TAIWANESE JOURNAL OF MATHEMATICS
Vol. 18, No. 5, pp. 1663-1678, October 2014
DOI: 10.11650/tjm.18.2014.3759
This paper is available online at http://journal.taiwanmathsoc.org.tw

WEIGHTED HARDY SPACES ASSOCIATED TO SELF-ADJOINT OPERATORS AND $BMO_{L,w}$

Suying Liu, Kai Zhao* and Shujuan Zhou

Abstract. Let L be a non-negative self-adjoint operator satisfying a pointwise Guassian estimate for its heat kernel. Let w be some A_s weight on $\mathbb{R}^n$. In this paper, we obtain a weighted $(p, q)-$atomic decomposition with $q \geq s$ for the weighted Hardy spaces $H^p_{L,w}(\mathbb{R}^n)$, $0 < p \leq 1$. We also introduce the suitable weighted BMO spaces $BMO^p_{L,w}$. Then the duality between $H^1_{L,w}(\mathbb{R}^n)$ and $BMO_{L,w}$ is established.

1. INTRODUCTION

One of the central part of modern harmonic analysis is the theory of Hardy spaces which was initiated by Stein, Fefferman and Weiss [26, 14]. It is known that the classical weighted spaces H^p_w, which are associated to Laplacian, have been extensively studied by Garcia-Cuerva [15] and Strömberg and Torchinsky [27], where w is a Muckenhoupt's weight A_p.

Since there are some important situations in which the theory of classical Hardy spaces is not applicable, many authors begin to study Hardy spaces that are adapted to a linear operator L. For example, Auscher, Duong and McIntosh [1], and then Duong and Yan [12, 13], introduced the unweighted Hardy and BMO spaces adapted to an operator L which satisfies the Gaussian heat kernel upper bounds. For more results, we refer to [3, 2, 20, 19, 18] and the references therein.

Recently, Song and Yan [24] discussed the weighted theory of Hardy space $H^1_{L,w}$ associated to Schrödinger operators, for $w \in A_1 \cap \mathrm{RH}_2$. Bui and Duong [4] improved the results of [24] to $H^p_{L,w}$, $0 < p \leq 1$, and obtained the atomic and molecular characterizations of the elements of $H^p_{L,w}$. In [5], they studied the weighted BMO spaces

Received September 6, 2013, accepted May 27, 2014.
Communicated by Chin-Cheng Lin.
2010 *Mathematics Subject Classification*: Primary 42B20, 42B30; Secondary 47F05.
Key words and phrases: Weighted Hardy space, Self-adjoint operator, Weighted atom, $BMO_{L,w}$.
This work is supported by NNSF-China Grant No. 11041004, Tianyuan Fund for Mathematics No. 11326093 and NSF of Shandong Province China No. ZR2010AM032.
*Corresponding author.

associated to operators, and obtained that the dual space of $H^1_L(X, w)$ in [24] was $BMO_{L^*}(X, w)$ associated to the adjoint operator L^*. As we know, the decompositions of function spaces are very critical in harmonic analysis. The first author and Song in [21] improved the results of [24]. They gave a new atomic decomposition (different from that of [24]) for weighted function space $H^1_{L,w}(\mathbb{R}^n)$. Comparing with [24], the condition $w \in A_1 \cap \mathrm{RH}_2$ was weakened to $w \in A_1$.

One of our purpose in this paper is to extend the results of [21] to $H^p_{L,w}(\mathbb{R}^n),\ 0 < p \le 1$, by the theory of Littlewood-Paley functions and semigroup properties. And then define an adapted weighted BMO space, and establish its duality with $H^1_{L,w}(\mathbb{R}^n)$.

The layout of the paper is as follows. In section 2, we prepare some notations and preliminary lemmas. In section 3, we introduce weighted Hardy spaces $H^p_{L,w}(\mathbb{R}^n)$ associated to a non-negative self-adjoint operator with Gaussian upper bounds on its heat kernel, and obtain an atomic decomposition. In section 4, we study $BMO^p_{L,w}$ spaces associated to operators and establish the duality between $H^1_{L,w}(\mathbb{R}^n)$ and $BMO_{L,w}$.

Throughout this paper, the letter "C" or "c" will denote (possibly different) constants that are independent of the essential variables.

2. Notations and Preliminaries

2.1. Preliminaries

Suppose that L is a non-negative self-adjoint operator on $L^2(\mathbb{R}^n)$ and that each of the heat semigroup e^{-tL} generated by $-L$, has the kernel $p_t(x, y)$ which satisfies the following Gaussian upper bounds, i.e., there exist constants $C, c > 0$ such that

$$(GE) \qquad |p_t(x,y)| \le \frac{C}{t^{n/2}} \exp\Big(-\frac{|x-y|^2}{c\,t}\Big).$$

We note that such estimates are typical for elliptic or sub-elliptic differential operators of second order (see for instance, [9] and [11]).

Now we introduce the following useful lemma, refer to [9] and [23].

Lemma 2.1. *Let L be a non-negative self-adjoint operator satisfying (GE). For every $k = 0, 1, \ldots$, there exist two positive constants C_k, c_k such that the kernel $p_{t,k}(x, y)$ of the operator $(t^2L)^k e^{-t^2L}$ satisfies*

$$(2.1) \qquad |p_{t,k}(x,y)| \le \frac{C_k}{(4\pi t)^n} \exp\Big(-\frac{|x-y|^2}{c_k t^2}\Big),$$

for all $t > 0$ and almost every $x, y \in \mathbb{R}^n$.

Suppose that F is a closed set in $\mathbb{R}^n$, $\gamma \in (0, 1)$ is fixed. We set

$$F^* := \Big\{x \in \mathbb{R}^n : \text{ for every ball } B(x) \text{ in } \mathbb{R}^n \text{ centered at } x, \frac{|F \cap B(x)|}{|B(x)|} \ge \gamma\Big\},$$

and every x as above is called a point having global γ-density with respect to F. One can see that F^* is closed and $F^* \subset F$. Also,

$$^cF^* = \{x \in \mathbb{R}^n : \mathcal{M}(\chi_{^cF})(x) > 1 - \gamma\},$$

where $\mathcal{M}$ is Hardy-Littlewood maximal function, which implies $|^cF^*| \leq C|F|$ with C depending on γ and the dimension only. We define a saw-tooth region $\mathcal{R}(F) = \bigcup_{x \in F} \Gamma(x)$, where $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x - y| < t\}$.

