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ON GÖLLNITZ-GORDON TYPE IDENTITIES AND DURFEE DISSECTION

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Abstract. By using the Durfee rectangle and Durfee dissection we give combinatorial interpretations for Göllnitz-Gordon type identities. Also, we give a generalization of an identity due to G. E. Andrews.

1. Introduction

The Durfee square has been utilized not only to give new proofs for some famous identities but also to extract nontrivial combinational results from known identities. We give some examples here. The identity of Euler [8]

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n^2}=\frac{1}{(q;q)_\infty} \tag{1}$$

owns an elegant combinatorial proof through the use of the Durfee square (see [11, pp. 280-281] or [6, p. 27]). Here, we have adopted the notation $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ and $(a;q)_\infty = \lim_{n\to\infty}(a;q)_n$ for $|q|<1$. Also, by convention, we define $(a;q)_0 = 1$. Sylvester [13] also applied the concept of the Durfee square to prove that

$$1+\sum_{n=1}^{\infty}\frac{x^n q^{(3n-1)/2}(1+xq^{2n})(-xq;q)_{n-1}}{(q;q)_n}=(-xq)_\infty, \tag{2}$$

from which Euler's pentagonal number theorem [1, Theorem 14-4] may be deduced by setting $x=-1$.

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The utilization of Durfee squares has been studied extensively by G. E. Andrews [2-4], in which he established generalizations of (1) and (2). In particular, in [4], Andrews introduced the concept of successive Durfee squares and rectangles and defined the "(k,a)-Durfee dissection" of a partition in order to give a combinatorial interpretation for the identity

$$\prod_{\substack{n=1 \\ n\not\equiv 0,\pm a (\mathrm{mod}\, 2k+1)}}^{\infty} \frac{1}{(1-q^n)} = \sum_{n_1,\ldots,n_{k-1}\geqslant 0} \frac{q^{N_1^2+\ldots+N_{k-1}^2+N_a+\ldots+N_{k-1}}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}}, \tag{3}$$

where $k \geq 2$ and $N_i = n_i + \cdots + n_{k-1}$ for $i = 1, 2, \ldots, k-1$. Note that the identity (3) reduces to the famous Rogers-Ramanujan identities when $k = a = 2$ and $k = a + 1 = 2$.

Motivated by the work of Andrews in [4], we naturally try to derive some combinatorial results from Göllnitz-Gordon type identities which are perfect analogue of Rogers-Ramanujan type identities. We will do this in the next section by extending Andrews's "(k,a)-Durfee dissection". At the end, we give a generalization of an identity due to Andrews [4, 5].

2. Durfee Dissection and the Main Results

The Göllnitz-Gordon identities may be stated analytically as

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n^2} = \frac{1}{(q;q^8)_\infty (q^4;q^8)_\infty (q^7;q^8)_\infty} \quad \text{and} \tag{4}$$

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n^2+2n} = \frac{1}{(q^3;q^8)_\infty (q^4;q^8)_\infty (q^5;q^8)_\infty}. \tag{5}$$

The identities (4) and (5) were discovered independently by H. Göllnitz [9] and B. Gordon [10]. D. Bressoud [7] and P. Paule [12] generalized (4) and (5) to

$$\begin{aligned} &\prod_{\substack{n=1 \\ n\not\equiv 2 \pmod 4 \\ n\not\equiv 0,\pm(2k-1) (\mathrm{mod}\, 4k)}}^{\infty} \frac{1}{(1-q^n)} \\ &= \sum_{n_1,\ldots,n_{k-1}\geqslant 0} \frac{q^{2(N_1^2+\ldots+N_{k-1}^2)}(-q^{2N_{k-1}+1};q^2)_\infty}{(q^2;q^2)_{n_1}\cdots(q^2;q^2)_{n_{k-1}}} \end{aligned} \tag{6}$$

(see Bressoud [7, (3.6) with $r = l = 1$] or Paule [12, (54)]) and

$$\prod_{\substack{n=1 \\ n\not\equiv 2 \pmod 4 \\ n\not\equiv 0,\pm 1 \pmod{4k}}}^{\infty} \frac{1}{(1-q^n)} = \sum_{n_1,\dots,n_{k-1}\geqslant 0} \frac{q^{2(N_1^2+\dots+N_{k-1}^2+N_1+\dots+N_{k-1})}(-q^{2N_{k-1}+3};q^2)_\infty}{(q^2;q^2)_{n_1}\cdots(q^2;q^2)_{n_{k-1}}} \tag{7}$$

