1}^{\alpha-1}(x) \right) J'_{n,k+1}(x) \\ &= \sum_{k=0}^{\infty} J_{n,k}^{\alpha-1}(x) \left(J'_{n,k}(x) - v'_{n,k}(x) \right) - \sum_{k=0}^{\infty} J_{n,k+1}^{\alpha-1}(x) J'_{n,k+1}(x) \\ &\le \sum_{k=1}^{\infty} J_{n,k}^{\alpha-1}(x) J'_{n,k}(x) - \sum_{k=0}^{\infty} J_{n,k+1}^{\alpha-1}(x) J'_{n,k+1}(x) \\ &\quad + \sum_{k=0}^{\infty} J_{n,k}^{\alpha-1}(x) |v'_{n,k}(x)| = J_2. \end{aligned}$$

We get

$$\delta_n(x) J_1 \le C\sqrt{n}. \tag{3.4}$$

Combining (3.2)-(3.4) we get for $p = \infty$

$$\left\| \delta_n(x) V'_{n\alpha}(f, x) \right\|_{\infty} \le C\sqrt{n} \|f\|_{\infty}. \tag{3.5}$$

For $p = 1$, let $a_k(f) = n \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt$, we write

$$\begin{aligned} \left| V'_{n,\alpha}(f, x) \right| &\le \sum_{k=0}^{\infty} |a_k(f)| \left[J_{n,k}^{\alpha-1}(x) - J_{n,k+1}^{\alpha-1}(x) \right] J'_{n,k+1}(x) \\ &+ \sum_{k=0}^{\infty} |a_k(f)| J_{n,k}^{\alpha-1}(x) |v'_{n,k}(x)| \\ &=: \left(\widetilde{J}_1 + \widetilde{J}_2 \right). \end{aligned} \tag{3.6}$$

Writing

$$\int_0^{\infty} \left| \delta_n(x) V'_{n\alpha}(f, x) \right| dx = \left(\int_{E_n^c} + \int_{E_n} \right) \delta_n(x) \left(\widetilde{J}_1 + \widetilde{J}_2 \right) dx. \tag{3.7}$$

Now we estimate the four parts in the right of (3.7):

$$\int_{E_n^c} \delta_n(x)\widetilde{J}_2 dx \le \int_{E_n^c} \delta_n(x) \sum_{k=1}^{\infty} |a_k(f)| n \left(v_{n+1,k-1}(x) + v_{n+1,k}(x)\right) dx$$
$$+ \int_{E_n^c} \delta_n(x) |a_0(f)| n v_{n+1,0}(x) dx.$$

For $x \in E_n^c,\ \delta_n(x) \le \frac{3}{\sqrt{n}}$, noting $\displaystyle\int_0^\infty v_{n,k}(x)dx = \frac{1}{n-1}$, we have

$$\int_{E_n^c} \delta_n(x)\widetilde{J}_2 dx \le \frac{6}{\sqrt{n}} \sum_{k=1}^{\infty} n \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(t)|dt + \frac{3n}{\sqrt{n}} \int_0^{\frac{1}{n}} |f(t)|dt \le 6\sqrt{n}\|f\|_1. \tag{3.8}$$

Since $J_{n,k}^{\alpha-1}(x) - J_{n,k+1}^{\alpha-1}(x) \le 1,\ J_{n,k+1}'(x) = n v_{n+1,k}(x)$, we have

$$\int_{E_n^c} \delta_n(x)\widetilde{J}_1 dx \le \int_{E_n^c} \delta_n(x) \sum_{k=0}^{\infty} |a_k(f)| n v_{n+1,k}(x) dx \le 3\sqrt{n}\|f\|_1. \tag{3.9}$$

To estimate $\int_{E_n} \delta_n(x)\widetilde{J}_2 dx$, we will need the relation [4, pp.129 (9.4.15)]:

$$\int_{E_n} \frac{(\frac{k}{n} - x)^2}{\varphi^2(x)} v_{n,k}(x)dx \le Cn^{-2}.$$