From now on, the paper we denote by ^cF the complement of F.

The following lemma is very important for our main result (see [7]).

Lemma 2.2. *Suppose that Φ is a non-negative function on $\mathbb{R}^{n+1}_+$. There exists $\gamma \in (0,1)$, sufficiently close to 1, such that for every closed set F whose complement has finite measure the following inequality holds:*

$$\iint_{\mathcal{R}(F^*)} \Phi(y,t) t^n \, dydt \leq C_\gamma \int_F \iint_{\Gamma(x)} \Phi(y,t) \, dydtdx. \tag{2.2}$$

2.2. Muckenhoupt weights

We review some background on Muckenhoupt weights. We use the notation

$$\fint_E h(x)\, dx = \frac{1}{|E|} \int_E h(x)\, dx.$$

A weight w is a non-negative locally integrable function on $\mathbb{R}^n$. It is said that $w \in A_p$, $1 < p < \infty$, if there exists a constant C such that for every ball $B \subseteq \mathbb{R}^n$,

$$\Big(\fint_B w\, dx\Big)\Big(\fint_B w^{-1/(p-1)}\, dx\Big)^{p-1} \leq C.$$

For $p = 1$, $w \in A_1$ means that there is a constant C such that for every ball $B \subseteq \mathbb{R}^n$,

$$\fint_B w(y)\, dy \leq Cw(x) \quad \text{ for a.e. } x \in B.$$

Let $w \in A_p$, for $1 \leq p < \infty$. The weighted Lebesgue spaces L^p_w can be defined by $\big\{f : \int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx < \infty\big\}$ with norm $\|f\|_{L^p_w} := \big(\int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx\big)^{1/p}$.

We summarize some of the properties of classes in the following results, for more details, see [10], [16], [25] references therein.

Lemma 2.3. *Denote $w(E) := \int_E w(x)\, dx$ for any set $E \subseteq \mathbb{R}^n$. For $1 \leq p \leq \infty$, denote p' the adjoint number of p, i.e. $1/p + 1/p' = 1$. We have the following properties:*

(*i*) $A_1 \subseteq A_p \subseteq A_q$, *for* $1 \leq p \leq q < \infty$.

(*ii*) *If* $w \in A_p$, $1 < p < \infty$, *then there exists* $1 < q < p$ *such that* $w \in A_q$.

(*iii*) $A_\infty = \bigcup_{1 \le p < \infty} A_p$.

(*iv*) *If* $1 < p < \infty$, $w \in A_p$ *if and only if* $w^{1-p'} \in A_{p'}$.

(*v*) *Let* $w \in A_p$, $p \ge 1$. *Then for any ball* B *and* $\lambda > 1$, *we have that*

$$w(\lambda B) \le C \lambda^{np} w(B),$$

for some constant C *independent of* B *and* λ.

2.3. Finite speed propagation for the wave equation

Let L be an operator satisfying (GE), $E_L(\lambda)$ denote its spectral decomposition. Then for every bounded Borel function $F : [0, \infty) \to \mathbb{C}$, one defines the operator $F(L) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by the formula

$$F(L) := \int_0^\infty F(\lambda)\, dE_L(\lambda). \tag{2.3}$$

In particular, the operator $\cos(t\sqrt{L})$ is well-defined on $L^2(\mathbb{R}^n)$. Moreover, it follows from Theorem 3 of [8] that there exists a constant c_0 such that the Schwartz kernel $K_{\cos(t\sqrt{L})}(x, y)$ of $\cos(t\sqrt{L})$ satisfies

$$\mathrm{supp} K_{\cos(t\sqrt{L})}(x, y) \subseteq \big\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \le c_0 t\big\}. \tag{2.4}$$

See also [6]. By the Fourier inversion formula, whenever F is an even bounded Borel function with $\hat{F} \in L^1(\mathbb{R})$, we can write $F(\sqrt{L})$ in terms of $\cos(t\sqrt{L})$. More precisely, by recalling (2.3), we have

$$F(\sqrt{L}) = (2\pi)^{-1} \int_{-\infty}^{\infty} \hat{F}(t) \cos(t\sqrt{L})\, dt,$$

which, combined with (2.4), gives

$$K_{F(\sqrt{L})}(x, y) = (2\pi)^{-1} \int_{|t| \ge c_0^{-1}|x-y|} \hat{F}(t) K_{\cos(t\sqrt{L})}(x, y)\, dt. \tag{2.5}$$

Lemma 2.4. *Let* $\varphi \in C_0^\infty(\mathbb{R})$ *be even and supp* $\varphi \subseteq [-c_0^{-1}, c_0^{-1}]$. *Let* Φ *denote the Fourier transform of* φ. *Then for each* $k = 0, 1, \dots$, *and every* $t > 0$, *the kernel* $K_{(t^2L)^k \Phi(t\sqrt{L})}(x, y)$ *of* $(t^2L)^k \Phi(t\sqrt{L})$ *satisfies*

$$\mathrm{supp}\, K_{(t^2L)^k \Phi(t\sqrt{L})} \subseteq \big\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \le t\big\} \tag{2.6}$$

and

$$|K_{(t^2L)^k \Phi(t\sqrt{L})}(x, y)| \le C t^{-n}, \tag{2.7}$$

for all $t > 0$ *and* $x, y \in \mathbb{R}^n$.

