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GENERALIZED KKM THEOREM WITH APPLICATIONS TO GENERALIZED MINIMAX INEQUALITIES AND GENERALIZED EQUILIBRIUM PROBLEMS

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Abstract. In this paper, a generalized version of the famous KKM theorem is obtained by using the concept of generalized KKM mappings introduced by Chang and Zhang [5]. By employing our generalized KKM theorem, we obtain a generalized minimax inequality which includes several existing ones as special cases. Further, by applying our generalized minimax inequality we establish an existence result for the saddle-point problem under general setting. Finally, we also derive some existence results for generalized equilibrium problems and generalized variational inequalities.

1. Introduction

It is well known that the famous Fan-Knaster-Kuratowski-Mazurkiewicz theorem (i.e., FanKKM theorem) and Fan's minimax inequality have played very important roles in the study of modern nonlinear analysis. Moreover, a great deal of effort has gone into the theory and applications of the FanKKM theorem and Fan's minimax inequality; see [1-3, 5, 7, 8, 17].

Let E be a Hausdorff topological vector space and let X be a nonempty subset of E. A multivalued mapping $G : X \to 2^E$ is called a KKM mapping if $\mathrm{co}\{x_1, ..., x_n\} \subset \cup_{i=1}^n G(x_i)$ for each finite subset $\{x_1, ..., x_n\} \subset X$ where

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$\mathrm{co}\{x_1, ..., x_n\}$ denotes the convex hull of the set $\{x_1, ..., x_n\}$. In Ref. 1, Ky Fan gave the following famous infinite-dimensional generalization of Knaster, Kuratowski, and Mazurkiewicz's classical finite-dimensional result [12].

Theorem 1.1. (FanKKM theorem). *See Lemma 1 in [9]. Let E be a Hausdorff topological vector space, X be a nonempty subset of E, and $G : X \to 2^E$ be a* KKM *mapping with nonempty closed values. If there exists an $x_0 \in X$ such that $G(x_0)$ is a compact set of E, then $\cap_{x \in X} G(x) \neq \emptyset$.*

On the other hand, in [10], Ky Fan also proved the following famous minimax inequality.

Theorem 1.2. (Fan's minimax inequality). *See Theorem 1 in [10]. Let E be a Hausdorff topological vector space and X be a nonempty compact convex subset of E. If a function $\varphi : X \times X \to (-\infty, +\infty)$ satisfies the following conditions:*

(i) *$\varphi(\cdot, y) : X \to (-\infty, +\infty)$ is upper semicontinuous for each $y \in X$,*

(ii) *$\varphi(x, \cdot) : X \to (-\infty, +\infty)$ is quasiconvex for each $x \in X$, then there exists a point $x^* \in X$ such that*

$$\inf_{y \in X} \varphi(x^*, y) \geq \inf_{x \in X} \varphi(x, x).$$

In this paper, we will derive a generalized version of the KKM theorem by using the concept of generalized KKM mappings introduced by Chang and Zhang [5]. Then by employing our generalized KKM theorem, we obtain a generalized minimax inequality which includes several existing ones as special cases. Further, by applying our generalized minimax inequality we establish an existence result for the saddle-point problem under general setting. Finally, we also derive some existence results for generalized equilibrium problems and generalized variational inequalities.

2. Generalized KKM Theorem

First, we recall the following definition due to Chang and Zhang [5].

Definition 2.1. Let X be a nonempty subset of a topological vector space E. A multivalued mapping $F : X \to 2^E$ is called a generalized KKM mapping if, for any finite set $\{x_1, ..., x_n\} \subset X$, there exists a finite subset $\{y_1, ..., y_n\} \subset E$ such that, for any subset $\{y_{i_1}, ..., y_{i_k}\} \subset \{y_1, ..., y_n\}$, $1 \leq k \leq n$, we have

$$\mathrm{co}\{y_{i_1}, ..., y_{i_k}\} \subset \bigcup_{j=1}^{k} F(x_{i_j}).$$

Motivated and inspired by the above concept of generalized KKM mapping, we introduce the following concept.

Definition 2.2. Let X be a nonempty subset of a topological vector space E. A multivalued mapping $F : X \to 2^E$ is called a generalized KKM mapping with respect to K if there exists a nonempty subset $K \subset E$, and for any finite set $\{x_1, ..., x_n\} \subset X$ there is a finite subset $\{y_1, ..., y_n\} \subset K$ such that for any subset $\{y_{i_1}, ..., y_{i_k}\} \subset \{y_1, ..., y_n\}$, $1 \le k \le n$, we have

$$\mathrm{co}\{y_{i_1}, ..., y_{i_k}\} \subset \bigcup_{j=1}^{k} F(x_{i_j}).$$

Remark 2.1. It is easy to see that every generalized KKM mapping is a generalized KKM mapping with respect to E. Obviously, every generalized KKM mapping with respect to K is a generalized KKM mapping.

We shall need the following result, which is used for proving the main result of this section.

