TAIWANESE JOURNAL OF MATHEMATICS
Vol. 18, No. 4, pp. 1071-1087, August 2014
DOI: 10.11650/tjm.18.2014.2860
This paper is available online at http://journal.taiwanmathsoc.org.tw

SOME EXISTENCE RESULTS OF SEMILINEAR SINGULARLY PERTURBED NONLOCAL BOUNDARY VALUE PROBLEMS

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Abstract. By standard barrier solution method associated with Schauder fixed point theorem, we establish an existence theory for nonlinear second order nonlinear multi-point boundary value problem (1.3), (1.4). Through the pervious existence theorem, we mainly work out some asymptotic behaviors of solutions for semilinear singularly perturbed three-point boundary value problem (1.1), (1.2). Barrier solutions will be constructed explicitly when the boundary or interior layers occur, respectively.

1. INTRODUCTION

In the past 20 years, extensive researches have come a long way on the existence and asymptotic estimates of singular perturbation problems. For instance, in 2004, Bukzhalev [1] establishes an existence theorem of Dirichlet boundary value problems

$$\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx}),\ y(a) = y^0,\ y(b) = y^1,$$

when assuming the existence of barrier solutions. By a construction of barrier solutions explicitly, he justifies an asymptotic representation for a solution of

$$\epsilon^4 \frac{d^2y}{dx^2} = \epsilon\frac{dy}{dx}A(\epsilon^3\frac{dy}{dx}, y, x) + B(\epsilon^3\frac{dy}{dx}, y, x),\ y(0, \epsilon) = y^0,\ y(1, \epsilon) = y^1,$$

where A and B are functions satisfying some sufficient conditions related with degenerate equation. Vrábel' et. al. [2, 3, 4] investigate the singularly perturbed semilinear differential equations

$$\epsilon y'' + ky = f(t, y),\ k < 0,$$

Received January 15, 2013, accepted December 6, 2013.
Communicated by Yingfei Yi.
2010 *Mathematics Subject Classification*: 34B15, 34B16, 34E20.
Key words and phrases: Singular perturbation, Barrier solutions, Boundary layer, Interior layer, Multi-point boundary value problem.

subject to the following nonlocal boundary constraints

$$y'(a) = 0,\ y(b) - y(c) = 0,\ a < c < b.$$

They consider the right boundary layer phenomenon, that is, the layer occurs at $t = b$, and study not only the asymptotic behaviors of solutions but also the estimate of the derivative. Lin and Liu [5] in 2009 deal with the three-point boundary value problem for nonlinear differential systems $\epsilon^2\mathbf{x}'' = \mathbf{f}(\mathbf{t}, \mathbf{x}, \mathbf{x}')$, $0 < t < 1$, together with $\mathbf{x}(\mathbf{0}, \epsilon) = \mathbf{0}$, $\mathbf{x}(\mathbf{1}, \epsilon) = \mathbf{P}(\eta, \epsilon)$, where $\mathbf{x}$ and $\mathbf{f}$ are n-dimensional vectors, $\mathbf{P}$ =diag$(p_1, \cdots, p_n)$. Under some sufficient conditions, an asymptotic behavior of solutions in the right boundary layer case are also obtained. There are other excellent results related with this technique(barrier method), for example, to consider the third order singularly perturbed (two-point or nonlocal) boundary value problems [6, 7]. We refer the readers to more interesting contributions [8, 9, 10, 11, 12, 13, 14, 15, 16, 17] involved with diverse numerical approaches, method of descent, and so on.

Motivated by the above mentioned, we observe that there still exists many materials about singular perturbations to study, especially on interior layer phenomena. In this paper, when the layer occurs at the boundary or interior points, we respectively discuss the existence and asymptotic behavior of solutions for semilinear singularly perturbed equation

$$\epsilon u''(t) = f(t, u(t)),\ t \in (0, 1), \tag{1.1}$$

with three-point boundary condition

$$u(0) = A_0,\ u(1) = B_0 + \delta u(\eta), \tag{1.2}$$

where A_0, $B_0 \in \mathbb{R}$, $0 < \eta < 1$ and $\delta \geq 0$. In order to attend the achievement, we first study the existence of solutions for the following nonlinear boundary value problem

$$u''(t) = f(t, u(t), u'(t)),\ t \in (0, 1), \tag{1.3}$$

$$u(0) = A_0 + \sum_{i=1}^{n} \gamma_i u(\zeta_i),\ u(1) = B_0 + \sum_{j=1}^{m} \delta_j u(\eta_j), \tag{1.4}$$

where $n, m > 0$ are integers, $0 < \zeta_1 < \zeta_2 < \cdots < \zeta_n < 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_m < 1$, A_0, $B_0 \in \mathbb{R}$ and $\gamma_i, \delta_j \geq 0$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. In the mathematical literature a number of works have appeared on multi-point boundary value problems. This topic recently still engages many researchers and has been studied extensively via various schemes. The upper and lower solutions approach is a powerful one. Du, Kong, Khan, Minghe, Wang, Guo, et. al. [18, 19, 20, 21, 22, 23] have lots of essential contributions by means of this way.