(see Bressoud [7, (3.7) with $r = 1$] or Paule [12, (53)]).
Also, Bressoud proved the identity

$$\prod_{\substack{n=1 \\ n\not\equiv 2 \pmod 4 \\ n\not\equiv 0,\pm(2a-1) \pmod{4k}}}^{\infty} \frac{1}{(1-q^n)} = \sum_{n_1,\dots,n_{k-1}\geqslant 0} \frac{q^{2(N_1^2+\dots+N_{k-1}^2+N_a+\dots+N_{k-1})}(-q^{1-2N_1};q^2)_{N_1}}{(q^2;q^2)_{n_1}\cdots(q^2;q^2)_{n_{k-1}}} \tag{8}$$

(see Bressoud [7, (3.9)]). Here we have $k \geq 2$ and $N_i = n_i + \cdots + n_{k-1}$ for $i = 1, 2, \dots, k-1$.

According to the similarity between the identities (6)-(8) and (3), we are motivated to find combinatorial meanings for these Göllnitz-Gordon type identities. To do so, we will extend Andrews's "(k, a)-Durfee dissection" of a partition by introducing an additional parameter r and defining the "(k, a, r)-Durfee dissection" of a partition. First, we determine the first $a - 1$ successive maximal rectangles of the partition such that, for each such rectangle, the number of dots on the horizontal side is r times the number of dots on the vertical side. Next, we determine $k - a$ maximal rectangles such that the number of dots on the horizontal side is one less than r times the number of dots on the vertical side. We will demonstrate the concept of the "(k, a, r)-Durfee dissection" in details in the proofs of Theorems 1-3.

As for "(k, a)-admissible" defined in [4], we shall say that a partition is "(k, a, r)-admissible" if it has no parts below the last rectangle in its (k, a, r)-Durfee dissection and furthermore the lower edge of each of the final $k - a$ rectangles of the (k, a, r)-Durfee dissection is actually a part of the partition.

For simplicity, we adopt the standard notation $\begin{bmatrix} M \\ N \end{bmatrix}_q$ for the Gaussian polynomial which is defined by

$$\begin{bmatrix} M \\ N \end{bmatrix}_q = \frac{(q;q)_M}{(q;q)_N(q;q)_{M-N}},$$

It is known that $\begin{bmatrix} M \\ N \end{bmatrix}_q$ is the generating function for the number of partitions of n into at most $M - N$ parts, all parts not exceeding N [6, p. 33, Theorem 3.1]. Hence, for any positive integer k, $\begin{bmatrix} M \\ N \end{bmatrix}_{q^k}$ serves as the generating function for the number of partitions of n into at most $M - N$ parts, all parts $\equiv 0 \pmod k$ and not exceeding kN. This fact will be used in the proofs of the theorems below.

In the following, for any partition λ, we introduce the partitions λ_e and λ_o such that λ_e is the subpartition collecting all the even parts of λ and λ_o is the subpartition collecting all the odd parts of λ. For example, if $\lambda = (7, 7, 5, 4, 4, 2, 1)$, a partition of 20, then $\lambda_e = (4, 4, 2)$ and $\lambda_o = (7, 7, 5, 1)$.

Theorem 1. *The number of partitions of n into parts $\not\equiv 0, \pm(2k-1) \pmod{4k}$ and $\not\equiv 2 \pmod 4$ equals the number of partitions λ of n satisfying that λ_e is $(k, k, 2)$-admissible and all parts of λ_o are distinct and greater than the horizontal size of the smallest Durfee rectangle in the $(k, k, 2)$-dissection of λ_e.*

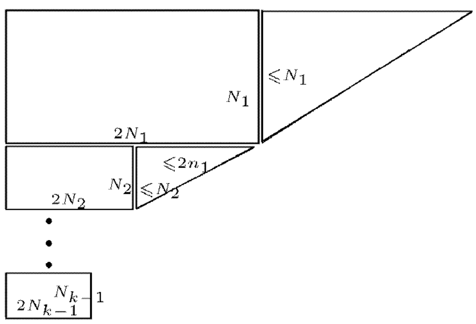
Proof. The left side of (6) is clearly the generating function for the number of partitions of n into parts $\not\equiv 0, \pm(2k-1) \pmod{4k}$ and $\not\equiv 2 \pmod 4$. To prove this theorem, we will show that the right side of (6) is actually the generating function for the number of partitions of n of the second type in the theorem.