Lemma 2.1. *See Theorem 3.1 in [5]. Let X be a nonempty convex subset of a Hausdorff topological vector space E. Let $F : X \to 2^E$ be a multivalued mapping such that, for each $x \in X$, $F(x)$ is finitely closed; that is, for every finite-dimensional subspace L in E, $F(x) \cap L$ is closed in the Euclidean topology in L. Then the family of sets $\{F(x) : x \in X\}$ has the finite intersection property if and only if $F : X \to 2^E$ is a generalized* KKM *mapping.*

Definition 2.3. See [17]. Let Y and Z be two topological spaces. A multivalued mapping $F : Y \to 2^Z$ is said to be transfer closed-valued on Y if, for every $x \in Y$, $y \notin F(x)$, there exists an element $x' \in Y$ such that $y \notin \overline{F(x')}$, where $\bar{A}$ denotes the closure of a subset A of a topological space.

It has been shown in [6, 17] that F is a transfer closed-valued mapping if and only if

$$\bigcap_{x \in Y} F(x) = \bigcap_{x \in Y} \overline{F(x)}.$$

Now we state and prove the main result of this section which will be used in the sequel.

Theorem 2.1. *Let X be a nonempty subset of a Hausdorff topological vector space E. Let $F : X \to 2^E$ be a transfer closed-valued mapping such that $\bar{F}$:*

$X \to 2^E$ *is a generalized* KKM *mapping with respect to* K. *If* $\overline{\text{co}}K$ *is compact where* $\overline{\text{co}}K$ *denotes the closure of convex hull of* K, *then*

$$\bigcap_{x\in X} F(x) \neq \emptyset.$$

Proof. Since $\bar{F} : X \to 2^E$ is defined by $\bar{F}(x) = \overline{F(x)}$ for each $x \in X$, we know that $\bar{F}$ is a generalized KKM mapping with closed values. For each $x \in X$ we define

$$G(x) = \overline{\text{co}}K \cap \overline{F(x)}.$$

Consider any finite subset $\{x_1, ..., x_n\}$ of X. From Definition 2.2 it follows that there is a finite subset $\{y_1, ..., y_n\} \subset K$ such that for any subset $\{y_{i_1}, ..., y_{i_k}\} \subset \{y_1, ..., y_n\},\ 1 \leq k \leq n$,

$$\text{co}\{y_{i_1}, ..., y_{i_k}\} \subset \bigcup_{j=1}^{k} \overline{F(x_{i_j})}.$$

Observe that

$$\text{co}\{y_{i_1}, ..., y_{i_k}\} \subset \overline{\text{co}}K.$$

Hence it follows from these facts that

$$\text{co}\{y_{i_1}, ..., y_{i_k}\} \subset \overline{\text{co}}K \cap \bigcup_{j=1}^{k} \overline{F(x_{i_j})}$$

which implies that

$$\text{co}\{y_{i_1}, ..., y_{i_k}\} \subset \overline{\text{co}}K \cap \bigcup_{j=1}^{k} \overline{F(x_{i_j})} = \bigcup_{j=1}^{k} G(x_{i_j}).$$

This shows that $G : X \to 2^E$ is a generalized KKM mapping with respect to K and so it is a generalized KKM mapping. Since $G(x)$ is closed for each $x \in X$, from Lemma 2.1 we deduce that the family of sets $\{G(x) : x \in X\}$ has the finite intersection property. Note that $G(x)$ is also compact for each $x \in X$. Thus we have

$$\overline{\text{co}}K \cap \bigcap_{x\in X} \overline{F(x)} = \bigcap_{x\in X} \overline{\text{co}}K \cap \overline{F(x)} = \bigcap_{x\in X} G(x) \neq \emptyset$$

which implies that

$$\bigcap_{x\in X} \overline{F(x)} \neq \emptyset.$$

Since F is a transfer closed-valued mapping, we have

$$\bigcap_{x\in X} F(x) = \bigcap_{x\in X} \overline{F(x)} \neq \emptyset. \qquad \blacksquare$$

3. Generalized Minimax Inequality

First we recall some definitions. Let E be a topological vector space and let X and Y be two nonempty subsets of E. The following Definitions 3.1-3.3 can be found in [18].

Definition 3.1. A function $\phi : X \to (-\infty, +\infty)$ is said to be quasiconvex if, for each $\lambda \in (-\infty, +\infty)$, the set $\{x \in X : \phi(x) \leq \lambda\}$ is convex; ϕ is said to be quasiconcave if $-\phi$ is quasiconvex.

Note that ϕ is quasiconcave [resp., quasiconvex] if and only if, for each $\lambda \in (-\infty, +\infty)$, the set $\{x \in X : \phi(x) > \lambda\}$ [resp., $\{x \in X : \phi(x) < \lambda\}$] is convex [resp., concave].

Definition 3.2. Let X be a nonempty convex subset of E. A function $\phi(x, y) : X \times X \to (-\infty, +\infty)$ is said to be diagonally quasiconvex in y if, for any finite subset $\{y_1, ..., y_n\} \subset X$ and any $y_0 \in \mathrm{co}\{y_1, ..., y_n\}$, we have

$$\phi(y_0, y_0) \leq \max_{1\leq i\leq n} \phi(y_0, y_i);$$

$\phi(x, y)$ is said to be diagonally quasiconcave in y if $-\phi(x, y)$ is diagonally quasiconvex in y.