The layout of this article is as follows. Section 2 contains an existence theorem of solutions for the multi-point boundary value problem (1.3), (1.4) established by barrier solutions method with Schauder fixed point theorem. In Section 3, by applying the previous theorem in Section 2, we construct explicit forms of upper and lower barriers of (1.1), (1.2) and get some behaviors of solutions when boundary layer occurs. The interior layer phenomena are considered in Section 4.

2. EXISTENCE THEOREM OF (1.3), (1.4)

In this section, we establish an existence result of (1.3), (1.4) via the upper and lower solution method. The first step is to introduce barrier solutions needed as follows:

Definition 2.1. *A function $\alpha \in C[0,1]$ is a C^2-lower solution of (1.3), (1.4) if*

(a) $\alpha(0) - \sum_{i=1}^{n} \gamma_i \alpha(\zeta_i) \le A_0$, $\alpha(1) - \sum_{j=1}^{m} \delta_j \alpha(\eta_j) \le B_0$,

(b) *for any $t_0 \in (0,1)$, either $D^- \alpha(t_0) < D_+ \alpha(t_0)$ or there exists an open interval $I_0 \subset (0,1)$ with $t_0 \in I_0$ and a function $\alpha_0 \in C^1(I_0)$ such that*

(i) $\alpha(t_0) = \alpha_0(t_0)$*and* $\alpha(t) \ge \alpha_0(t)$, *for any* $t \in I_0$;

(ii) $\alpha_0''(t_0)$ *exists and* $\alpha_0''(t_0) \ge f(t_0, \alpha_0(t_0), \alpha_0'(t_0))$.

Definition 2.2. *A function $\beta \in C[0,1]$ is a C^2-upper solution of (1.3), (1.4) if*

(a) $\beta(0) - \sum_{i=1}^{n} \gamma_i \beta(\zeta_i) \ge A_0$, $\beta(1) - \sum_{j=1}^{m} \delta_j \beta(\eta_j) \ge B_0$,

(b) *for any $t_0 \in (0,1)$, either $D_- \beta(t_0) > D^+ \beta(t_0)$ or there exists an open interval $I_0 \subset (0,1)$ with $t_0 \in I_0$ and a function $\beta_0 \in C^1(I_0)$ such that*

(i) $\beta(t_0) = \beta_0(t_0)$*and* $\beta(t) \le \beta_0(t)$, *for any* $t \in I_0$;

(ii) $\beta_0''(t_0)$ *exists and* $\beta_0''(t_0) \le f(t_0, \beta_0(t_0), \beta_0'(t_0))$.

We note that if $D^- \alpha(t_0) \ge D_+ \alpha(t_0)$ for some $t_0 \in (0,1)$, from the definition, there exists $\alpha_0(t_0) \in C^1(I_0)$ such that $\alpha(t_0) = \alpha_0(t_0)$ and $\alpha(t) \ge \alpha_0(t)$ on I_0. It follows from

$$D^- \alpha(t_0) \le \alpha_0'(t_0) \le D_+ \alpha(t_0) \le D^- \alpha(t_0)$$

that α has a derivative at t_0 and $\alpha'(t_0) = \alpha_0'(t_0)$. Similarly the case $D_- \beta(t_0) \le D^+ \beta(t_0)$ can imply β has a derivative at t_0 and $\beta'(t_0) = \beta_0'(t_0)$.

The main existence theorem for solutions of (1.3), (1.4) is now listed in the following.

Theorem 2.3. *Assume α and $\beta \in C[0.1]$ be C^2-lower and upper solution of problem (1.3), (1.4) such that $\alpha \le \beta$. Define $A \subset [0,1]$ (resp. $B \subset [0,1]$) to be the set of points where α (resp. β) is derivable. Let*

$$E := \{(t,u,v) \in [0,1] \times \mathbb{R} \times \mathbb{R} \mid \alpha(t) \le u \le \beta(t)\},$$

$\psi : \mathbb{R}^+ \to \mathbb{R}$ be a positive continuous function satisfying

$$\int_r^{\infty} \frac{s}{\psi(s)} ds > \max_t \beta(t) - \min_t \alpha(t),$$

where $r = \max\{\beta(1) - \alpha(0), \beta(0) - \alpha(1)\}$ and let f is continuous on E which satisfy

$$\forall (t, u, v) \in E, \; |f(t, u, v)| \le \psi(|v|),$$

that is, the Nagumo's condition (p. 46, [24]). Assume there exists $N > 0$ such that for all $t \in A$ (resp. for all $t \in B$)

$$f(t, \alpha(t), \alpha'(t)) \ge -N, \; (\textit{resp. } f(t, \beta(t), \beta'(t)) \le N).$$