Proof. For the proof, we refer the reader to Lemma 3.5 in [18]. ∎

For $s > 0$, we define

$$\mathbb{F}(s) := \Big\{\psi : \mathbb{C} \to \mathbb{C} \text{ measurable}: \quad |\psi(z)| \le C\frac{|z|^s}{(1+|z|^{2s})}\Big\}.$$

Then for any non-zero function $\psi \in \mathbb{F}(s)$, we have that $\{\int_0^\infty |\psi(t)|^2 \frac{dt}{t}\}^{1/2} < \infty$. Denote $\psi_t(z) = \psi(tz)$. It follows from the spectral theory in [28] that for any $f \in L^2(\mathbb{R}^n)$,

$$\begin{aligned}\Big\{ \textstyle\int_0^\infty \|\psi(t\sqrt{L})f\|^2_{L^2(\mathbb{R}^n)}\frac{dt}{t}\Big\}^{1/2} &= \Big\{ \int_0^\infty \big\langle\, \overline{\psi}(t\sqrt{L})\,\psi(t\sqrt{L})f, f\big\rangle \frac{dt}{t}\Big\}^{1/2} \\ &= \Big\{\Big\langle \int_0^\infty |\psi|^2(t\sqrt{L})\frac{dt}{t} f, f\Big\rangle\Big\}^{1/2} \\ &= \kappa\|f\|_{L^2(\mathbb{R}^n)},\end{aligned} \tag{2.8}$$

where $\kappa = \big\{ \int_0^\infty |\psi(t)|^2\, dt/t\big\}^{1/2}$. The estimate will be used repeatedly in this paper.

3. Atomic Characterization of Weighted Hardy Spaces

3.1. Weighted Hardy spaces and weighted atoms

Suppose $w \in A_\infty$ and $0 < p \le 1$. We define Hardy spaces $H^p_{L,w}(\mathbb{R}^n)$ as the completion of $\{f \in L^2(\mathbb{R}^n) : \|S_L(f)\|_{L^p_w(\mathbb{R}^n)} < \infty\}$ with respect to L^p_w-norm of the square function; e.g.,

$$\big\|f\big\|_{H^p_{L,w}(\mathbb{R}^n)} := \big\|S_L(f)\big\|_{L^p_w(\mathbb{R}^n)},$$

where

$$S_L(f)(x) := \Big(\iint_{|y-x|<t} \big|t^2Le^{-t^2L}f(y)\big|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2}, \qquad x \in \mathbb{R}^n.$$

The (p, q, M, w)-atom associated to the operator L is defined as follows.

Definition 3.1. Suppose that M is a positive integer, $w \in A_s, 1 \le s < \infty$ and $0 < p \le 1$. A function $a(x) \in L^2(\mathbb{R}^n)$ is called a (p, q, M, w)-atom associated to an operator L, $1 < q < \infty$, if there exist a function $b \in \mathcal{D}(L^M)$, the domain of an operator L, and a ball B of $\mathbb{R}^n$ such that

(i) $a = L^M b$;

(ii) supp $L^k b \subseteq B,\ k = 0, 1, \cdots, M$;

(iii) $\big\|(r_B^2 L)^k b\big\|_{L^q_w(\mathbb{R}^n)} \le r_B^{2M} w(B)^{1/q-1/p},\ k = 0, 1, \cdots, M.$

Remark 3.2. It follows directly from Hölder's inequality that (p, q_1, M, w)-atom is also (p, q_2, M, w)-atom whenever $q_1 > q_2$.

Definition 3.3. Let M, w and p be the same as above. The weighted atomic Hardy spaces $H^{p,q,M}_{L,w}(\mathbb{R}^n)$ are defined as follows. We will say that $f = \sum \lambda_j a_j$ is an atomic (p, q, M, w)-representation (of f) if $\{\lambda_j\}_{j=0}^{\infty} \in \ell^1$, each a_j is a (p, q, M, w)-atom, and the sum converges in $L^2(\mathbb{R}^n)$. Set

$$\mathbb{H}^{p,q,M}_{L,w}(\mathbb{R}^n) := \Big\{ f : f \text{ has an atomic } (p, q, M, w)\text{-representation} \Big\},$$

with the norm $\|f\|_{\mathbb{H}^{p,q,M}_{L,w}(\mathbb{R}^n)}$ given by

$$\inf \Big\{ \big(\sum_{j=0}^{\infty} |\lambda_j|^p \big)^{1/p} : f = \sum_{j=0}^{\infty} \lambda_j a_j \ \text{ is an atomic } (p, q, M, w)\text{-representation} \Big\}.$$

The spaces $H^{p,q,M}_{L,w}(\mathbb{R}^n)$ are then defined as the completion of $\mathbb{H}^{p,q,M}_{L,w}(\mathbb{R}^n)$ with respect to this norm.

3.2. Atomic characterization of weighted Hardy spaces

The definition of $g^*_{\mu,\Psi}$ function is given as

$$g^*_{\mu,\Psi}(f)(x) = \bigg(\iint_{\mathbb{R}^{n+1}_+} \Big(\frac{t}{t+|x-y|} \Big)^{n\mu} \big| \Psi(t\sqrt{L}) f(y) \big|^2 \ \frac{dy\,dt}{t^{n+1}} \bigg)^{1/2}, \mu > 1, \tag{3.1}$$

where φ and Φ are the same as in Lemma 2.4, and $\Psi(x) := x^{2s}\Phi^3(x)$, $s \geq n+1$, $x \in \mathbb{R}^n$. The following lemma was proved in Lemma 5.1 of [17].

Lemma 3.4. *Let L be a non-negative self-adjoint operator such that the corresponding heat kernel satisfies condition (GE). There exists a constant $C > 0$ such that for all $w \in A_p$, $1 < p < \infty$, $\mu > 3$, the following estimate holds:*

$$\big\| g^*_{\mu,\Psi}(f) \big\|_{L^p_w(\mathbb{R}^n)} + \big\| S_L(f) \big\|_{L^p_w(\mathbb{R}^n)} \leq C \big\| f \big\|_{L^p_w(\mathbb{R}^n)}.$$

Then we have the following main result.