The right side of (6) may be rewritten as

$$\sum_{n_1,\ldots,n_{k-1}\geqslant 0} \frac{q^{2N_1\times N_1}}{(q^2;q^2)_{N_1}} \cdot q^{2N_2\times N_2} \begin{bmatrix} N_1 \\ n_1 \end{bmatrix}_{q^2} \cdots q^{2N_{k-1}\times N_{k-1}} \begin{bmatrix} N_{k-2} \\ n_{k-2} \end{bmatrix}_{q^2} \cdot (-q^{2N_{k-1}+1}; q^2)_\infty .$$

Note that $\frac{q^{2N_1\times N_1}}{(q^2;q^2)_{N_1}}$ is the generating function for the Durfee rectangle of horizontal side $2N_1$ and vertical side N_1, with an attached partition which has at most N_1 parts, each even. In the following graph, the largest rectangle and triangle are, respectively, the Durfee rectangle and the attached partition mentioned above.

For $2 \leqslant j \leqslant k-1$, $\begin{bmatrix} N_{j-1} \\ n_{j-1} \end{bmatrix}_{q^2}$ generates the partition which has at most $(N_{j-1} - n_{j-1}) = N_j$ parts, each even and $\leqslant 2n_{j-1} = 2(N_{j-1} - N_j)$. So, $q^{2N_j\times N_j} \begin{bmatrix} N_{j-1} \\ n_{j-1} \end{bmatrix}_{q^2}$



generates those parts of the partition with a j-th Durfee rectangle of horizontal side $2N_j$ and vertical side N_j and the partition attached to the j-th Durfee rectangle which has at most N_j parts, each even and $\leqslant 2(N_{j-1} - N_j)$. Thus

$$\frac{q^{2N_1\times N_1}}{(q^2;q^2)_{N_1}} \cdot q^{2N_2\times N_2} \begin{bmatrix} N_1 \\ n_1 \end{bmatrix}_{q^2} \cdots q^{2N_{k-1}\times N_{k-1}} \begin{bmatrix} N_{k-2} \\ n_{k-2} \end{bmatrix}_{q^2}$$

generates the partition λ_e with all parts even and $(k,k,2)$-admissible. On the other hand, $(-q^{2N_{k-1}+1};q^2)_\infty$ clearly generates distinct odd parts, each $> 2N_{k-1}$, the horizontal size of the smallest Durfee rectangle in the $(k,k,2)$-dissection of λ_e.

Hence the right side of (6) is the generating function for the number of partitions of n of the second type in the theorem. We complete the proof of the theorem. ■

In Theorem 1, let $k=2, n=10$, then the partitions of 10 into parts $\equiv 1,4,7 \pmod 8$ are $9+1$, $7+1+1+1$, $4+4+1+1$, $4+1+1+1+1+1+1$, and $1+1+1+1+1+1+1+1+1+1$; and the partitions $\lambda = \lambda_e + \lambda_o$ of 5 satisfying that λ_e is $(2,2,2)$-admissible and all parts of λ_o are distinct and greater than the horizontal size of the smallest Durfee rectangle in the $(2,2,2)$-dissection of λ_e are $9+1$, $7+3$, $5+3+2$, $6+4$, and 10.

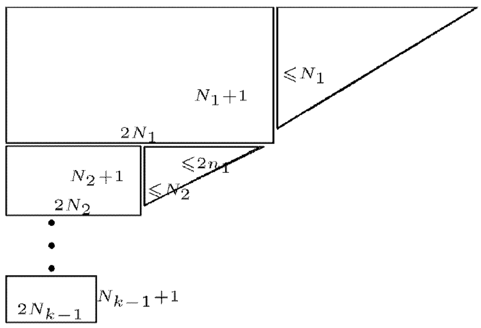
Theorem 2. *The number of partitions of n into parts* $\not\equiv 0, \pm 1 \pmod{4k}$ *and* $\not\equiv 2 \pmod 4$ *equals the number of partitions* $\lambda = \lambda_e + \lambda_o$ *of n satisfying that* λ_e *is* $(k,1,2)$*-admissible and all parts of* λ_o *are distinct and greater than 2 plus the smallest even part of* λ_e.