Then, the problem (1.3), (1.4) has at least one solution $u \in C^2(0,1) \cap C[0,1]$ such that, for all $t \in [0,1]$,

$$\alpha(t) \le u(t) \le \beta(t).$$

Proof. Choose $R > 0$ be large enough so that

$$\int_r^{R} \frac{s}{\psi(s)} ds > \max_t \beta(t) - \min_t \alpha(t),$$

and increase the value N if necessary, we can assume $N \ge \max_{[0,R]} \psi(v)$. Consider the modified problem

$$\text{(2.1)} \quad \begin{cases} u'' = \overline{f}(t, u, u') + u - \omega(t, u), \\ u(0) = A_0 + \sum_{i=1}^{n} \gamma_i \omega(\zeta_i, u(\zeta_i)), \; u(1) = B_0 + \sum_{j=1}^{m} \delta_j \omega(\eta_j, u(\eta_j)), \end{cases}$$

where $\overline{f} := \max\{\min\{f(t, \omega(t,u), v), N\}, -N\}$ and $\omega : [0,1] \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\omega(t, u) = \begin{cases} \alpha(t), & \text{if } u < \alpha(t), \\ u, & \text{if } \alpha(t) \le u \le \beta(t), \\ \beta(t), & \text{if } u > \beta(t). \end{cases}$$

Step 1. Take the inverse of the operator $Lu = (u'', u(0), u(1))$, one can transform (2.1) into a fixed point problem in C^1. By means of Schauder's fixed point theorem we can prove existence of a fixed point u, which is also a solution of (2.1).

Step 2. The solution u is such that $\alpha(t) \le u(t) \le \beta(t)$ on $[0,1]$. If $u(t_0) - \alpha(t_0) = \min_t (u(t) - \alpha(t)) < 0$ for $t_0 \in (0,1)$, then

$$D^- \alpha(t_0) \ge D_+ \alpha(t_0).$$

Hence, there exists $\alpha_0 \in C^1(I_0)$ as in Definition 2.1. It follows that t_0 is a minimum of $u - \alpha_0$, $(u - \alpha_0)'(t_0) = 0$ and $(u - \alpha_0)''(t_0) \geq 0$. We also have $\alpha'(t_0) = \alpha_0'(t_0)$ by the note after Definition 2.2. Hence $t_0 \in A$ and

$$\overline{f}(t_0, u(t_0), u'(t_0)) \leq f(t_0, \alpha(t_0), \alpha'(t_0)) = f(t_0, \alpha_0(t_0), \alpha_0'(t_0)).$$

Thus, we obtain the contradiction

$$\begin{aligned} 0 \leq u''(t_0) - \alpha_0''(t_0) &= \overline{f}(t_0, u(t_0), u'(t_0)) + u(t_0) - \alpha_0(t_0) - \alpha_0''(t_0) \\ &\leq f(t_0, \alpha_0(t_0), \alpha_0'(t_0)) - \alpha_0''(t_0) + u(t_0) - \alpha_0(t_0) < 0. \end{aligned}$$

If $t_0 = 0$, that is, $\min_t(u(t) - \alpha(t)) = u(0) - \alpha(0) < 0$, then

$$\begin{aligned} A_0 = u(0) - \sum_{i=1}^{n} \gamma_i \omega(\zeta_i, u(\zeta_i)) &\leq u(0) - \sum_{i=1}^{n} \gamma_i \alpha(\zeta_i) \\ &< \alpha(0) - \sum_{i=1}^{n} \gamma_i \alpha(\zeta_i) \leq A_0, \end{aligned}$$

which is a contradiction. Similar argument for $t_0 = 1$ can be obtained and hence, we get the desired result in this step. As a consequence, u satisfies

$$(2.2) \qquad \begin{cases} u'' = \max\{\min\{f(t, u, u'), N\}, -N\}, \\ u(0) = A_0 + \sum_{i=1}^{n} \gamma_i u(\zeta_i), \ u(1) = B_0 + \sum_{j=1}^{m} \delta_j u(\eta_j). \end{cases}$$

Step 2. The solution u satisfies $||u'||_\infty < R$. Observe that for all $(t, u, v) \in E$,

$$\max\{\min\{f(t, u, u'), N\}, -N\} \leq \psi(|v|).$$

From Proposition 4.4 (p. 47, [24]), every solution $u \in [\alpha, \beta]$ of (2.2) is such that

$$||u'||_\infty < R.$$

Therefore, $|f(t, u(t), u'(t))| \leq N$ and the function u is a solution of (1.3), (1.4). ∎