Definition 3.3. Let X be a nonempty convex subset of E. A function $\phi(x, y) : X \times X \to (-\infty, +\infty)$ is said to be γ-diagonally quasiconvex in y for some $\gamma \in (-\infty, +\infty)$ if, for any finite subset $\{y_1, ..., y_n\} \subset X$ and any $y_0 \in \mathrm{co}\{y_1, ..., y_n\}$, we have

$$\gamma \leq \max_{1\leq i\leq n} \phi(y_0, y_i);$$

$\phi(x, y)$ is said to be γ-diagonally quasiconcave in y for some $\gamma \in (-\infty, +\infty)$ if $-\phi(x, y)$ is $-\gamma$-diagonally quasiconvex in y.

In 1991, Chang and Zhang [5] introduced the following concepts.

Definition 3.4. See Definition 2.2 in [5]. Let X be a nonempty convex subset of E. A function $\phi(x, y) : X \times Y \to (-\infty, +\infty)$ is said to be γ-generalized quasiconvex in y for some $\gamma \in (-\infty, +\infty)$ if, for any finite subset $\{y_1, ..., y_n\} \subset Y$, there is a finite subset $\{x_1, ..., x_n\} \subset X$ such that, for any subset $\{x_{i_1}, ..., x_{i_k}\} \subset \{x_1, ..., x_n\}$ and any $x_0 \in \mathrm{co}\{x_{i_1}, ..., x_{i_k}\}$, we have

$$\gamma \leq \max_{1 \leq j \leq k} \phi(x_0, y_{i_j});$$

$\phi(x, y)$ is said to be γ-generalized quasiconcave in y for some $\gamma \in (-\infty, +\infty)$ if $-\phi(x, y)$ is $-\gamma$-generalized quasiconvex in y.

Motivated and inspired by Definition 2.2 in [5], we introduce the following concepts.

Definition 3.5. A function $\phi(x, y) : X \times Y \to (-\infty, +\infty)$ is said to be γ-generalized quasiconvex in y with respect to K for some $\gamma \in (-\infty, +\infty)$ and some convex subset $K \subset X$ if for any finite subset $\{y_1, ..., y_n\} \subset Y$, there is a finite subset $\{x_1, ..., x_n\} \subset K$ such that, for any subset $\{x_{i_1}, ..., x_{i_k}\} \subset \{x_1, ..., x_n\}$ and any $x_0 \in \mathrm{co}\{x_{i_1}, ..., x_{i_k}\}$, we have

$$\gamma \leq \max_{1 \leq j \leq k} \phi(x_0, y_{i_j});$$

$\phi(x, y)$ is said to be γ-generalized quasiconcave in y with K for some $\gamma \in (-\infty, +\infty)$ and some convex subset $K \subset X$ if $-\phi(x, y)$ is $-\gamma$-generalized quasiconvex in y with K.

Remark 3.1. It is easy to see that whenever $K = X$, every γ-generalized quasiconvex [resp., quasiconcave] function $\phi(x, y)$ in y is γ-generalized quasiconvex [resp., quasiconcave] in y with respect to K.

In [5, Proposition 2.1], Chang and Zhang gave the relation between generalized KKM mappings and γ-generalized convexity (concavity). Motivated and inspired by their result, we have the following proposition.

Proposition 3.1. *Let X and Y be two nonempty subsets of a topological vector space E. Let K be a nonempty convex subset of X, $\phi(x, y) : X \times Y \to (-\infty, +\infty)$ and $\gamma \in (-\infty, +\infty)$. Then, the following statements are equivalent:*

(i) *The mapping $G : Y \to 2^X$ defined by*

$$G(y) = \{x \in X : \phi(x, y) \leq \gamma\} \text{ [resp., } G(y) = \{x \in X : \phi(x, y) \geq \gamma\}]$$

is a generalized KKM *mapping with respect to K;*

(ii) *$\phi(x, y)$ is γ-generalized quasiconcave in y [resp., quasiconvex] with respect to K.*

Proof. By the careful analysis of the proof of Proposition 2.1 in [5], we can readily see that Proposition 3.1 is valid. ■

Let Y and Z be two topological spaces. A function $\phi(x, y) : Y \times Z \to (-\infty, +\infty)$ is said to be a γ-transfer lower semicontinuous function in x for some $\gamma \in (-\infty, +\infty)$ [17] if, for all $x \in Y$ and $y \in Z$ with $\phi(x, y) > \gamma$, there exist some point $y' \in Z$ and some neighborhood $N(x)$ of x such that $\phi(z, y') > \gamma$ for all $z \in N(x)$.

We now state and prove the following generalized minimax inequality.