By Hölder inequality and (2.4), we get

$$\begin{aligned}
\int_{E_n} \delta_n(x)\widetilde{J}_2 dx &\le 2\sum_{k=0}^{\infty} |a_k(f)| \int_{E_n} \varphi(x) \cdot \frac{n}{\varphi^2(x)} \left|\frac{k}{n} - x\right| v_{n,k}(x)dx \\
&\le 2n \sum_{k=0}^{\infty} |a_k(f)| \left(\int_{E_n} \frac{\left(\frac{k}{n} - x\right)^2}{\varphi^2(x)} v_{n,k}(x)dx\right)^{\frac{1}{2}} \cdot \left(\int_{E_n} v_{n,k}(x)dx\right)^{\frac{1}{2}} \\
&\le C\sqrt{n} \sum_{k=0}^{\infty} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(t)|dt = C\sqrt{n}\|f\|_1.
\end{aligned} \tag{3.10}$$

In order to estimate $\int_{E_n} \delta_n(x)\widetilde{J}_1 dx$, we consider two cases $\alpha \ge 2$ and $1 < \alpha < 2$ (when $\alpha = 1,\ \widetilde{J}_1 = 0$). For $\alpha \ge 2$, $J_{n,k}^{\alpha-1}(x) - J_{n,k+1}^{\alpha-1}(x) \le (\alpha - 1)v_{n,k}(x)$.

Using integration by parts, we can deduce

$$
\begin{aligned}
&\int_{E_n}\delta_n(x)\widetilde{J}_1dx \le C\sum_{k=0}^{\infty}|a_k(f)|\int_{E_n}\varphi(x)v_{n,k}(x)J'_{n,k+1}(x)dx\\
&= C\sum_{k=0}^{\infty}|a_k(f)|\left(\varphi(x)v_{n,k}(x)J_{n,k+1}(x)|_{\frac{1}{n}}^{\infty} - \int_{\frac{1}{n}}^{\infty}J_{n,k+1}(x)d(\varphi(x)v_{n,k}(x))\right)\\
&= C\sum_{k=0}^{\infty}|a_k(f)|\varphi(x)v_{n,k}(x)J_{n,k+1}(x)|_{\frac{1}{n}}^{\infty}\\
&\quad -C\sum_{k=0}^{\infty}|a_k(f)|\left(\int_{\frac{1}{n}}^{\infty}J_{n,k+1}(x)\frac{1+2x}{2\sqrt{x(1+x)}}v_{n,k}(x)dx\right.\\
&\qquad\left. -\int_{\frac{1}{n}}^{\infty}J_{n,k+1}(x)\varphi(x)v'_{n,k}(x)dx\right).
\end{aligned}
$$

Noting that $\varphi(x)v_{n,k}(x)J_{n,k+1}(x)|_{\frac{1}{n}}^{\infty} < 0$, $\int_{\frac{1}{n}}^{\infty}J_{n,k+1}(x)\frac{1+2x}{2\sqrt{x(1+x)}}v_{n,k}(x)dx > 0$ and (2.4), we have

$$
\begin{aligned}
&\int_{E_n}\delta_n(x)\widetilde{J}_1dx \le C\sum_{k=0}^{\infty}|a_k(f)|\int_{\frac{1}{n}}^{\infty}|J_{n,k+1}(x)\varphi(x)v'_{n,k}(x)|dx\\
&\le C\sum_{k=0}^{\infty}|a_k(f)|\left(\int_{E_n}\frac{\left(\frac{k}{n}-x\right)^2}{\varphi^2(x)}v_{n,k}(x)dx\right)^{\frac{1}{2}}\cdot\left(\int_{E_n}v_{n,k}(x)dx\right)^{\frac{1}{2}}\\
&\le C\sqrt{n}\sum_{k=0}^{\infty}\int_{\frac{k}{n}}^{\frac{k+1}{n}}|f(t)|dt = C\sqrt{n}\|f\|_1.
\end{aligned}
\tag{3.11}
$$

When $1<\alpha<2$, using the differential intermediate value theorem, we know

$$J_{n,k}^{\alpha-1}(x) - J_{n,k+1}^{\alpha-1}(x) = (\alpha-1)(\xi_k(x))^{\alpha-2}v_{n,k}(x)$$

where $J_{n,k+1}(x) < \xi_k(x) < J_{n,k}(x)$ and $\alpha-2<0$, then

$$J_{n,k}^{\alpha-1}(x) - J_{n,k+1}^{\alpha-1}(x) \le (\alpha-1)J_{n,k+1}^{\alpha-2}(x)v_{n,k}(x).$$