We here notice that Theorem 2.3 is still crucial although there has already been plenty of results involving with multi-point boundary value problems. One can compare it with two essentially recent works. Figueroa[25] considers second order functional differential equations with very general boundary value conditions which contain (1.4). However, Theorem 2.3 concludes the existence of classical solutions differing from their results in weak sense. Secondly, in [26], Graef, Kong, Minhós and Fialho discuss some general higher order functional boundary value problems by use of upper and lower solutions method. Their barrier solutions α or β must belong the class $C^1[0,1]$ (as $n = 2$ in Definition 2 of [26]) and their existence theorem depends strongly on it. This sufficient condition restrict the construction of barrier solutions. The barriers constructed in section 3 and 4 contain points which are not differentiable.

3. Existence Results of (1.1), (1.2): Boundary Layer Phenomena

As mentioned in the introduction, many authors have made significant strides in studying various singularly perturbed nonlocal boundary value problems via diverse schemes. In this section and the next, we systematically deal with the singular perturbation equation

$$\epsilon u''(t) = f(t, u(t)),\ t \in (0,1) \tag{1.1}$$

equipped with three-point boundary condition

$$u(0) = A_0,\ u(1) = B_0 + \delta u(\eta), \tag{1.2}$$

where $\eta \in (0,1)$, $A_0, B_0 \in \mathbb{R}$, and $\delta \geq 0$, which is not yet considered.

Let $u = u_0 \in C^2[0,1]$ be a certain solution of the reduced equation

$$f(t, u) = 0,\ 0 \leq t \leq 1, \tag{3.1}$$

and

$$F = \{(t,u) \mid 0 \leq t \leq 1,\ |u - u_0(t)| \leq d(t)\}, \tag{3.2}$$

where $d(t)$ is a positive continuous function defined below

$$d(t) := \begin{cases} |A_0 - u_0(0)| + \mu, & \text{for } 0 \leq t \leq \frac{\mu}{2}, \\ \mu, & \text{for } \mu \leq t \leq 1 - \mu \\ |B_0 - u_0(1)| + \mu, & \text{for } 1 - \frac{\mu}{2} \leq t \leq 1, \end{cases} \tag{3.3}$$

here $\mu > 0$ is a small constant. This section describes the case where boundary layers take place at boundary points as the following several theorems.

Theorem 3.1. *Let q be a nonnegative integer. Assume that $u_0 \in C^2[0,1]$ is a solution of the reduced equation (3.1) and there exists $m > 0$ such that*

$$D_2^j f(t, u_0(t)) \equiv 0 \text{ for } 0 \leq t \leq 1 \text{ and } 0 \leq j \leq 2q, \tag{3.4}$$

and

$$D_2^{2q+1} f(t,u) \geq m > 0 \text{ in } F. \tag{3.5}$$

Then for $\epsilon > 0$ *small enough, there exists a solution* u_ϵ *of* (1.1), (1.2) *such that*

$$|u_\epsilon(t) - u_0(t)| \le w_L(t,\epsilon) + w_R(t,\epsilon) + \Gamma_\epsilon(t),$$

where

$$w_L(t,\epsilon) = \frac{|A_0 - u_0(0)|\big[(e^{-\sqrt{\frac{m}{\epsilon}}} - \delta e^{-\sqrt{\frac{m}{\epsilon}}\eta})e^{\sqrt{\frac{m}{\epsilon}}t} - (e^{\sqrt{\frac{m}{\epsilon}}} - \delta e^{\sqrt{\frac{m}{\epsilon}}\eta})e^{-\sqrt{\frac{m}{\epsilon}}t}\big]}{\Delta} \text{ if } q = 0,$$

$$w_L(t,\epsilon) = |A_0 - u_0(0)|\big(1 + \frac{\sigma_1|A_0 - u_0(0)|^q}{\sqrt{\epsilon}}t\big)^{-1/q} \text{ if } q \ge 1,$$

$$w_R(t,\epsilon) = \frac{|B_0 - u_0(1) + \delta u_0(\eta)|(e^{\sqrt{\frac{m}{\epsilon}}t} - e^{\sqrt{\frac{m}{\epsilon}}}t)}{\Delta} \text{ if } q = 0,$$

$$w_R(t,\epsilon) = |B_0 - u_0(1) + \delta u_0(\eta)|\big(1 + \frac{\sigma_1|B_0 - u_0(1) + \delta u_0(\eta)|^q}{\sqrt{\epsilon}}(1-t)\big)^{-1/q} \text{ if } q \ge 1,$$

$\Delta = e^{-\sqrt{\frac{m}{\epsilon}}} - e^{\sqrt{\frac{m}{\epsilon}}} + \delta(e^{\sqrt{\frac{m}{\epsilon}}\eta} - e^{-\sqrt{\frac{m}{\epsilon}}\eta})$, $\sigma_1 = q\sqrt{\frac{m}{(q+1)(2q+1)!}}$ *and* Γ_ϵ *is a function determined as* (3.12).