Proof. The left side of (7) is clearly the generating function for the number of partitions of n into parts $\not\equiv 0, \pm 1 \pmod{4k}$ and $\not\equiv 2 \pmod 4$. To prove this theorem, we will show that the right side of (7) is the generating function for the number of partitions of n of the second type in the theorem.

The right side of (7) may be rewritten as

$$\sum_{n_1,\ldots,n_{k-1}\geqslant 0} \frac{q^{2N_1\times(N_1+1)}}{(q^2;q^2)_{N_1}} \cdot q^{2N_2\times(N_2+1)} \begin{bmatrix} N_1 \\ n_1 \end{bmatrix}_{q^2} \cdots q^{2N_{k-1}\times(N_{k-1}+1)} \begin{bmatrix} N_{k-2} \\ n_{k-2} \end{bmatrix}_{q^2} \cdot (-q^{2N_{k-1}+3};q^2)_\infty.$$

For $2 \leqslant j \leqslant k-1$, $q^{2N_j\times(N_j+1)} \begin{bmatrix} N_{j-1} \\ n_{j-1} \end{bmatrix}_{q^2}$ generates those parts of the partition with the j-th Durfee rectangle of horizontal side $2N_j$ and vertical side N_j+1 whose lower edge is actually a part since the partition attached to the j-th Durfee rectangle has at most N_j parts, each even and $\leqslant 2(N_{j-1} - N_j)$ (see the graph below).



Thus

$$\frac{q^{2N_1\times(N_1+1)}}{(q^2;q^2)_{N_1}}\cdot q^{2N_2\times(N_2+1)}\begin{bmatrix}N_1\\ n_1\end{bmatrix}_{q^2}\cdots q^{2N_{k-1}\times(N_{k-1}+1)}\begin{bmatrix}N_{k-2}\\ n_{k-2}\end{bmatrix}_{q^2}$$

generates the partition λ_e with all parts even and $(k,1,2)$-admissible. On the other hand, $(-q^{2N_{k-1}+3};q^2)_\infty$ clearly generates those parts of the partition with distinct odd parts, each $> 2N_{k-1}+2$. Moreover, $2N_{k-1}$ is the horizontal size of the smallest Durfee rectangle in the $(k,1,2)$-dissection of λ_e and is indeed the smallest part of λ_e, and so $(-q^{2N_{k-1}+3};q^2)_\infty$ generates those parts of the partition with distinct odd parts, each greater than 2 plus the smallest even part of λ_e. This completes the proof of the theorem. ∎

In Theorem 2, let $k=2, n=10$, then the partitions of 10 into parts $\equiv 3,4,5 \pmod 8$ are $5+5$ and $4+3+3$, and the partitions $\lambda=\lambda_e+\lambda_o$ of 10 satisfying that λ_e is $(2,1,2)$-admissible and all parts of λ_o are distinct and greater than the smallest even part of λ_e plus 2 are $7+3$ and $8+2$.

Theorem 3. *The number of partitions of n into parts* $\not\equiv 0, \pm(2a-1) \pmod{4k}$ *and* $\not\equiv 2 \pmod 4$ *equals the number of representations of n into* $e_1+e_2+\cdots+e_i-o_1-o_2-\cdots-o_j$ *where* $e_1\geqslant e_2\geqslant\cdots\geqslant e_i$ *are all even and* $(k,a,2)$*-admissible,* $o_1>o_2>\cdots>o_j$ *are all odd and smaller than the number of dots in the horizontal side of the first Durfee rectangle of the dissection of the partition* $e_1+e_2+\cdots+e_i$.

Proof. The left side of (8) is clearly the generating function for the number of partitions of n into parts $n\not\equiv 0,\pm(2a-1) (\mathrm{mod}\ 4k)$ and $\not\equiv 2 \pmod 4$. To prove this theorem, we will show that the right side of (8) is the generating function for the number of representations of n into $e_1+e_2+\cdots+e_i-o_1-o_2-\cdots-o_j$ satisfying all the conditions described in the theorem.