For $1<\alpha<2$, we get from the procedure of (3.11)

$$
\begin{aligned}
&\int_{E_n} \delta_n(x)\widetilde{J}_1 dx \\
&\le C\int_{E_n} \varphi(x)\sum_{k=0}^{\infty}|a_k(f)|v_{n,k}(x)(\alpha-1)J_{n,k+1}^{\alpha-2}(x)J_{n,k+1}^{'}(x)dx \\
&= C\sum_{k=0}^{\infty}|a_k(f)|\int_{E_n}\varphi(x)v_{n,k}(x)dJ_{n,k+1}^{\alpha-1}(x) \\
&\le C\sum_{k=0}^{\infty}|a_k(f)|\int_{\frac{1}{n}}^{\infty}\varphi(x)|v_{n,k}'(x)|dx \le C\sqrt{n}\|f\|_1.
\end{aligned}
\tag{3.12}
$$

Combining (3.11) and (3.12), we get for $\alpha \ge 1$

$$
\int_{E_n} \delta_n(x)\widetilde{J}_1 dx \le C\sqrt{n}\|f\|_1. \tag{3.13}
$$

From (3.6)-(3.10) and (3.13), we obtain for $p = 1$

$$
\int_0^{\infty} \delta_n(x)|V_{n\alpha}'(f,x)|dx \le C\sqrt{n}\|f\|_1. \tag{3.14}
$$

By (3.5) and (3.14), Lemma 3.1 holds. ■

Lemma 3.2. *For $f \in W_p$, $\varphi(x) = \sqrt{x(1+x)}$, $\delta_n(x) = \varphi(x) + \frac{1}{\sqrt{n}}$, one has*

$$
\left\|\delta_n(x)V_{n\alpha}^{'}(f,x)\right\|_p \le \tag{3.15}
$$

Proof. By the Riesz-Thorin interpolation theorem, we will show Lemma 3.2 for $p=\infty$ and $p=1$. For $f \in W_p$ and noting that $J_{n,0}^{'}(x) = 0$, we write

$$
\begin{aligned}
V_{n,\alpha}^{'}(f,x) &= \alpha\left[\sum_{k=1}^{\infty} n\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt J_{n,k}^{\alpha-1}(x)J_{n,k}'(x)\right. \\
&\qquad \left. -\sum_{k=0}^{\infty} n\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt J_{n,k+1}^{\alpha-1}(x)J_{n,k+1}'(x)\right] \\
&= \alpha\sum_{k=1}^{\infty} n\left(\int_0^{\frac{1}{n}} f(\frac{k}{n}+t)dt - \int_0^{\frac{1}{n}} f(\frac{k-1}{n}+t)dt\right) J_{n,k}^{\alpha-1}(x)J_{n,k}'(x) \\
&= \alpha\sum_{k=1}^{\infty} n\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}} f'(\frac{k-1}{n}+u+t)dudt J_{n,k}^{\alpha-1}(x)J_{n,k}'(x),
\end{aligned}
$$

then

$$\begin{aligned}\left|V'_{n,\alpha}(f,x)\right| &\leq \alpha\sum_{k=1}^{\infty}\int_0^{\frac{2}{n}}\left|f'(\frac{k-1}{n}+v)\right|dvJ'_{n,k}(x)\\ &= \alpha\sum_{k=0}^{\infty}\int_0^{\frac{2}{n}}\left|f'(\frac{k}{n}+v)\right|dvJ'_{n,k+1}(x)\\ &= \alpha\left(\int_0^{\frac{2}{n}}|f'(v)|dvJ'_{n,1}(x)+\sum_{k=1}^{\infty}\int_0^{\frac{2}{n}}\left|f'(\frac{k}{n}+v)\right|dvJ'_{n,k+1}(x)\right)\\ &=:\alpha\left(Q_1+Q_2\right).\end{aligned} \tag{3.16}$$

I. $p=\infty$**.** We estimate (3.16) in two cases.

1°. $x\in E_n^c$. Noting that $\delta_n(x)\sim\frac{1}{\sqrt{n}}$, we write

$$\begin{aligned}\left|\delta_n(x)V'_{n\alpha}(f,x)\right| &\leq \frac{\alpha}{\sqrt{n}}\sum_{k=0}^{\infty}\int_0^{\frac{2}{n}}\left|f'(\frac{k}{n}+v)\right|dvJ'_{n,k+1}(x)\\ &\leq C\|\delta_n f'\|_\infty\sum_{k=0}^{\infty}v_{n+1,k}(x).\end{aligned}$$