Proof. We will exhibit, by construction, the existence of the upper and lower solutions of (1.1) and (1.2) with the required properties as in Definition 2.1 and 2.2 and then, this theorem will follows from Theorem 2.3. For $q = 0$, we set $w_R(t,\epsilon)$ and $w_L(t,\epsilon)$ are the solutions of

$$\text{(3.6)} \qquad \begin{cases} \epsilon w'' = mw, \\ w(0) = 0,\ w(1) - \delta w(\eta) = |B_0 - u_0(1) + \delta u_0(\eta)| \end{cases}$$

and

$$\text{(3.7)} \qquad \begin{cases} \epsilon w'' = mw, \\ w(0) = |A_0 - u_0(0)|,\ w(1) = \delta w(\eta), \end{cases}$$

respectively. For $q \ge 1$, we set $w_R(t,\epsilon)$ and $w_L(t,\epsilon)$ are the solutions of

$$\text{(3.8)} \qquad \begin{cases} \epsilon w'' = \frac{m}{(2q+1)!}w^{2q+1}, \\ w(1) = |B_0 - u_0(1) + \delta u_0(\eta)|,\ w'(1) = \sqrt{\frac{m}{\epsilon(q+1)(2q+1)!}}|B_0 - u_0(1) + \delta u_0(\eta)|^{q+1} \end{cases}$$

and

$$\text{(3.9)} \qquad \begin{cases} \epsilon w'' = \frac{m}{(2q+1)!}w^{2q+1}, \\ w(0) = |A_0 - u_0(0)|,\ w'(0) = -\sqrt{\frac{m}{\epsilon(q+1)(2q+1)!}}|A_0 - u_0(0)|^{q+1}, \end{cases}$$

respectively. Thus, the explicit forms of w_L and w_R are as stated in this theorem in case $q = 0$ or $q \geq 1$. Moreover, for $q \geq 1$, w_L (resp. w_R) is nonnegative and decrease to the right (resp. left).

Choose $K_2 > 0$ such that $\frac{m}{(2q+1)!}K_2^{2q+1} > ||u_0''||_\infty$, $K_1 > K_2$, and define $k(t)$ to be a convex quadratic function on $[\eta, 1]$ such that $k(\eta) = K_1$, $\min_{t\in[\eta,1]} k(t) = K_2$ and $k(1) \geq \delta K_1 + 1$. We set, for $t \in [0, 1]$,

$$\alpha(t, \epsilon) = u_0(t) - w_L(t, \epsilon) - w_R(t, \epsilon) - \Gamma_\epsilon(t), \tag{3.10}$$

and

$$\beta(t, \epsilon) = u_0(t) + w_L(t, \epsilon) + w_R(t, \epsilon) + \Gamma_\epsilon(t), \tag{3.11}$$

where

$$\Gamma_\epsilon(t) = \begin{cases} \epsilon^{\frac{1}{2q+1}} k(t), & t \geq \eta, \\ \epsilon^{\frac{1}{2q+1}} K_1, & t < \eta. \end{cases} \tag{3.12}$$

For $q = 0$, from (3.6) and (3.7), it is obvious that $\alpha \leq \beta$,

$$\alpha(0, \epsilon) \leq A_0 \leq \beta(0, \epsilon),$$

$$\alpha(1, \epsilon) - \delta\alpha(\eta, \epsilon) = u_0(1) - \delta u_0(\eta) - |B_0 - u_0(1) + \delta u_0(\eta)| - \epsilon[k(1) - \delta k(\eta)] \leq B_0,$$

and $\beta(1, \epsilon) - \delta\beta(\eta, \epsilon) \geq B_0$. For $q \geq 1$, it follows from (3.8), (3.9) and (3.12) that