The right side of (8) may be rewritten as

$$\sum_{n_1,\ldots,n_{k-1}\geqslant 0} \frac{q^{2N_1\times N_1}}{(q^2;q^2)_{N_1}} \cdot q^{2N_2\times N_2} \begin{bmatrix} N_1 \\ n_1 \end{bmatrix}_{q^2} \cdots q^{2N_{a-1}\times N_{a-1}} \begin{bmatrix} N_{a-2} \\ n_{a-2} \end{bmatrix}_{q^2}$$
$$\cdot\, q^{2N_a\times(N_a+1)} \begin{bmatrix} N_{a-1} \\ n_{a-1} \end{bmatrix}_{q^2} \cdots q^{2N_{k-1}\times(N_{k-1}+1)} \begin{bmatrix} N_{k-2} \\ n_{k-2} \end{bmatrix}_{q^2} \cdot (-q^{1-2N_1};q^2)_{N_1}.$$

By similar arguments used in the proofs of theorem (1) and (2), we know that

$$\begin{aligned} &\frac{q^{2N_1\times N_1}}{(q^2;q^2)_{N_1}} \cdot q^{2N_2\times N_2} \begin{bmatrix} N_1 \\ n_1 \end{bmatrix}_{q^2} \cdots q^{2N_{a-1}\times N_{a-1}} \begin{bmatrix} N_{a-2} \\ n_{a-2} \end{bmatrix}_{q^2} \\ &\cdot q^{2N_a\times(N_a+1)} \begin{bmatrix} N_{a-1} \\ n_{a-1} \end{bmatrix}_{q^2} \cdots q^{2N_{k-1}\times(N_{k-1}+1)} \begin{bmatrix} N_{k-2} \\ n_{k-2} \end{bmatrix}_{q^2} \end{aligned} \tag{9}$$

is the generating function of $(k,a,2)$-admissible partitions. And, $(-q^{1-2N_1};q^2)_{N_1}$ is the generating function for the number of representations of a negative integer into distinct parts taken from $-1,-3,\ldots,-(2N_1-1)$. Note that the odd numbers $1,3,\ldots,(2N_1-1)$ are all smaller than $2N_1$ which turns out to be the horizontal side of the first Durfee rectangle appearing in (9). Therefore, the right side of (8) is the generating function for the number of representations of n of the second type described in the theorem. This completes the proof of the theorem. ■

Remark In the statement of Theorem 3, let $o_1 = 2m-1$ for some positive integer m, then we have $e_1+e_2+\cdots+e_i = n+o_1+o_2+\cdots+o_j \leqslant n+m^2$ and $(2m)\cdot m \leqslant e_1+e_2+\cdots+e_i$. Thus $2m^2 \leqslant n+m^2$ and so $m \leqslant \sqrt{n}$, which means $o_1 \leqslant 2\sqrt{n}-1$. In the case $a=1$, we can do slightly better, that is $m \leqslant \sqrt{n+1}-1$, since $e_1+e_2+\cdots+e_i \geqslant (2m)(m+1)$ in this case.

In Theorem 3, let $n=10, a=1, k=2$, then the partitions of 10 into parts$\equiv 3,4,5 \pmod 8$ are $5+5$ and $4+3+3$, and the representations of 10 into $e_1+e_2+\cdots+e_i-o_1-o_2-\cdots-o_j$ in which $e_1 \geqslant e_2 \geqslant \cdots \geqslant e_i$ are all even and $(k,a,2)$-admissible, $o_1 > o_2 > \cdots > o_j$ are all odd and smaller than the number of dots in the horizontal side of the first Durfee rectangle of the dissection of the partition $e_1+e_2+\cdots+e_i$ are $8+2$ and $6+4+4-3-1$.