Theorem 3.1. *Let X and Y be two nonempty subsets of a Hausdorff topological vector space E. Let $K \subset X$ be a nonempty convex subset such that $\overline{K}$ is compact. Let $\gamma \in (-\infty, +\infty)$ be a given number, and let $\phi, \psi : X \times Y \to (-\infty, +\infty)$ satisfy the following conditions:*

- (i) *For any fixed $y \in Y$, $\phi(x, y)$ is a γ-transfer lower semicontinuous function in x;*
- (ii) *For any fixed $x \in X$, $\psi(x, y)$ is a γ-generalized quasiconcave function in y with respect to K;*
- (iii) *$\phi(x, y) \le \psi(x, y)$, for all $(x, y) \in X \times Y$.*

Then there exists $\bar{x} \in X$ such that

$$\phi(\bar{x}, y) \le \gamma, \quad \text{for all } y \in Y.$$

In particular, we have

$$\inf_{x \in X} \sup_{y \in Y} \phi(x, y) \le \gamma.$$

Proof. Define two multivalued mappings $T, G : Y \to 2^X$ by

$$T(y) = \{x \in X : \psi(x, y) \le \gamma\} \quad \text{and} \quad G(y) = \{x \in X : \phi(x, y) \le \gamma\},$$

for all $y \in Y$. Condition (i) implies that G is a transfer closed-valued mapping on Y. Indeed, if $x \notin G(y)$, then $\phi(x, y) > \gamma$. Since $\phi(x, y)$ is γ-transfer lower semicontinuous in x, there is a $y' \in Y$ and a neighborhood $N(x)$ of x such that

$$\phi(z, y') > \gamma, \quad \text{for all } z \in N(x).$$

Then $G(y') \subset X \setminus N(x)$. Hence $x \notin \overline{G(y')}$. Thus G is transfer closed-valued. From condition (ii), T is a generalized KKM mapping with respect to K. From condition (iii), we have that

$$T(y) \subset G(y), \quad \text{for all } y \in Y,$$

and hence G is also a generalized KKM mapping with K. So, $\bar{G}$ is also a generalized KKM mapping with K. Since $K \subset X$ is a nonempty convex subset such that $\overline{K}$ is compact, $\overline{\text{co}}K = \overline{K}$ is compact. From Theorem 2.1, we get

$$\bigcap_{y \in Y} G(y) \neq \emptyset.$$

As a result, there exists $\bar{x} \in X$ such that

$$\phi(\bar{x}, y) \leq \gamma, \quad \text{for all } y \in Y.$$

In particular, we have

$$\inf_{x \in X} \sup_{y \in Y} \phi(x, y) \leq \gamma. \qquad \blacksquare$$

Remark 3.1. If for every fixed $y \in Y$, the function $\phi(x, y)$ is lower semicontinuous in x, then condition (i) of Theorem 3.1 is satisfied immediately.

As an application of Theorem 3.1, we derive the following existence result for the saddle-point problem.

Theorem 3.2. *Let X and Y be two nonempty subsets of a Hausdorff topological vector space E. Let $\gamma \in (-\infty, +\infty)$ be a given number and let $\phi : X \times Y \to (-\infty, +\infty)$ satisfy the following conditions:*

- (i) *$\phi(x, y)$ is γ-transfer lower semicontinuous in x and γ-generalized quasiconcave in y with respect to K where $K \subset X$ is a nonempty convex subset such that $\overline{K}$ is compact;*
- (ii) *$\phi(x, y)$ is γ-transfer upper semicontinuous in y and γ-generalized quasiconvex in x with respect to M where $M \subset Y$ is a nonempty convex subset such that $\overline{M}$ is compact.*
  Then there exists a saddle point of $\phi(x, y)$; that is, there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that

$$\phi(\bar{x}, y) \leq \phi(\bar{x}, \bar{y}) \leq \phi(x, \bar{y}), \quad \text{for all } (x, y) \in X \times Y.$$

Moreover, we also have

$$\inf_{x \in X} \sup_{y \in Y} \phi(x, y) = \sup_{y \in Y} \inf_{x \in X} \phi(x, y) = \gamma.$$

Proof. By Theorem 3.1 with $\phi = \psi$, there exists $\bar{x} \in X$ such that

$$(1) \qquad \phi(\bar{x}, y) \leq \gamma, \quad \text{for all } y \in Y.$$

Let $f : Y \times X \to (-\infty, +\infty)$ be defined as $f(y, x) = -\phi(x, y)$. By assumption (ii), $f(y, x)$ is γ-transfer lower semicontinuous in y and $-\gamma$-generalized quasiconcave in x with respect to M where $M \subset Y$ is a nonempty convex subset such that $\overline{M}$ is compact. Therefore again by Theorem 3.1, there exists $\bar{y} \in Y$ such that

$$f(\bar{y}, x) = -\phi(x, \bar{y}) \le -\gamma, \quad \text{for all } x \in X$$

which implies that

$$\phi(x, \bar{y}) \ge \gamma, \quad \text{for all } x \in X. \tag{2}$$

Combining (1) with (2), we have $\phi(\bar{x}, \bar{y}) = \gamma$ and

$$\phi(\bar{x}, y) \le \phi(\bar{x}, \bar{y}) \le \phi(x, \bar{y}), \quad \text{for all } (x, y) \in X \times Y. \tag{3}$$

Finally, again from (1)-(3), we deduce that

$$\begin{aligned} &\sup\nolimits_{y\in Y} \inf\nolimits_{x\in X} \phi(x, y) \le \inf\nolimits_{x\in X} \sup\nolimits_{y\in Y} \phi(x, y) \le \sup\nolimits_{y\in Y} \phi(\bar{x}, y) \\ &\le \phi(\bar{x}, \bar{y}) \le \inf\nolimits_{x\in X} \phi(x, \bar{y}) \le \sup\nolimits_{y\in Y} \inf\nolimits_{x\in X} \phi(x, y). \end{aligned}$$

Consequently,

$$\inf_{x\in X} \sup_{y\in Y} \phi(x, y) = \sup_{y\in Y} \inf_{x\in X} \phi(x, y) = \gamma,$$

and the proof is completed. ∎

Remark 3.2. For results related to Theorem 3.2, see for example [1, 5, 10, 14, 16].