$$\alpha(0, \epsilon) \leq A_0 \leq \beta(0, \epsilon),$$

$$\begin{aligned}
&\alpha(1, \epsilon) - \delta\alpha(\eta, \epsilon) \\
&= \big(u_0(1) - \delta u_0(\eta)\big) - \big(\Gamma_\epsilon(1) - \delta\Gamma_\epsilon(\eta)\big) \\
&\quad - \big(w_L(1, \epsilon) - \delta w_L(\eta, \epsilon)\big) - \big(w_R(1, \epsilon) - \delta w_R(\eta, \epsilon)\big) \\
&= \big(u_0(1) - \delta u_0(\eta)\big) - \epsilon^{\frac{1}{2q+1}} \big(k(1) - \delta k(\eta)\big) \\
&\quad - |A_0 - u_0(0)|[\big(1 + \frac{\sigma_1|A_0 - u_0(0)|^q}{\sqrt{\epsilon}}\big)^{-\frac{1}{q}} - \delta\big(1 + \frac{\sigma_1|A_0 - u_0(0)|^q}{\sqrt{\epsilon}}\eta\big)^{-\frac{1}{q}}] \\
&\quad - |B_0 - u_0(1) + \delta u_0(\eta)|[1 - \delta\big(1 + \frac{\sigma_1|B_0 - u_0(1) + \delta u(\eta)|^q}{\sqrt{\epsilon}}(1 - \eta)\big)^{-\frac{1}{q}}] \\
&= \big(u_0(1) - \delta u_0(\eta)\big) - |B_0 - u_0(1) + \delta u_0(\eta)| - O(\epsilon^{\frac{1}{2q+1}}) + o(\epsilon^{\frac{1}{2q+1}}) \leq B_0,
\end{aligned}$$

and $\beta(1, \epsilon) - \delta\beta(\eta, \epsilon) \geq B_0$ similarly.

We now show that $\epsilon\alpha'' \geq f(t, \alpha)$. By Taylor's theorem, (3.4) and (3.5), we have

$$
\begin{aligned}
f(t,\alpha(t,\epsilon)) &= f(t,\alpha(t,\epsilon)) - f(t,u_0(t)) \\
&= \sum_{n=1}^{2q} \frac{1}{n!} D_2^n f(t,u_0(t))[\alpha(t,\epsilon)-u_0(t)]^n \\
&\quad + \frac{1}{(2q+1)!} D_2^{2q+1} f(t,\xi(t))[\alpha(t,\epsilon)-u_0(t)]^{2q+1} \\
&= -\frac{1}{(2q+1)!} D_2^{2q+1} f(t,\xi(t))[w_L + w_R + \Gamma_\epsilon]^{2q+1},
\end{aligned}
$$

where $(t,\xi(t))$ is some intermediate point between $(t,\alpha(t,\epsilon))$ and $(t,u_0(t))$ which lies in F for sufficiently small ϵ. Since w_L, w_R and Γ_ϵ are positive functions, we have

$$
-f(t,\alpha(t,\epsilon)) \geq \frac{m}{(2q+1)!}\big(w_L^{2q+1} + w_R^{2q+1} + \Gamma_\epsilon^{2q+1}\big).
$$

Hence, for $t < \eta$,

$$
\begin{aligned}
\epsilon\alpha'' - f(t,\alpha(t,\epsilon)) &\geq \epsilon u_0'' - \epsilon w_L'' - \epsilon w_R'' + \frac{m}{(2q+1)!}\big(w_L^{2q+1} + w_R^{2q+1} + \Gamma_\epsilon^{2q+1}\big) \\
&\geq -\epsilon ||u_0''||_\infty + \frac{m}{(2q+1)!}\epsilon K_1^{2q+1} > 0.
\end{aligned}
$$

and for $t > \eta$,

$$
\begin{aligned}
\epsilon\alpha'' - f(t,\alpha(t,\epsilon)) &\geq \epsilon u_0'' - \epsilon w_L'' - \epsilon w_R'' + \frac{m}{(2q+1)!}\big(w_L^{2q+1} + w_R^{2q+1} + \Gamma_\epsilon^{2q+1}\big) \\
&\geq -\epsilon ||u_0''||_\infty - \epsilon\epsilon^{\frac{1}{2q+1}} k''(t) + \frac{m}{(2q+1)!}\epsilon (k(t))^{2q+1} > 0 \\
&\geq -\epsilon ||u_0''||_\infty - o(\epsilon) + \frac{m}{(2q+1)!}\epsilon K_2^{2q+1} > 0.
\end{aligned}
$$

Moreover, as $t = \eta$,

$$
D^-\alpha(\eta,\epsilon) = u_0'(\eta) - w_L' - w_R' < u_0'(\eta) - w_L' - w_R' - k'(\eta^-) - D_+\alpha(\eta,\epsilon)
$$

because of the convexity of $k(t)$ on $[\eta,1]$. One can also follow the above steps to prove that $\epsilon\beta'' \leq f(t,\beta)$ in $(0,1)$ and therefore, α and β defined as (3.10) and (3.11) are lower and upper solutions of (1.1), (1.2), respectively. By means of Theorem 2.3, we complete this proof. ∎

If $A_0 \geq u_0(0)$ and $B_0 \geq u_0(1) - \delta u_0(\eta)$, we define

$$
F_1 := \{(t,u) \mid 0 \leq t \leq 1,\ 0 \leq u - u_0(t) \leq d(t)\}, \tag{3.13}
$$

where $d(t)$ is as (3.3) and have the following conclusion.