Theorem 3.5. *Suppose that $w \in A_s, 1 \leq s < \infty$, and $0 < p \leq 1$.*

(i) *Let $M \in \mathbb{N}$ and $1 < q < \infty$. If $f \in H^p_{L,w}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then there exist a family of (p, q, M, w)-atoms $\{a_i\}_{i=0}^{\infty}$ and a sequence of numbers $\{\lambda_i\}_{i=0}^{\infty}$ such that f can be represented in the form $f = \sum_{i=0}^{\infty} \lambda_i a_i$, and the sum converges in the sense of $L^2(\mathbb{R}^n)$-norm. Moreover,*

$$\big(\sum_{i=0}^{\infty} |\lambda_i|^p \big)^{1/p} \leq C \big\| f \big\|_{H^p_{L,w}(\mathbb{R}^n)}.$$

(ii) *Suppose that* $M \in \mathbb{N}$, $M > \frac{(s-p)n}{2p}$ *and* $q > s$. *Let* $f = \sum_{i=0}^{\infty} \lambda_i a_i$, *where* $\{\lambda_i\} \in \ell^p$, $a_i, i = 0, 1, 2, \dots$, *be* (p, q, M, w)*-atoms, and the sum converges in* $L^2(\mathbb{R}^n)$. *Then* $f \in H^p_{L,w}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ *and*

$$\Big\| \sum_{i=0}^{\infty} \lambda_i a_i \Big\|_{H^p_{L,w}(\mathbb{R}^n)} \le C \Big(\sum_{i=0}^{\infty} |\lambda_i|^p \Big)^{1/p}.$$

Remark 3.6. If $s > 1$, then $w \in A_s$ implies $w \in A_{s-\epsilon}$ for some $\epsilon > 0$. Thus Theorem 3.5 (ii) holds for $q \ge s > 1$.

Proof.

Step 1. Let $f \in H^p_{L,w}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Here we will apply the similar idea with [21] to obtain the weighted atomic decomposition.

Let φ and Φ be the same as in Lemma 2.4. Set $\Psi(x) := x^{2\alpha}\Phi^3(x)$ and $\alpha = M + n + 1$. By L^2-functional calculus ([22]), for every $f \in L^2(\mathbb{R}^n)$, one can write

$$\begin{aligned} f(x) &= c_\Psi \int_0^\infty \Psi(t\sqrt{L}) t^2 L e^{-t^2 L} f(x) \frac{dt}{t} \\ &= \lim_{N\to\infty} c_\Psi \int_{1/N}^{N} \Psi(t\sqrt{L}) t^2 L e^{-t^2 L} f(x) \frac{dt}{t} \end{aligned} \tag{3.2}$$

with the integral converging in $L^2(\mathbb{R}^n)$.

Now for each $k \in \mathbb{Z}$, we define $O_k = \{x \in \mathbb{R}^n : S_L(f)(x) > 2^k\}$ and $O_k^* = \{x \in \mathbb{R}^n : \mathcal{M}(\chi_{O_k})(x) > 2^{-(n+1)}\}$. Then we know that $O_k \subseteq O_k^*$ and $|O_k^*| \le C|O_k|$ for every $k \in \mathbb{Z}$. Let $\{Q_j^k\}_j$ be a Whitney decomposition of O_k^*, and $\widehat{O_k^*}$ be a tent region, that is

$$\widehat{O_k^*} := \Big\{ (x,t) \in \mathbb{R}^n \times (0,\infty) : \ \mathrm{dist}(x, {}^c O_k^*) \ge t \Big\}.$$

Choose a large constant c. Let B_j^k denote the ball with the same center as Q_j^k, but c times its diameter. Then for every $i, j \in \mathbb{Z}$, we define

$$T_j^k = \widehat{B_j^k} \cap \big(Q_j^k \times (0, +\infty) \big) \cap \big(\widehat{O_k^*} \backslash \widehat{O_{k+1}^*} \big), \tag{3.3}$$

and $\lambda_j^k = 2^k w(B_j^k)^{1/p}$. Note that $\mathbb{R}^{n+1}_+ = \cup_{j,k} T_j^k$ and T_j^k are disjoint for different j or k. Then one can write

$$\begin{aligned} f(x) &= \sum_{j,k\in\mathbb{Z}} c_\Psi \int_0^\infty \Psi(t\sqrt{L}) \Big(\chi_{T_j^k} t^2 L e^{-t^2 L} \Big) f(x) \frac{dt}{t} \\ &=: \sum_{j,k\in\mathbb{Z}} \lambda_j^k a_j^k, \end{aligned} \tag{3.4}$$

where $a_j^k = L^M b_j^k$ and

$$b_j^k = (\lambda_j^k)^{-1} c_\Psi \int_0^\infty t^{2\alpha} L^{n+1} \Phi^3(t\sqrt{L}) \Big(\chi_{T_j^k} t^2 L \mathrm{e}^{-t^2 L} \Big) f(x) \, \frac{dt}{t}.$$

We claim that, up to a normalization by a multiplicative constant, a_j^k are (p, q, M, w)-atoms. Once the claim is established, we shall have

$$\begin{aligned} \sum_{j,k} |\lambda_j^k|^p &= \sum_{j,k} 2^{kp} w(B_j^k) \le C \sum_{j,k} 2^{kp} w(Q_j^k) \le C \sum_k 2^{kp} w(O_k^*) \\ &\le C \sum_k 2^{kp} w(O_k) \le C \|f\|^p_{H^p_{L,w}(\mathbb{R}^n)} \end{aligned}$$

as desired.

Let us now prove the claim. By Remark 3.2, it suffices to show that for every $k \in \mathbb{Z}$ and $q > s$, the function $C^{-1} a_k$ is a (p, q, M, w)-atom associated with the ball B_j^k, for some constant C. From Lemma 2.4, the integral kernel $K_{(t^2 L)^i \Phi^3(t\sqrt{L})}(x, y)$ of the operator $(t^2 L)^i \Phi^3(t\sqrt{L})$ satisfies

$$\text{supp } K_{(t^2 L)^i \Phi^3(t\sqrt{L})}(x, y) \subseteq \big\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \le 3t \big\}.$$

This, together with the fact $(x, t) \in T_j^k \subseteq \widehat{B_j^k}$, implies that for every $i = 0, 1, \cdots, M$,

$$\text{supp}\big(L^i b_j^k\big) \subseteq 3B_j^k.$$

To continue, for any $s < q < \infty$ and every ball B_j^k we consider some $g \in L^{q'}_{w^{-q'/q}}(3B_j^k)$ such that $\|g\|_{L^{q'}_{w^{-q'/q}}} \le 1$. Then for every $i = 0, 1, \cdots, M$, we have

$$\begin{aligned} &\Big| \int \big(r^2_{B_j^k} L\big)^i b_j^k(x) g(x) \, dx \Big| \\ &= \frac{c_\Psi}{\lambda_j^k} \left| \iint_{T_j^k} r^{2i}_{B_j^k} t^{2\alpha} L^{n+1+i} \Phi^3(t\sqrt{L}) g(y) t^2 L e^{-t^2 L} f(y) \, \frac{dydt}{t} \right| \qquad (3.5) \\ &\le r^{2M}_{B_j^k} \frac{c_\Psi}{\lambda_j^k} \iint_{\widehat{B_j^k} \backslash \widehat{O^*_{k+1}}} \big| (t^2 L)^{n+1+i} \Phi^3(t\sqrt{L}) g(y) \big| \big| t^2 L e^{-t^2 L} f(y) \big| \, \frac{dydt}{t}, \end{aligned}$$

where in the inequality above we have used the fact $0 < t \le r_{B_j^k}$.