3. A Generalization of Andrews's Identity

Andrews [5] derives the identity

$$\frac{1}{(zq;q)_\infty} = \sum_{n_1,\ldots,n_{l-1}\geqslant 0} \frac{q^{N_1^2+\cdots+N_{l-1}^2} z^{N_1+\cdots+N_{l-1}}}{(q;q)_{n_1}\cdots(q;q)_{n_{l-1}}(zq;q)_{n_{l-1}}}, \tag{10}$$

where $l \geqslant 2$ and $N_i = n_i + \cdots + n_{l-1}$ for $i = 1, 2, \ldots, l-1$. Letting $l = 2$ in (10), we have

$$\frac{1}{(zq;q)_\infty} = \sum_{n \geqslant 0} \frac{q^{n^2} z^n}{(q;q)_n (zq;q)_n}, \tag{11}$$

which is originally due to Cauchy and can be found in [6, p. 20, Corollary 2.6]. Andrews [2] also obtained a generalization for (11) by establishing the identity

$$\frac{1}{(zq;q)_\infty} = \sum_{(i,j)} \sum_{N=0}^{\infty} \frac{q^{(hN+i)(kN+j)} z^{hN+i}}{(q;q)_{hN+i\delta(j,k)} (zq;q)_{kN+j-1+\delta(i,0)+\delta(i,h)}}, \tag{12}$$

where h, k are positive integers, $\delta(a, b)$ is defined to be zero if $a \neq b$ and 1 if $a = b$, and the sum $\sum_{(i,j)}$ is restricted to those pairs (i, j) such that

(R) either $i = j = 0$, or $1 \leqslant i \leqslant h, 1 \leqslant j \leqslant k$ with $(i,k) \neq (h,k)$.

Now, by observing the way that (12) generalizes (11), one might naturally conjecture an extension for (10), namely,

$$\frac{1}{(zq;q)_\infty} = \sum_{\substack{(i_1,j_1) \\ \vdots \\ (i_{l-1},j_{l-1})}} \sum_{n_1,\ldots,n_{l-1} \geqslant 0} \frac{q^{(hN_1+i_1)(kN_1+j_1)+\cdots+(hN_{l-1}+i_{l-1})(kN_{l-1}+j_{l-1})} z^{(hN_1+i_1)+\cdots+(hN_{l-1}+i_{l-1})}}{(q;q)_{hn_1+i_1\delta(j_1,k)} \cdots (q;q)_{hn_{l-1}+i_{l-1}\delta(j_{l-1},k)} (zq;q)_{kn_{l-1}+j_{l-1}-1+\delta(i_{l-1},0)+\delta(i_{l-1},h)}}, \tag{13}$$

where $l \geqslant 2$, h, k are positive integers, $N_i = n_i + \cdots + n_{l-1}$ for $i = 1, 2, \ldots, l-1$, and $(i_1, j_1), \ldots, (i_{l-1}, j_{l-1})$ are ordered pairs satisfying the restriction (**R**).

Although it implies both (10) (with $h = k = 1$) and (12) (with $l = 2$), the conjecture (13) turns out to be incorrect. To see this, one only needs to equate the coefficients of zq on both sides of (13) with $h = 1, k = 2$, and $l = 3$. The valid generalization we are looking for is indeed akin to (13) but a bit more complicated and will be derived in the following.

From Andrews's result [2, (2.13)]

$$\frac{1}{(zq;q)_M} = \sum_{(i,j)} \sum_{N \geqslant 0} \frac{q^{(hN+i)(kN+j)} z^{hN+i}}{(zq;q)_{kN+j-1+\delta(i,0)+\delta(i,h)}} \begin{bmatrix} M + hN + i\delta(j,k) - kN - j \\ hN + i\delta(j,k) \end{bmatrix}_q$$

where the ordered pair (i, j) satisfies the restriction (**R**) and the inner sum satisfies that $kN + j \leqslant M$, we can rewrite (12) as

$$\begin{aligned}
&\frac{1}{(zq;q)_\infty}\\
&=\sum_{\substack{(i_1,j_1)\\(i_2,j_2)}}\sum_{N_1\geqslant N_2\geqslant 0}\frac{q^{(hN_1+i_1)(kN_1+j_1)}z^{hN_1+i_1}}{(q;q)_{hN_1+i_1\delta(j_1,k)}}\frac{q^{(hN_2+i_2)(kN_2+j_2)}z^{hN_2+i_2}}{(zq;q)_{kN_2+j_2-1+\delta(i_2,0)+\delta(i_2,h)}}\\
&\qquad\cdot\begin{bmatrix}kN_1+j_1-1+\delta(i_1,0)+\delta(i_1,h)+hN_2+i_2\delta(j_2,k)-kN_2-j_2\\ hN_2+i_2\delta(j_2,k)\end{bmatrix}_q
\end{aligned}$$