Theorem 3.2. *Let* $n \geq 2$*. Assume that* $u_0 \in C^2[0,1]$ *is a solution of the reduced equation* (3.1)*,* $A_0 \geq u_0(0)$*,* $B_0 \geq u_0(1) - \delta u_0(\eta)$*,* $u_0'' \geq 0$ *in* $(0,1)$ *and there exists* $m > 0$ *such that*

(3.14) $$D_2^j f(t, u_0(t)) \geq 0 \text{ for } 0 \leq t \leq 1 \text{ and } 1 \leq j \leq n-1,$$

and

(3.15) $$D_2^n f(t,u) \geq m > 0 \text{ in } F_1.$$

Then for $\epsilon > 0$ *small enough, there exists a solution* u_ϵ *of* (1.1)*,* (1.2) *such that*

$$0 \leq u_\epsilon(t) - u_0(t) \leq w_L(t,\epsilon) + w_R(t,\epsilon) + \Delta_\epsilon(t),$$

where

$$w_L(t,\epsilon) = |A_0 - u_0(0)|\big(1 + \frac{\sigma_2 |A_0 - u_0(0)|^{\frac{n-1}{2}}}{\sqrt{\epsilon}} t\big)^{-\frac{2}{n-1}},$$

$$w_R(t,\epsilon) = |B_0 - u_0(1) + \delta u_0(\eta)|\big(1 + \frac{\sigma_2 |B_0 - u_0(1) + \delta u_0(\eta)|^{\frac{n-1}{2}}}{\sqrt{\epsilon}}(1-t)\big)^{-\frac{2}{n-1}},$$

$\sigma_2 = (n-1)\sqrt{\frac{m}{2(n+1)!}}$ *and* Δ_ϵ *is a function determined as* (3.18)*.*

Proof. The proof of this theorem follows in the same manner the proof of Theorem 3.1 once we notice that $w_R \geq 0$ is now the solution of

$$\begin{cases} \epsilon w'' = \frac{m}{n!} w^n, \\ w(1) = B_0 - u_0(1) + \delta u_0(\eta), \ w'(1) = \sqrt{\frac{2m}{\epsilon(n+1)!}}\big(B_0 - u_0(1) + \delta u_0(\eta)\big)^{\frac{n+1}{2}}, \end{cases}$$

and $w_L \geq 0$ is now the solution of

$$\begin{cases} \epsilon w'' = \frac{m}{n!} w^n, \\ w(0) = A_0 - u_0(0), \ w'(0) = -\sqrt{\frac{2m}{\epsilon(n+1)!}}\big(A_0 - u_0(0)\big)^{\frac{n+1}{2}}. \end{cases}$$

We then choose $K_2 > 0$ such that $\frac{m}{n!} K_2^n > ||u_0''||_\infty$, $K_1 > K_2$, and define $k(t)$ to be a convex quadratic function on $[\eta, 1]$ such that $k(\eta) = K_1$, $\min_{t \in [\eta,1]} k(t) = K_2$ and $k(1) \geq \delta K_1 + 1$. For $t \in [0,1]$, set

(3.16) $$\alpha(t,\epsilon) = u_0(t),$$

(3.17) $$\beta(t,\epsilon) = u_0(t) + w_L(t,\epsilon) + w_R(t,\epsilon) + \Delta_\epsilon(t),$$

where

$$\Delta_\epsilon(t) = \begin{cases} \epsilon^{\frac{1}{n}} k(t), & t \geq \eta, \\ \epsilon^{\frac{1}{n}} K_1, & t < \eta. \end{cases} \tag{3.18}$$

All details to show that $\alpha(t, \epsilon)$ and $\beta(t, \epsilon)$ are barriers of (1.1), (1.2) are much similar to the demonstration in the proof of Theorem 3.1. Hence, we omit them except noting that the convexity of u_0 implies $\epsilon\alpha'' - f(t, \alpha) = \epsilon u_0'' - f(t, \alpha) = \epsilon u_0'' \geq 0$. ∎

The next theorem is the analog of Theorem 3.2 when the solution u_0 of the reduced equation (3.1) satisfies $A_0 \leq u_0(0)$ and $B_0 \leq u_0(1) - \delta u_0(\eta)$.