By Lemma 2.2 and estimate (3.5), we obtain

$$\begin{aligned} &\Big| \int \big(r^2_{B_j^k} L\big)^i b_j^k(x) g(x) dx \Big| \\ &\le \frac{C}{\lambda_j^k} r^{2M}_{B_j^k} \int_{{}^c O_{k+1}} \left(\iint_{\Gamma(x)} \chi_{\widehat{B_j^k}}(y, t) \big| (t^2 L)^{n+1+i} \Phi^3(t\sqrt{L}) g(y) \big| \big| t^2 L e^{-t^2 L} f(y) \big| \, \frac{dydt}{t^{n+1}} \right) dx \\ &= \frac{C}{\lambda_j^k} r^{2M}_{B_j^k} \int_{B_j^k \cap {}^c O_{k+1}} \left(\iint_{\Gamma(x)} \big| (t^2 L)^{n+1+i} \Phi^3(t\sqrt{L}) g(y) \big| \big| t^2 L e^{-t^2 L} f(y) \big| \, \frac{dydt}{t^{n+1}} \right) dx. \end{aligned}$$

We observe that if $|x-y|<t$, then $\big(\frac{t}{|x-y|+t}\big)^{n\mu}\geq C$. By Hölder's inequality, we have

$$
\begin{aligned}
&\Big|\int \big(r_{B_j^k}^2 L\big)^i b_j^k(x)g(x)dx\Big| \\
&\leq \frac{C}{\lambda_j^k} r_{B_j^k}^{2M}\int_{B_j^k\cap{}^cO_{k+1}}\Big(\iint_{\Gamma(x)}|(t^2L)^{n+1+i}\Phi^3(t\sqrt{L})g(y)|^2\,\frac{dydt}{t^{n+1}}\Big)^{1/2} \\
&\quad S_L(f)(x)\,dx \\
&\leq \frac{C}{\lambda_j^k} r_{B_j^k}^{2M}\int_{B_j^k\cap{}^cO_{k+1}} g^*_{\mu,\Psi}(g)(x)S_L(f)(x)\,dx \qquad (3.6)\\
&\leq \frac{C}{\lambda_j^k} r_{B_j^k}^{2M}\Big(\int\big(g^*_{\mu,\Psi}(g)(x)\big)^{q'}w^{-q'/q}(x)dx\Big)^{1/q'} \\
&\quad \Big(\int_{B_j^k\cap{}^cO_{k+1}}\big(S_L(f)(x)\big)^q w(x)\,dx\Big)^{1/q}.
\end{aligned}
$$

Note that

$$
\int_{B_j^k\cap{}^cO_{k+1}}\big(S_L(f)(x)\big)^q w(x)\,dx\leq C2^{kq}w(B_j^k). \tag{3.7}
$$

Since $s<q$, then $w\in A_s$ implies $w\in A_q$. By (iv) of Lemma 2.3, we have $w^{-q'/q}\in A_{q'}$. Together with Lemma 3.4, we obtain

$$
\Big(\int\big(g^*_{\mu,\Psi}(g)(x)\big)^{q'}w^{-q'/q}(x)\,dx\Big)^{1/q'}\leq C\big\|g\big\|_{L^{q'}_{w^{-q'/q}}}\leq C. \tag{3.8}
$$

Combing estimates (3.6)-(3.8) and the definition of λ_j^k, we have

$$
\Big|\int\big(r_{B_j^k}^2L\big)^i b_j^k(x)g(x)dx\Big|\leq Cr_{B_j^k}^{2M}w(B_j^k)^{1/q-1/p},
$$

which implies that a_j^k are (p,q,M,w)-atoms for $s<q<\infty$, and thus for $1<q<\infty$.

To prove Theorem 3.5 (ii), we need the following lemma (see [4]).

Lemma 3.7. *Fix $M\in\mathbb{N}$, $0<p\leq 1$ and $w\in A_\infty$. Assume that T is a non-negative sublinear operator, satisfying the weak-type (2,2)*

$$
\mu\{x\in\mathbb{R}^n:|Tf(x)|>\eta\}\leq C_T\,\eta^{-2}\|f\|^2_{L^2(\mathbb{R}^n)},\quad \textit{for all } \eta>0,
$$

and that for every (p,q,M,w)-atom a, we have

$$
\|Ta\|_{L^p_w(\mathbb{R}^n)}\leq C
$$

with constant C independent of a. Then T is bounded from $H^p_{L,w}(\mathbb{R}^n)$ to $L^p_w(\mathbb{R}^n)$, and

$$\|Tf\|_{L^p_w(\mathbb{R}^n)} \leq C\|f\|_{H^p_{L,w}(\mathbb{R}^n)}.$$

Step 2. By Lemma 3.7, it is enough to establish a uniform L^p_w, $0 < p \leq 1$, bound on any (p, q, M, w)-atom. That is to say, there exists a constant $C > 0$ such that

$$\|S_L(a)\|_{L^p_w(\mathbb{R}^n)} \leq C, \tag{3.9}$$

where a is a (p, q, M, w)-atom associated to a ball $B = B(x_B, r_B)$.