where (i_1, j_1) and (i_2, j_2) satisfy (**R**) and the restriction $N_1 \geqslant N_2 \geqslant 0$ follows from the inequality

$$kN_2 + j_2 \leqslant kN_1 + j_1 - 1 + \delta(i_1, 0) + \delta(i_1, h)$$

or equivalently,

$$N_1 - N_2 \geqslant \frac{1}{k}(j_2 - j_1 + 1 - \delta(i_1, 0) - \delta(i_1, h)) \geqslant 0.$$

Repeating the process above for $l - 2$ times, we arrive at

$$\begin{aligned}
&\frac{1}{(zq;q)_\infty}\\
&=\sum_{\substack{(i_1,j_1)\\ \vdots\\ (i_{l-1},j_{l-1})}}\sum_{N_1\geq\cdots\geq N_{l-1}\geq 0}\frac{q^{\sum_{s=1}^{l-1}(hN_s+i_s)(kN_s+j_s)}z^{\sum_{s=1}^{l-1}(hN_s+i_s)}}{(q;q)_{hN_1+i_1\delta(j_1,k)}(zq;q)_{kN_{l-1}+j_{l-1}-1+\delta(i_{l-1},0)+\delta(i_{l-1},h)}}\\
&\cdot\prod_{s=1}^{l-2}\begin{bmatrix}kN_s+j_s-1+\delta(i_s,0)+\delta(i_s,h)+hN_{s+1}+i_{s+1}\delta(j_{s+1},k)-kN_{s+1}-j_{s+1}\\ hN_{s+1}+i_{s+1}\delta(j_{s+1},k)\end{bmatrix}_q.
\end{aligned}$$

To simplify the notation, we let $\alpha_s = kn_s + j_s - 1 + \delta(i_s, 0) + \delta(i_s, h)$, $\Gamma_s = hN_s + i_s\delta(j_s, k)$, and $\gamma_s = hn_s + i_s\delta(j_s, k)$ for $s = 1, \cdots, l-1$. Note that $\Gamma_{l-1} = \gamma_{l-1}$. With the help of simplified notation, we can rewrite the last equality as

$$\frac{1}{(zq;q)_\infty}$$

$$
(14)\quad
\begin{aligned}
&= \sum_{\substack{(i_1,j_1)\\ \vdots \\ (i_{l-1},j_{l-1})}} \sum_{N_1\geq\cdots\geq N_{l-1}\geq 0} \frac{q^{\sum_{s=1}^{l-1}(hN_s+i_s)(kN_s+j_s)} z^{\sum_{s=1}^{l-1}(hN_s+i_s)}}{(q;q)_{\Gamma_1}(zq;q)_{\alpha_{l-1}}} \\
&\qquad \prod_{s=1}^{l-2} \begin{bmatrix} (q;q)_{\alpha_s+\Gamma_{s+1}-j_{s+1}} \\ \Gamma_{s+1} \end{bmatrix}_q \\
&= \sum_{\substack{(i_1,j_1)\\ \vdots \\ (i_{l-1},j_{l-1})}} \sum_{n_1,\ldots,n_{l-1}\geq 0} \frac{q^{\sum_{s=1}^{l-1}(hN_s+i_s)(kN_s+j_s)} z^{\sum_{s=1}^{l-1}(hN_s+i_s)}}{\left(\prod_{s=1}^{l-1}(q;q)_{\gamma_s}\right)(zq;q)_{\alpha_{l-1}}} \\
&\qquad \prod_{s=1}^{l-2} \frac{(q;q)_{\alpha_s+\Gamma_{s+1}-j_{s+1}}(q;q)_{\gamma_s}}{(q;q)_{\alpha_s-j_{s+1}}(q;q)_{\Gamma_s}}.
\end{aligned}
$$

Observe the difference between (14) and our previously false conjecture (13). On the other hand, note that (14) reduces to (12) with $l = 2$ and reduces to (10) with $h = k = 1$ since, in the latter case, we have $i_1 = j_1 = \cdots = i_{l-1} = j_{l-1} = 0$.

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