4. Generalized Equilibrium Problems

In this section, we shall employ Theorem 2.1 to derive some existence results for generalized equilibrium problems. Let Y and Z be two topological spaces. The multivalued mapping $T : Y \to 2^Z$ is said to be upper semicontinuous at $x_0 \in Y$ [4] if, for any open set V in Z containing $T(x_0)$, there is an open neighborhood U of x_0 in Y such that $T(x) \subset V$, for all $x \in U$. We say that T is upper semicontinuous in Y [4] if it is upper semicontinuous at each point of Y and if also $T(x)$ is a compact set for each $x \in Y$. For any topological vector space E over real or complex numbers, E^* denotes the vector space of all linear continuous functionals on E and $\langle u, v\rangle$ denotes the pairing between $u \in E^*$ and $v \in E$.

Theorem 4.1. *Let X be a nonempty subset of a Hausdorff topological vector space E. Let $\Phi : E^* \times X \times X \to (-\infty, +\infty)$ and $f : X \times X \to (-\infty, +\infty)$ be* *two functions. Let* $T : X \to 2^{E^*}$ *be a multivalued mapping and* $\gamma \in (-\infty, +\infty)$ *be a given number. For any* $(x, y) \in X \times X$*, define* $\phi, \psi : X \times X \to (-\infty, +\infty)$ *as follows:*

$$\phi(x, y) = \inf_{u \in T(x)} \Phi(u, x, y) + f(x, y),$$

$$\psi(x, y) = \sup_{w \in T(y)} \Phi(w, x, y) + f(x, y).$$

Suppose that the following conditions are satisfied:

(i) $\psi(x, y)$ is lower semicontinuous in x and $\phi(x, y)$ is γ-generalized quasiconcave in y with respect to K such that $\phi(x, x) \leq \gamma$ for all $x \in X$ where $K \subset X$ is a nonempty convex subset such that $\overline{K}$ is compact;

(ii) $T : X \to 2^{E^*}$ has nonempty values such that for each fixed $y \in X$,

$$\{x \in X : \phi(x, y) \leq \gamma\} \subset \{x \in X : \psi(x, y) \leq \gamma\}.$$

Then there exists $\bar{x} \in X$ such that

$$\sup_{w \in T(y)} \Phi(w, \bar{x}, y) + f(\bar{x}, y) \leq \gamma, \quad \text{for all } y \in X.$$

Proof. We define two multivalued mappings $G, F : X \to 2^X$ by

$$G(y) = \{x \in X : \psi(x, y) \leq \gamma\}, \quad \text{and} \quad F(y) = \{x \in X : \phi(x, y) \leq \gamma\},$$

for all $y \in X$. Then from condition (ii) we have $F(y) \subset G(y)$ for all $y \in X$. Since $\phi(x, x) \leq \gamma$ for all $x \in X$, it is known that

$$y \in F(y) \subset G(y) \text{ for all } y \in X.$$

As $\psi(x, y)$ is lower semicontinuous in x, $G(y)$ is closed for each $y \in X$. So G is transfer closed-valued. Note that $\phi(x, y)$ is γ-generalized quasiconcave in y with respect to K. As a result, F is a generalized KKM mapping with respect to K by Proposition 3.1. Since $F(y) \subset G(y)$ for all $y \in X$, G is also a generalized KKM mapping with respect to K. So $\bar{G}$ is also a generalized KKM mapping with respect to K. Also note that $\overline{\text{co}}K = \overline{K}$ is compact. Therefore by Theorem 2.1,

$$\bigcap_{y \in X} G(y) \neq \emptyset.$$

That is, there exists $\bar{x} \in X$ such that

$$\sup_{w \in T(y)} \Phi(w, \bar{x}, y) + f(\bar{x}, y) \leq \gamma, \quad \text{for all } y \in X.$$

The proof is now completed. ∎

By adopting the argument of Shih and Tan [15, Lemma 1], we derive the following lemma which will be used for proving the next theorem.

Lemma 4.1. *Let B be a reflexive Banach space, and let X be a nonempty convex subset of B. Let $T : X \to 2^{B^*}$ be upper semicontinuous from the line segments in X to the weak topology of B^*, let $f : X \times X \to (-\infty, +\infty)$ be a convex and lower semicontinuous function in the first variable, and let $\gamma \in (-\infty, +\infty)$ be a given number. Suppose that the function $\Phi : B^* \times X \times X \to (-\infty, +\infty)$ satisfies the following conditions:*

(i) $\Phi(u, x, y)$ *is weakly continuous in* u*;*

(ii) *For every bounded subset $D \subset B^*$ and each $\{x_n\} \subset X$ such that $x_n \to x_0 \in X$, there holds*

$$\lim_{n\to\infty} \sup_{u\in D} |\Phi(u, x_n, y) - \Phi(u, x_0, y)| = 0, \quad \text{for all } y \in X.$$

Then for each fixed $y \in X$, the intersection of the following set:

$$A = \{x \in X : \inf_{u\in T(x)} \Phi(u, x, y) + f(x, y) \le \gamma\}$$

with any line segment is closed in X.