We can write

$$\begin{aligned}\int \big(S_L(a)(x)\big)^p w(x)dx \;&=\; \int_{2B} \big(S_L(a)(x)\big)^p w(x)dx + \int_{^c(2B)} \big(S_L(a)(x)\big)^p w(x)dx \\ &=: \mathrm{I}_1 + \mathrm{I}_2.\end{aligned}$$

To estimate term I_1, note that if $w \in A_s$ and $1 \leq s < q$, then $w \in A_q$. Thus, we use Hölder's inequality and Lemma 3.4 to obtain that

$$\begin{aligned}\mathrm{I}_1 \;&\leq\; \Big(\int_{2B} \big(S_L(a)(x)\big)^q w(x)\,dx\Big)^{p/q}\Big(\int_{2B} w(x)\,dx\Big)^{1-p/q} \\ &\leq C\|S_L(a)\|^p_{L^q_w(\mathbb{R}^n)} w(2B)^{1-p/q} \\ &\leq C\|a\|^p_{L^q_w(\mathbb{R}^n)} w(2B)^{1-p/q} \\ &\leq Cw(B)^{p(1/q-1/p)} w(B)^{1-p/q} \\ &\leq C,\end{aligned}$$

where in the fourth inequality above we have used the definition of (p, q, M, w) -atom a.

It remains to estimate term I_2. For any $x \in^c (2B)$, we write

$$\begin{aligned}S^2_L(a)(x) \;&=\; \Big(\int_0^{r_B} + \int_{r_B}^{+\infty}\Big) \int_{|x-y|<t} \big|t^2 L e^{-t^2 L} a(y)\big|^2 \frac{dydt}{t^{n+1}} \\ &=: \mathrm{I}_{21} + \mathrm{I}_{22}.\end{aligned}$$

Observe that $x \in^c (2B),\ z \in B$ and $|x - y| < t$ imply $r_B \leq |x - z| < t + |y - z|$. Thus, $|x - x_B| \leq |x - y| + |y - z| + |z - x_B| < 3(t + |y - z|)$, which, combined with Lemma 2.1, implies that for all $N > 0$,

$$
\begin{aligned}
\mathrm{I}_{21} &\leq C\int_0^{r_B}\int_{|x-y|<t}\Big(\int \frac{t^N}{(t+|y-z|)^{n+N}}|a(z)|\,dz\Big)^2\frac{dydt}{t^{n+1}}\\
&\leq C\frac{1}{|x-x_B|^{2n+2N}}\int_0^{r_B} t^{2N-1}\,dt\,\|a\|^2_{L^1(\mathbb{R}^n)}\\
&\leq C\frac{r_B^{2N}}{|x-x_B|^{2n+2N}}\,\|a\|^2_{L^1(\mathbb{R}^n)}.
\end{aligned}\tag{3.10}
$$

Consider the term I_{22}. Noting that $a = L^M b$, applying Lemma 2.1, we obtain

$$
\begin{aligned}
\mathrm{I}_{22} &= \int_{r_B}^{\infty}\int_{|x-y|<t}\big|t^2Le^{-t^2L}(L^Mb)(y)\big|^2\frac{dydt}{t^{n+1}}\\
&= \int_{r_B}^{\infty}\int_{|x-y|<t}\big|(t^2L)^{M+1}e^{-t^2L}b(y)\big|^2\,\frac{dydt}{t^{4M+n+1}}\\
&\leq C\int_{r_B}^{\infty}\int_{|x-y|<t}\Big(\int_B \frac{t^N}{(t+|y-z|)^{n+N}}|b(z)|\,dz\Big)^2\,\frac{dydt}{t^{4M+n+1}}\\
&\leq C\frac{1}{|x-x_B|^{2n+2N}}\int_{r_B}^{\infty}\frac{dt}{t^{4M-2N+1}}\,\|b\|^2_{L^1(\mathbb{R}^n)}\\
&\leq C\frac{r_B^{2N-4M}}{|x-x_B|^{2n+2N}}\|b\|^2_{L^1(\mathbb{R}^n)},
\end{aligned}\tag{3.11}
$$

whenever $M > N/2$.

Therefore, combing (3.10) and (3.11), we have

$$
\mathrm{I}_2 \leq C\int_{^c(2B)}\frac{r_B^{pN}}{|x-x_B|^{(n+N)p}}w(x)\,dx\,\Big(\|a\|^p_{L^1(\mathbb{R}^n)} + r_B^{-2M}\|b\|^p_{L^1(\mathbb{R}^n)}\Big).
$$

Using Hölder's inequality and the definition of (p, q, M, w)-atom and $w \in A_q$, we obtain

$$
\begin{aligned}
\|a\|_{L^1(\mathbb{R}^n)} &\leq \Big(\|a\|_{L^q_w(\mathbb{R}^n)}\Big)\Big(\int_B w(x)^{-1/(q-1)}\,dx\Big)^{1-1/q}\\
&\leq Cw(B)^{1/q-1/p}w(B)^{-1/q}|B|\\
&\leq Cw(B)^{-1/p}|B|.
\end{aligned}\tag{3.12}
$$

Similarly, one also can have

$$
r_B^{-2M}\|b\|_{L^1(\mathbb{R}^n)} \leq Cw(B)^{-1/p}|B|.\tag{3.13}
$$

Thus,

$$
\begin{aligned}
\int_{{}^c(2B)} \frac{r_B^{pN}}{|x-x_B|^{(n+N)p}} w(x)\,dx &\le \sum_{k=1}^{\infty} \int_{2^{k+1}B\setminus 2^k B} \frac{r_B^{pN}}{|x-x_B|^{(n+N)p}} w(x)\,dx \\
&\le C\sum_{k=1}^{\infty} \frac{r_B^{pN}}{(2^k r_B)^{p(n+N)}}\, w(2^{k+1}B) \\
&\le C\sum_{k=1}^{\infty} \frac{r_B^{pN}}{(2^k r_B)^{p(n+N)}} 2^{(k+1)sn}\, w(B) \\
&\le Cw(B)|B|^{-p},
\end{aligned}
\tag{3.14}
$$

where in the third inequality above we have used (v) of Lemma 2.3. The condition that $M > \frac{(s-p)n}{2p}$ ensures us to find N such that $2M > N > \frac{n(s-p)}{p}$.