Since $\sum_{k=0}^{\infty}v_{n+1,k}(x)=1$, one has

$$\left|\delta_n(x)V'_{n\alpha}(f,x)\right|\leq C\|\delta_n f'\|_\infty. \tag{3.17}$$

2°. $x\in E_n$, $\delta_n(x)\sim\varphi(x)$. We write

$$\begin{aligned}\varphi(x)Q_1 &\leq \|\varphi f'\|_\infty\varphi(x)\int_0^{\frac{2}{n}}\varphi^{-1}(v)dvJ'_{n,1}(x)\\ &\leq 2\|\varphi f'\|_\infty\varphi(x)\sqrt{\frac{2}{n}}nv_{n+1,0}(x)\leq 4\|\varphi f'\|_\infty,\end{aligned}$$

here we have used the relation $\frac{\varphi(x)\sqrt{n}}{(1+x)^{n+1}}\leq 1$.

Similarly, noticing that $\sum_{k=1}^{\infty}\varphi^{-2}(\frac{k}{n})v_{n+1,k}(x)\leq C\varphi^{-2}(x)$, we have

$$\begin{aligned}\varphi(x)Q_2 &\leq \|\varphi f'\|_\infty\varphi(x)\sum_{k=1}^{\infty}\int_0^{\frac{2}{n}}\varphi^{-1}(\frac{k}{n}+v)dvJ'_{n,k+1}(x)\\ &\leq 2\|\varphi f'\|_\infty\varphi(x)\sum_{k=1}^{\infty}\varphi^{-1}(\frac{k}{n})v_{n+1,k}(x)\\ &\leq 2\|\varphi f'\|_\infty\varphi(x)\left(\sum_{k=1}^{\infty}\varphi^{-2}(\frac{k}{n})v_{n+1,k}(x)\right)^{\frac{1}{2}}\leq C\|\varphi f'\|_\infty.\end{aligned}$$

Then for $x \in E_n$, we get from (3.17)

$$\left\|\delta_n(x)V'_{n\alpha}(f,x)\right\|_{\infty} \le C\|\delta_n f'\|_{\infty}. \tag{3.18}$$

II. $p = 1$**.** First we will estimate $\int_0^\infty \delta_n(x)Q_2dx$. For $k \ge 1, 0 \le v \le \frac{2}{n}$, one has

$$\int_0^{\frac{2}{n}} \left|f'(\frac{k}{n}+v)\right| dv \le \varphi^{-1}(\frac{k}{n}) \int_0^{\frac{2}{n}} \varphi(\frac{k}{n}+v) \left|f'(\frac{k}{n}+v)\right| dv.$$

Therefore we have

$$\begin{aligned}
&\int_0^\infty \delta_n(x)Q_2dx \le \sum_{k=1}^\infty \int_{\frac{k}{n}}^{\frac{k+2}{n}} \varphi(u)\left|f'(u)\right| dun \int_0^\infty \varphi^{-1}(\frac{k}{n})\delta_n(x)v_{n+1,k}(x)dx \\
&\le \sum_{k=1}^\infty \int_{\frac{k}{n}}^{\frac{k+2}{n}} \varphi(u)\left|f'(u)\right| du \cdot n \left(\int_0^\infty \varphi^{-2}(\frac{k}{n})\delta_n^2(x)v_{n+1,k}(x)dx\right)^{\frac{1}{2}} \frac{1}{\sqrt{n}}.
\end{aligned}$$

Noticing $k \ge 1, \frac{n}{k}\frac{n}{n+k}v_{n+1,k}(x) = \frac{(k+1)n}{k(n-1)x(1+x)}v_{n-1,k+1}(x)$, then

$$\begin{aligned}
&\int_0^\infty \frac{n}{k}\frac{n}{n+k}\left(\varphi(x)+\frac{1}{\sqrt{n}}\right)^2 v_{n+1,k}(x)dx \\
\le\ & 4\int_0^\infty \frac{n}{k}\frac{n}{n+k}\left(\varphi^2(x)+\frac{1}{n}\right) v_{n+1,k}(x)dx \\
\le\ & 4\left(\int_0^\infty \varphi^2(x)\frac{n}{k}\frac{n}{n+k}v_{n+1,k}(x)dx + \int_0^\infty \frac{1}{k}\frac{n}{n+k}v_{n+1,k}(x)dx\right) \\
\le\ & 4\left(4\int_0^\infty v_{n-1,k+1}(x)dx + \frac{1}{n}\right) = \frac{C}{n}.
\end{aligned}$$