Proof. For $x_1, x_2 \in X$, let $[x_1, x_2]$ denote the line segment

$$[x_1, x_2] = \{tx_1 + (1-t)x_2 : t \in [0, 1]\}.$$

Let $\{x_n\}$ be a sequence in $A \cap [x_1, x_2]$ such that $x_n \to x_0 \in [x_1, x_2]$. Since $\Phi(u, x, y)$ is weakly continuous in u, and since for each n, $T(x_n)$ is weakly compact, there exists $u_n \in T(x_n)$ such that

$$\Phi(u_n, x_n, y) + f(x_n, y) = \inf_{u\in T(x_n)} \Phi(u, x_n, y) + f(x_n, y) \le \gamma.$$

As T is upper semicontinuous on $[x_1, x_2]$, which is compact, and since each $T(x_n)$ is weakly compact, the set $\cup_{x\in[x_1,x_2]} T(x)$ is also weakly compact [4, Theorem 3, p. 110]. By the Eberlein-Smulian theorem [11], $\cup_{x\in[x_1,x_2]} T(x)$ is weakly sequentially compact. Thus there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \to u_0$ in the weak topology for some $u_0 \in \cup_{x\in[x_1,x_2]} T(x)$. By the upper semicontinuity of T, $u_0 \in T(x_0)$. Note that condition (i) yields

$$\lim_{k\to\infty} \Phi(u_{n_k}, x_0, y) = \Phi(u_0, x_0, y),$$

since $u_{n_k} \to u_0$ in the weak topology. Also note that

$$\lim_{k\to\infty} \sup_{u\in D} |\Phi(u, x_{n_k}, y) - \Phi(u, x_0, y)| = 0,$$

since $D = \{u_{n_k}\}$ is bounded. Consequently,

$$\begin{aligned}
&\limsup_{k\to\infty} |\Phi(u_{n_k}, x_{n_k}, y) - \Phi(u_0, x_0, y)| \\
&\le \limsup_{k\to\infty} |\Phi(u_{n_k}, x_{n_k}, y) - \Phi(u_{n_k}, x_0, y)| \\
&\quad + \limsup_{k\to\infty} |\Phi(u_{n_k}, x_0, y) - \Phi(u_0, x_0, y)| \\
&\le \limsup_{k\to\infty} \sup_{u\in D} |\Phi(u, x_{n_k}, y) - \Phi(u, x_0, y)| \\
&\quad + \limsup_{k\to\infty} |\Phi(u_{n_k}, x_0, y) - \Phi(u_0, x_0, y)| \\
&= \lim_{k\to\infty} \sup_{u\in D} |\Phi(u, x_{n_k}, y) - \Phi(u, x_0, y)| \\
&\quad + \lim_{k\to\infty} |\Phi(u_{n_k}, x_0, y) - \Phi(u_0, x_0, y)| \\
&= 0,
\end{aligned}$$

so that

$$\lim_{k\to\infty} \Phi(u_{n_k}, x_{n_k}, y) = \Phi(u_0, x_0, y).$$

Since $f(x, y)$ is convex and lower semicontinuous in x, it is also lower semicontinuous in the weak topology of B. Hence we have

$$\begin{aligned}
f(x_0, y) &\le \liminf_{k\to\infty} f(x_{n_k}, y) \\
&\le \liminf_{k\to\infty} (-\Phi(u_{n_k}, x_{n_k}, y) + \gamma) \\
&= -\limsup_{k\to\infty} \Phi(u_{n_k}, x_{n_k}, y) + \gamma \\
&= -\Phi(u_0, x_0, y) + \gamma.
\end{aligned}$$

Therefore for each fixed $y \in X$,

$$\inf_{u\in T(x_0)} \Phi(u, x_0, y) + f(x_0, y) \le \Phi(u_0, x_0, y) + f(x_0, y) \le \gamma,$$

and hence

$$x_0 \in A \cap [x_1, x_2].$$

As a result, the set $A \cap [x_1, x_2]$ is closed and the proof is completed. ∎

Now we can derive the following result for generalized equilibrium problems.

Theorem 4.2. *Let X be a nonempty convex subset of a reflexive Banach space B. Let $T : X \to 2^{B^*}$ be upper semicontinuous from the line segments in X to the weak topology of B^*, and let $f : X \times X \to (-\infty, +\infty)$ be a convex and lower semicontinuous function in the first variable. Let $\Phi : B^* \times X \times X \to (-\infty, +\infty)$ be a function and let $\phi, \psi : X \times X \to (-\infty, +\infty)$ be define by*

$$\phi(x, y) = \inf_{u \in T(x)} \Phi(u, x, y) + f(x, y), \quad \text{for all } (x, y) \in X \times X,$$
$$\psi(x, y) = \sup_{w \in T(y)} \Phi(w, x, y) + f(x, y), \quad \text{for all } (x, y) \in X \times X.$$

Suppose that the following conditions are satisfied:

(i) $\psi(x, y)$ is lower semicontinuous in x and $\phi(x, y)$ is 0-generalized quasiconcave in y with respect to K where $K \subset X$ is a nonempty convex subset such that $\overline{K}$ is weakly compact;

(ii) $T : X \to 2^{B^*}$ has nonempty values such that, for each fixed $y \in X$,

$$\{x \in X : \phi(x, y) \le 0\} \subset \{x \in X : \psi(x, y) \le 0\};$$

(iii) For each $(x, y) \in X \times X$ and each $y_t = ty + (1 - t)x, t \in [0, 1]$, there holds

$$\sup_{w \in T(y_t)} \Phi(w, x, y_t) + f(x, y_t) \le 0 \;\Rightarrow\; \sup_{w \in T(y_t)} \Phi(w, x, y) + f(x, y) \le 0;$$

(iv) $\Phi(u, x, y)$ is weakly continuous in u and convex in x such that $\Phi(u, x, x) + f(x, x) \le 0$ for all $(u, x) \in B^* \times X$;

(v) For every bounded subset $D \subset B^*$ and each $\{x_n\} \subset X$ such that $x_n \to x_0 \in X$, there holds

$$\lim_{n \to \infty} \sup_{u \in D} |\Phi(u, x_n, y) - \Phi(u, x_0, y)| = 0, \quad \text{for all } y \in X.$$

Then there exists $\bar{x} \in X$ such that

$$\inf_{u \in T(\bar{x})} \Phi(u, \bar{x}, y) + f(\bar{x}, y) \le 0, \quad \text{for all } y \in X. \tag{4}$$

In addition if $\Phi(u, x, y)$ is convex in u and $T(\bar{x})$ is convex, then there exists $\bar{u} \in T(\bar{x})$ such that

$$\Phi(\bar{u}, \bar{x}, y) + f(\bar{x}, y) \le 0, \quad \text{for all } y \in X. \tag{5}$$

Proof. Since all the assumptions of Theorem 4.1 are satisfied, there exists $\bar{x} \in X$ such that

$$\sup_{w\in T(y)} \Phi(w,\bar{x},y)+f(\bar{x},y)\leq 0, \quad \text{for all } y\in X. \tag{6}$$

Now we claim that inequality (4) holds for $y \in X$. Indeed, suppose that there exists $\bar{y} \in X$ such that

$$\inf_{u\in T(\bar{x})} \Phi(u,\bar{x},\bar{y})+f(\bar{x},\bar{y})> 0. \tag{7}$$

Let

$$y_t = t\bar{y}+(1-t)\bar{x}\in X, \quad t\in[0,1].$$

By (6), we have that for each $t\in[0,1]$,

$$\sup_{w\in T(y_t)} \Phi(w,\bar{x},y_t)+f(\bar{x},y_t)\leq 0,$$

from which together with condition (iii), it follows that

$$\sup_{w\in T(y_t)} \Phi(w,\bar{x},\bar{y})+f(\bar{x},\bar{y})\leq 0, \quad \text{for all } t\in[0,1]. \tag{8}$$

Now by Lemma 4.1 and (7), the set

$$U=\{x\in X: \inf_{u\in T(x)} \Phi(u,x,\bar{y})+f(x,\bar{y})>0\}\cap[\bar{y},\bar{x}]$$

is open in $[\bar{y},\bar{x}]$ and contains $\bar{x}$. Since $y_t\to\bar{x}$ as $t\to 0^+$, there exists $t_0\in(0,1]$ such that $y_t\in U$ for all $t\in(0,t_0)$, so that

$$\inf_{w\in T(y_t)} \Phi(w,y_t,\bar{y})+f(y_t,\bar{y})>0, \quad \text{for all } t\in(0,t_0).$$

Since $f(x,y)$ is convex in x and $\Phi(u,x,y)$ is convex in x such that $\Phi(u,x,x)+f(x,x)\leq 0$, $\forall(u,x)\in B^*\times X$, it follows that for each $t\in(0,t_0)$,

$$\begin{aligned} 0 &< \inf_{w\in T(y_t)}[\Phi(w,y_t,\bar{y})+f(y_t,\bar{y})] \\ &\leq \inf_{w\in T(y_t)}[t(\Phi(w,\bar{y},\bar{y})+f(\bar{y},\bar{y}))+(1-t)(\Phi(w,\bar{x},\bar{y})+f(\bar{x},\bar{y}))] \\ &\leq \inf_{w\in T(y_t)}[(1-t)(\Phi(w,\bar{x},\bar{y})+f(\bar{x},\bar{y}))] \\ &= (1-t)[\inf_{w\in T(y_t)}(\Phi(w,\bar{x},\bar{y})+f(\bar{x},\bar{y})]. \end{aligned}$$

Consequently,

$$\inf_{w\in T(y_t)} \Phi(w,\bar{x},\bar{y})+f(\bar{x},\bar{y})>0, \quad \text{for all } t\in(0,t_0),$$

which contradicts (8). This shows that (4) is valid.