It follows from estimates (3.12)–(3.14) that $\mathrm{I}_2 \le C$, which completes the proof of (3.9) and the proof of Theorem 3.5. ∎

4. $BMO_{L,w}$: Duality with $H^1_{L,w}(\mathbb{R}^n)$ Spaces

In this section, we introduce and study the duality of the weighted Hardy space $H^1_{L,w}(\mathbb{R}^n)$. Following [13], we introduce the definition of the class of functions that the operator L act on. For any $\beta > 0$, a function $f \in L^2_{\rm loc}(\mathbb{R}^n)$ is said to be a function of β-type if f satisfies

$$
\Big(\int_{\mathbb{R}^n} \frac{|f(x)|^2}{1+|x|^{n+\beta}}\,dx\Big)^{1/2} \le c < \infty. \tag{4.1}
$$

Denote by $\mathcal{M}_\beta$ the collection of all functions of β-type. If $f \in \mathcal{M}_\beta$, the norm of f in $\mathcal{M}_\beta$ is defined by

$$
\|f\|_{\mathcal{M}_\beta} = \inf\{c \ge 0 : (4.1) \text{ holds}\}.
$$

Then, we give the definition of $BMO^p_{L,w}$, where $1 \le p < \infty$.

Definition 4.1. Let L be a non-negative self-adjoint operator such that the corresponding heat kernel satisfies condition (GE). For $w \in A_s, 1 \le s < \infty$ and $1 \le p < \infty$, an element $f \in \mathcal{M}_\beta$ is said to belong to $BMO^p_{L,w}$ if

$$
\|f\|_{BMO^p_{L,w}} =: \sup_{B\subset\mathbb{R}^n} \left(\frac{1}{w(B)}\int_B |(\mathbb{I}-(1+r_B^2L)^{-1})^M f|^p w^{1-p}\,dx\right)^{1/p} < \infty,
$$

where the sup is taken over all balls B in $\mathbb{R}^n$, $\mathbb{I}$ denotes the identity operator on $\mathbb{R}^n$. In particularly, for $p = 1$ denote $BMO^1_{L,w} =: BMO_{L,w}$.

We have the following theorem.

Theorem 4.2. $H^{1,q,M}_{L,w}(\mathbb{R}^n)^* = BMO^{q'}_{L,w}(\mathbb{R}^n)$, $q \ge 1$.

Proof. We begin by showing that each $f \in BMO^{q'}_{L,w}$ induces a bounded linear functional on $H^{1,q,M}_{L,w}(\mathbb{R}^n)$. Suppose that a is a $(1, q, M, w)$-atom in $H^{1,q,M}_{L,w}(\mathbb{R}^n)$, and let $f \in BMO^{q'}_{L,w}(\mathbb{R}^n)$. Then

$$\begin{aligned}\textstyle\int_B a(x)f(x)\,dx &= \int_B (\mathbb{I} - (1 + r_B^2 L)^{-1})^M a(x)f(x)\,dx\\ &\quad + \int_B \Big(\mathbb{I} - (\mathbb{I} - (1 + r_B^2 L)^{-1})^M\Big) a(x)f(x)\,dx\\ &=: \mathrm{J}_1 + \mathrm{J}_2.\end{aligned}$$

For the term J_1, by Hölder's inequality and the properties of atom,

$$\begin{aligned}\mathrm{J}_1 &\le \|a\|_{L^q_w} \Big(\int_B |(\mathbb{I} - (1 + r_B^2 L)^{-1})^M f(x)|^{q'} w^{1-q'}\,dx\Big)^{1/q'}\\ &\le C\|f\|_{BMO^{q'}_{L,w}} w(B)^{1/q-1} w(B)^{1/q'}\\ &\le C\|f\|_{BMO^{q'}_{L,w}}.\end{aligned}$$

To analyze J_2, by condition $a = L^M b$ and the fact that L is self-adjoint, we write

$$\begin{aligned}&\Big(\mathbb{I} - (\mathbb{I} - (1 + r_B^2 L)^{-1})^M\Big) a(x)\\ &= L^M \Big(\mathbb{I} - (\mathbb{I} - (1 + r_B^2 L)^{-1})^M\Big) b(x)\\ &= \sum_{k=1}^{M} \frac{M!}{(M-k)!k!} (r_B^{-2k} L^{M-k})(\mathbb{I} - (1 + r_B^2 L)^{-1})^M b(x).\end{aligned}$$

Thus,

$$\begin{aligned}\mathrm{J}_2 &\le \sum_{k=1}^{M} \frac{M!}{(M-k)!k!} \Big| r_B^{-2M} \int_B (r_B^2 L)^{M-k} b(x)(\mathbb{I} - (1 + r_B^2 L)^{-1})^M f(x)\,dx\Big|\\ &\le \sum_{k=1}^{M} \frac{M!}{(M-k)!k!} r_B^{-2M} \|(r_B^2 L)^{M-k} b\|_{L^q_w} \Big(\int_B |(\mathbb{I} - (1 + r_B^2 L)^{-1})^M f(x)|^{q'} w^{1-q'} dx\Big)^{1/q'}\\ &\le C r_B^{-2M} r_B^{2M} w(B)^{1/q-1} w(B)^{1-1/q} \|f\|_{BMO^{q'}_{L,w}}\\ &\le C\|f\|_{BMO^{q'}_{L,w}}.\end{aligned}$$

Therefore, for every $h = \sum_j \lambda_j a_j \in H^{1,q,M}_{L,w}(\mathbb{R}^n)$, where a_j are weighted atoms, we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} f(x)h(x)\,dx\Big| &\le \sum_j |\lambda_j|\Big|\int_{\mathbb{R}^n} f(x)a_j(x)\,dx\Big|\\ &\le C\sum_j |\lambda_j|\|f\|_{BMO^{q'}_{L,w}}\\ &\le C\|h\|_{H^1_{L,w,at,M}(\mathbb{R}^n)}\|f\|_{BMO^{q'}_{L,w}},\end{aligned}$$

and the assertion follows.

Conversely, suppose that $l \in H^1_{L,w}(\mathbb{R}^n)^*$. For any $g \in H^1_{L,w}(\mathbb{R}^n)\bigcap L^2(\mathbb{R}^n)$, which is dense in $H^1_{L,w}(\mathbb{R}^n)$, l can be represented by f in the form

$$l(g) = \int_{\mathbb{R}^n} g(x)f(x)\,dx.$$

For fixed ball B, let $\phi \in L^q_w(B)$ and $\|\phi\|_{L^q_w(B)} \le 1$. Set

$$a(x) = \frac{1}{w(B)^{1-1/q}}(\mathbb{I} - (1 + r_B^2 L)^{-1})^M \phi.$$

Then it is not difficult to check that a is a $(1, q, M, w)$-atom (see the Theorem 6.4 in [18]).