Combining the above relations, we can deduce

$$\int_0^\infty \delta_n(x)Q_2dx \le C\|\varphi f'\|_1. \tag{3.19}$$

Secondly, we consider Q_1. Noticing $\delta_n(u)\sqrt{n} \ge 1$, one has

$$\begin{aligned}
\delta_n(x)Q_1 &= \delta_n(x)\int_0^{\frac{2}{n}} |f'(u)|duJ'_{n,1}(x) \\
&\le \delta_n(x)\int_0^{\frac{2}{n}} \sqrt{n}\delta_n(u)|f'(u)|du \cdot nv_{n+1,0}(x) \le n^{\frac{3}{2}}\|\delta_n f'\|_1\delta_n(x)v_{n+1,0}(x),
\end{aligned}$$

then

$$
\begin{aligned}
&\int_0^\infty \delta_n(x) Q_1 dx \le n^{\frac{3}{2}} \|\delta_n f'\|_1 \int_0^\infty \left(\varphi(x) + \frac{1}{\sqrt{n}}\right) v_{n+1,0}(x) dx \\
&\le n^{\frac{3}{2}} \|\delta_n f'\|_1 \left[\left(\int_0^\infty \varphi^2(x) v_{n+1,0}(x) dx \right)^{\frac{1}{2}} \frac{1}{\sqrt{n}} + n^{-\frac{3}{2}} \right] \\
&= \|\delta_n f'\|_1 \left[n \left(\int_0^\infty \frac{1}{n-1} v_{n-1,1}(x) dx \right)^{\frac{1}{2}} + 1 \right] = 2\|\delta_n f'\|_1.
\end{aligned} \tag{3.20}
$$

Combining (2.23), (3.20), we obtain

$$
\int_0^\infty \delta_n(x) \left| V'_{n\alpha}(f,x) \right| dx \le C \|\delta_n f'\|_1. \tag{3.21}
$$

Therefore Lemma 3.2 can been proved by (3.18) and (3.20). ■

Using Lemma 3.1 and 3.2, we can prove the inverse theorem.

Theorem 3.3. *For* $f \in L_p[0,\infty)$ $(1 \le p \le \infty)$, $\varphi(x) = \sqrt{x(1+x)}$, $0 < \beta < 1$, *then*

$$
\|V_{n\alpha}(f,x) - f(x)\|_p = O\left(n^{-\frac{\beta}{2}}\right)
$$

implies $\omega_\varphi(f,t)_p = O\left(t^\beta\right)$.

Proof. Using Lemma 3.1, 3.2 and the same method in (cf. [4, pp.165, [5, pp.145],). For a suitable function g, we have

$$
\begin{aligned}
K_\varphi(f,t)_p &\le \|f - V_{n\alpha}(f)\|_p + t\|\varphi V'_{n\alpha}(f)\|_p \\
&\le Cn^{-\frac{\beta}{2}} + t\left(\|\delta_n V'_{n\alpha}(f-g)\|_p + \|\delta_n V'_{n\alpha}(g)\|_p\right) \\
&\le Cn^{-\frac{\beta}{2}} + t\sqrt{n}\left(\|f-g\|_p + \frac{1}{\sqrt{n}}\|\delta_n g'\|_p\right) \\
&\le Cn^{-\frac{\beta}{2}} + t\sqrt{n}\left(\|f-g\|_p + \frac{1}{\sqrt{n}}\|\varphi g'\|_p + \frac{1}{n}\|g'\|_p\right) \\
&\le C\left(n^{-\frac{\beta}{2}} + \frac{t}{n^{-\frac{1}{2}}}\overline{K}_\varphi(f, n^{-\frac{1}{2}})_p\right) \le C\left(n^{-\frac{\beta}{2}} + \frac{t}{n^{-\frac{1}{2}}}K_\varphi(f, n^{-\frac{1}{2}})_p\right).
\end{aligned}
$$

By the Berens-Lorentz Lemma, the above relation implies $K_\varphi(f,t)_p = O(t^\beta)$. $\omega_\varphi(f,t)_p = O(t^\beta)$ holds from (1.3), we see that the proof of Theorem 3.3 is completed. ■

Remark 1. From Theorem 2.1 and 3.1 we can deduce $(1.4) \Leftrightarrow (1.5)$.