In addition, suppose that $\Phi(u, x, y)$ is convex in u and $T(\bar{x})$ is convex. Then by the Kneser's minimax theorem [13], we have

$$\begin{aligned} &\inf_{u \in T(\bar{x})} \sup_{y \in X} \Phi(u, \bar{x}, y) + f(\bar{x}, y) \\ &= \sup_{y \in X} \inf_{u \in T(\bar{x})} \Phi(u, \bar{x}, y) + f(\bar{x}, y) \le 0. \end{aligned} \tag{9}$$

Note that the function

$$u \mapsto \sup_{y \in X} \Phi(u, \bar{x}, y) + f(\bar{x}, y)$$

is convex and weakly lower semicontinuous on B^*. Since $T(\bar{x})$ is weakly compact, it follows from (9) that there exists $\bar{u} \in T(\bar{x})$ such that

$$\Phi(\bar{u}, \bar{x}, y) + f(\bar{x}, y) \le 0, \quad \text{for all } y \in X,$$

and hence inequality (5) is proved. ■

Utilizing Theorem 4.2, we can derive the following result for generalized variational inequalities.

Corollary 4.1. *Let X be a nonempty bounded closed convex subset of a reflexive Banach space B. Suppose that the following conditions are satisfied:*

- (i) $f : X \times X \to (-\infty, +\infty)$ *is convex and lower semicontinuous in the first variable and concave in the second variable such that $f(x, x) = 0$, for all $x \in X$;*
- (ii) $T : X \to 2^{B^*}$ *is upper semicontinuous from the line segments in X to the weak topology of B^* such that, for each $x \in X$, $T(x)$ is a nonempty subset of B^* and, for each fixed $y \in X$,*

$$\begin{aligned} &\{x \in X : \inf_{u \in T(x)} \mathrm{Re}\langle u, x - y\rangle + f(x, y) \le 0\} \\ &\subset \{x \in X : \sup_{w \in T(y)} \mathrm{Re}\langle w, x - y\rangle + f(x, y) \le 0\}. \end{aligned}$$

Then there exists $\bar{x} \in X$ such that

$$\inf_{u \in T(\bar{x})} \mathrm{Re}\langle u, \bar{x} - y\rangle + f(\bar{x}, y) \le 0, \quad \text{for all } y \in X.$$

In addition, if $T(\bar{x})$ is convex, then there exists $\bar{u} \in T(\bar{x})$ such that

$$\mathrm{Re}\langle \bar{u}, \bar{x} - y\rangle + f(\bar{x}, y) \le 0, \quad \text{for all } y \in X.$$

Proof. Since B is reflexive and X is a nonempty bounded closed convex subset of B, X is compact in the weak topology. Put $K = X$ and define $\Phi : B^* \times X \times X \to (-\infty, +\infty)$ by

$$\Phi(u, x, y) = \mathrm{Re}\langle u, x - y\rangle, \quad \text{for all } (u, x, y) \in B^* \times X \times X.$$

Moreover, define $\phi, \psi : X \times X \to (-\infty, +\infty)$ as follows:

$$\phi(x, y) = \inf_{u \in T(x)} \mathrm{Re}\langle u, x - y\rangle, \quad \text{and} \quad \psi(x, y) = \sup_{w \in T(y)} \mathrm{Re}\langle w, x - y\rangle,$$

for all $(x, y) \in X \times X$. Next, we verify that conditions (i)-(v) in Theorem 4.2 are satisfied. Indeed since for each fixed $x \in X$, the mapping

$$y \mapsto \inf_{u \in T(x)} \mathrm{Re}\langle u, x - y\rangle$$

is concave and hence is 0-diagonally concave. Thus, $\phi(x, y)$ is 0-generalized quasiconcave in y. In particular, $\phi(x, y)$ is 0-generalized quasiconcave in y with respect to K. Also for each fixed $y \in X$, $\psi(x, y)$ is lower semicontinuous in x. This shows that condition (i) in Theorem 4.2 holds. Furthermore, it is easy to see that conditions (ii), (iv), (v) in Theorem 4.2 also hold. Finally, we shall verify that condition (iii) in Theorem 4.2 is valid. Suppose that for each $(x, y) \in X \times X$, $\sup_{w \in T(y_t)} \Phi(w, x, y_t) + f(x, y_t) \le 0, \forall y_t = ty + (1 - t)x$, $t \in [0, 1]$. Utilizing the facts that $f(x, y)$ is concave in y and $f(\bar{x}, \bar{x}) = 0$, we conclude that

$$\begin{aligned}
&t[\sup_{w \in T(y_t)} \mathrm{Re}\langle w, \bar{x} - \bar{y}\rangle + f(\bar{x}, \bar{y})] \\
&\le t \sup_{w \in T(y_t)} \mathrm{Re}\langle w, \bar{x} - \bar{y}\rangle + tf(\bar{x}, \bar{y}) + (1 - t)f(\bar{x}, \bar{x}) \\
&\le \sup_{w \in T(y_t)} \Phi(w, x, y_t) + f(\bar{x}, y_t) \le 0.
\end{aligned}$$

Consequently,

$$\sup_{w \in T(y_t)} \mathrm{Re}\langle w, \bar{x} - \bar{y}\rangle + f(\bar{x}, \bar{y}) \le 0, \quad \text{for all } t \in [0, 1].$$

This implies that condition (iii) in Theorem 4.2 holds. Therefore, the conclusion follows immediately from Theorem 4.2. ∎

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