Consequently,

$$\begin{aligned}\|l\| &\ge \|l(a)\|\\ &= \frac{1}{w(B)^{1-1/q}}\int_B (\mathbb{I} - (1 + r_B^2 L)^{-1})^M \phi f(x)\,dx\\ &= \frac{1}{w(B)^{1-1/q}}\int_B \phi(\mathbb{I} - (1 + r_B^2 L)^{-1})^M f(x)\,dx.\end{aligned}$$

Thus, by duality it readily follows that

$$\left(\frac{1}{w(B)}\int_B |(\mathbb{I} - (1 + r_B^2 L)^{-1})^M f(x)|^{q'} w^{1-q'} dx\right)^{1/q'} \le \|l\|,$$

which is what we wanted to show. ∎

Remark 4.3. By Theorem 3.5 and Theorem 4.2, we can obtain $BMO^p_{L,w} \sim BMO_{L,w}$ for $1 \le p < \infty$.

ACKNOWLEDGMENTS

The authors would like to thank the referees for their meticulous work and helpful suggestions, which improve the presentation of this paper.

References

1. P. Auscher, X. T. Duong and A. McIntosh, *Boundedness of Banach Space Valued Singular Integral Operators and Hardy Spaces*, Unpublished preprint, 2005.
2. P. Auscher, A. McIntosh and E. Russ, Hardy spaces of differential forms on Riemannian manifolds, *J. Geom. Anal.*, **18** (2008), 192-248.
3. P. Auscher and E. Russ, Hardy spaces and divergence operators on strongly Lipschitz domain of $\mathbb{R}^n$, *J. Funct. Anal.*, **201** (2003), 148-184.
4. A. Bui and X. T. Duong, *Weighted Hardy Spaces Associated to Operators and Boundedness of Singular Integrals*, arXiv:1202.2063.
5. A. Bui and X. T. Duong, *Weighted BMO Spaces Associated to Operators*, arXiv: 1201.5828.
6. J. Cheeger, M. Cromov and M. Taylor, Finite propagation speed, kernel estimates for functions of the Laplacian and the geometry of complete Riemannian manifolds, *J. Differential Geom.*, **17** (1982), 15-53.
7. R. R. Coifman, Y. Meyer and E. M. Stein, Some new functions and their applications to harmonic analysis, *J. Funct. Anal.*, **62** (1985), 304-335.
8. T. Coulhon and A. Sikora, Gaussian heat kernel upper bounds via Phragmén-Lindelöf theorem, *Proc. Lond. Math.*, **96** (2008), 507-544.
9. E. B. Davies, *Heat Kernels and Spectral Theory*, Cambridge Univ. Press, 1989.
10. J. Duoandikoetxea, *Fourier Analysis*, *Grad. Stud. Math.*, **29**, American Math. Soc., Providence, 2000.
11. X. T. Duong, E. M. Ouhabaz and A. Sikora, Plancherel-type estimates and sharp spectral multipliers, *J. Funct. Anal.*, **196** (2002), 443-485.
12. X. T. Duong and L. X. Yan, New function spaces of BMO type, John-Nirenberg inequality, interpolation and applications, *Comm. Pure Appl. Math.*, **58** (2005), 1375-1420.
13. X. T. Duong and L. X. Yan, Duality of Hardy and BMO spaces associated with operators with heat kernel bounds, *J. Amer. Math. Soc.*, **18** (2005), 943-973.
14. C. Fefferman and E. M. Stein, H^p spaces of several variables, *Acta Math.*, **129** (1972), 137-195.
15. J. Garcia-Cuerva, Weighted H^p spaces, *Dissertationes Math.*, **162** (1979), 1-63.
16. J. Garcia-Cuerva and J. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, Amsterdam, 1985.
17. R. M. Gong and L. X. Yan, *Weighted L^p Estimates for the Area Integral Associated to Self-adjoint Operators*, arxiv:1109.1662v1.
18. S. Hofmann, G. Lu, D. Mitrea, M. Mitrea and L. Yan, Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffeney estimates, *Memoirs Amer. Math. Soc.*, **214** (2011), no. 1007, 78 pp.

19. S. Hofmann and S. Mayboroda, Hardy and BMO spaces assocated to divergence form elliptic operators, *J. Funct. Anal.*, **258** (2010), 1167-1224.

20. R. Jiang and D. C. Yang, New Orlicz-Hardy spaces associated to divergence form elliptic operators, *Math. Ann.*, **344** (2009), 37-116.

21. S. Y. Liu and L. Song, An atomic decomposition of weighted Hardy spaces associated to self-adjoint operators, *J. Funct. Anal.*, **265** (2013), 2709-2723.

22. A. McIntosh, Operators which have an H_∞-calculus, *Miniconference on Operator Theory and Partial Differential Equations (North Ryde, 1986)*, 210-231, Proceedings of the Centre for Mathematical Analysis, ANU, Canberra, **14** (1986).

23. E. M. Ouhabaz, *Analysis of Heat Equations on Domains*, London Math. Soc. Monographs, Vol. **31**, Princeton Univ. Press, 2005.

24. L. Song and L. X. Yan, Riesz transforms associated to Schorödinger operators on weighted Hardy spaces, *J. Funct. Anal.*, **259** (2010), 1466-1490.

25. E. M. Stein, *Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton Univ. Press, Princeton, NJ, 1993.

26. E. M. Stein and G. Weiss, On the theory of harmonic functions of several variables. I. The theory of H^p-spaces, *Acta Math.*, **103** (1960) 25-62.

27. J.-O. Strömberg and A. Torchinsky, *Weighted Hardy Spaces, Lecture Notes in Mathematics*, **1381** Springer-Verlag, Berlin/New York, 1989.

28. K. Yosida, *Functional Analysis* (Fifth edition), Spring-Verlag, Berlin, 1978.

Suying Liu, Kai Zhao and Shujuan Zhou
College of Mathematics
Qingdao University
Shandong, 266071
P. R. China
E-mail: liusuying0319@126.com
zhkzhc@aliyun.com
xuanweishamo@163.com