Remark 2. In [6], the author also studied the Szász-Kantorovich-Bézier operators, using the above method, we can obtain the similar direct, inverse and equivalence theorem for the Szász-Kantorovich-Bézier operators without any difficulty.

4. A Note on the Second Order Modulus

In this section we will explain that the second order modulus can not be used.

Lemma 4.1. *If* $a_i,\ b_i > 0\ (i = 0, 1, \cdots, n-1;\ j = n, \cdots)$, $e_0 > e_1 > \cdots > e_{n-1} > e_n > \cdots > 0$ *and* $\sum_{i=0}^{n-1} a_i = \sum_{j=n}^{\infty} b_j$, *then we have*

$$\sum_{i=0}^{n-1} a_i e_i > \sum_{j=n}^{\infty} b_j e_j. \tag{4.1}$$

Proof. The inequality (4.1) is equivalent to $\sum_{i=0}^{n-1} a_i \frac{e_i}{e_{n-1}} > \sum_{j=n}^{\infty} b_j \frac{e_j}{e_{n-1}}$. Since $\frac{e_i}{e_{n-1}} \geq 1$ and $\frac{e_j}{e_n} < 1$, we have

$$\sum_{i=0}^{n-1} a_i \frac{e_i}{e_{n-1}} \geq \sum_{i=0}^{n-1} a_i = \sum_{j=n}^{\infty} b_j > \sum_{j=n}^{\infty} b_j \frac{e_j}{e_{n-1}}.$$

The proof of (4.1) is complete. ■

Now we explain that in the direct result $\omega_\varphi(f,t)$ can not be replaced by $\omega_\varphi^2(f,t)$.

Take $f(t) = t-1,\ \alpha = 2,\ x = 1$. Then for $t > 0$, $\omega_\varphi^2(f,t) = 0$. If $\omega_\varphi(f,t)$ can be replaced by $\omega_\varphi^2(f,t)$, it would be $|V_{n2}(f,1) - f(1)| = 0$, i.e.

$$V_{n2}(f,1) = 0. \tag{4.2}$$

But as

$$\begin{aligned} V_{n2}(f,1) &= n \sum_{k=0}^{\infty} \int_{\frac{k}{n}}^{\frac{k+1}{n}} (t-1)dt[J_{n,k}^2(1) - J_{n,k+1}^2(1)] \\ &= n \sum_{k=0}^{\infty} \int_{\frac{k}{n}}^{\frac{k+1}{n}} (t-1)dt v_{n,k}(1)[J_{n,k}(1) + J_{n,k+1}(1)] \end{aligned}$$

with (4.2) we have

$$
\begin{aligned}
I_1 =: & \sum_{i=0}^{n-1} \frac{2n-2i-1}{n} v_{n,i}(1)[J_{n,i}(1)+J_{n,i+1}(1)] \\
= & \sum_{j=n}^{\infty} \frac{2j+1-2n}{n} v_{n,j}(1)[J_{n,j}(1)+J_{n,j+1}(1)] =: I_2.
\end{aligned} \tag{4.3}
$$

Take $a_i = \frac{2n-2i-1}{n} v_{n,i}(1)$, $e_i = J_{n,i}(1)+J_{n,i+1}(1)$, $b_j = \frac{2j+1-2n}{n} v_{n,j}(1)$, $e_j = J_{n,j}(1)+J_{n,j+1}(1)$ $(i=0,\cdots,n-1;\ j=n,\cdots)$, since $V_{n1}(t-1,1)=0$, we have $\sum\limits_{i=0}^{n-1} a_i = \sum\limits_{j=n}^{\infty} b_j$. Obviously, $e_0 > e_1 > \cdots > e_{n-1} > e_n > \cdots > 0$, by Lemma 4.1, we get $I_1 > I_2$. This is contradictory with (4.3).

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