Theorem 3.3. *Let $n \geq 2$. Assume that $u_0 \in C^2[0, 1]$ is a solution of the reduced equation (3.1), $A_0 \leq u_0(0)$, $B_0 \leq u_0(1) - \delta u_0(\eta)$, $u_0'' \leq 0$ in $(0, 1)$ and there exists $m > 0$ such that*

$$D_2^{j_o(j_e)} f(t, u_0(t)) \geq 0 \ (\leq 0) \text{ for } 0 \leq t \leq 1 \text{ and } 1 \leq j_o, j_e \leq n - 1, \tag{3.19}$$

where $j_o(j_e)$ denotes an odd (even) integer, and

$$D_2^n f(t, u) \leq -m < 0 \ (\geq m > 0) \text{ in } F_2, \text{ if } n \text{ is even (odd)}, \tag{3.20}$$

where

$$F_2 := \{(t, u) \mid 0 \leq t \leq 1,\ -d(t) \leq u - u_0(t) \leq 0\}. \tag{3.21}$$

Then for $\epsilon > 0$ small enough, there exists a solution u_ϵ of (1.1), (1.2) such that

$$-w_L(t, \epsilon) - w_R(t, \epsilon) - \Delta_\epsilon(t) \leq u_\epsilon(t) - u_0(t) \leq 0,$$

where w_L, w_R and Δ_ϵ are the same as in Theorem 3.2.

Proof. We set that the lower and upper barriers of (1.1), (1.2) are, for $t \in [0, 1]$,

$$\alpha(t, \epsilon) = u_0(t) - w_L(t, \epsilon) - w_R(t, \epsilon) - \Delta_\epsilon(t), \tag{3.22}$$

$$\beta(t, \epsilon) = u_0(t), \tag{3.23}$$

and the rest is similar to the proof of Theorem 3.2. ∎

Remark. Notice that in Theorem 3.1, 3.2 and 3.3, the solution $u_\epsilon(t)$ of (1.1), (1.2) tends to $u_0(t)$ as $\epsilon \to 0$, uniformly on every compact subset of $(0, 1)$. The convergence is however nonuniform at the endpoint $t = 0$ and $t = 1$. This is boundary layer phenomena of solution u_ϵ.

4. Existence Results of (1.1), (1.2): Interior Layer Phenomena

The previous section deals with problem (1.1), (1.2) when the solution $u_0 = u_0(t)$ of reduced equation (3.1) is twice continuously differentiable in $[0, 1]$. In fact, the smoothness restriction imposed on u_0 can be weakened without alternating the validity of those results in Section 3.

Theorem 4.1. *Let q be a nonnegative integer. Assume that the reduced equation (3.1) has a solution $u_0 = u_0(t)$ of $C^2[0, 1]$, except at $t_* \in (0, 1)$ where $u_0'(t_*^-) \neq u_0'(t_*^+)$ and $|u_0''(t_*^\pm)| < \infty$. Furthermore, there exists $m > 0$ such that f and u_0 satisfy (3.4) and (3.5). Then for $\epsilon > 0$ small enough, there exists a solution u_ϵ of (1.1), (1.2) such that*

$$|u_\epsilon(t) - u_0(t)| \leq w_L(t, \epsilon) + w_R(t, \epsilon) + v_I(t, \epsilon) + \Gamma_\epsilon(t).$$

Here w_L, w_R, Γ_ϵ are as given in Theorem 3.1,

$$v_I(t, \epsilon) = \frac{1}{2}\sqrt{\frac{\epsilon}{m}}|u_0'(t_*^+) - u_0'(t_*^-)|e^{-\sqrt{\frac{m}{\epsilon}}|t-t_*|} \text{ if } q = 0,$$

and

$$v_I(t, \epsilon) = \tau_1\big(1 + q\sqrt{\frac{m}{\epsilon(2q+1)!}}\tau_1^q|t - t_*|\big)^{-\frac{1}{q}} \text{ if } q \geq 1,$$

where $\tau_1^{q+1} = \frac{1}{2}|u_0'(t_*^+) - u_0'(t_*^-)|\sqrt{\frac{m}{\epsilon(2q+1)!}}$.

We note that $u_0 = u_0(t)$ of $C^2[0, 1]$ except $t_* \in (0, 1)$, $|u_0''(t_*^\pm)| < \infty$ implies $||u_0''||_\infty < \infty$. Hence, the function Γ_ϵ as (3.12) is still well-defined.

Proof. We can suppose first that $u_0'(t_*^-) < u_0'(t_*^+)$. Then, for $t \in [0, 1]$, we define

$$\alpha(t, \epsilon) = u_0(t) - w_L(t, \epsilon) - w_R(t, \epsilon) - \Gamma_\epsilon(t), \tag{4.1}$$

$$\beta(t, \epsilon) = u_0(t) + w_L(t, \epsilon) + w_R(t, \epsilon) + v_I(t, \epsilon) + \Gamma_\epsilon(t). \tag{4.2}$$

Claim 1. $\alpha(t, \epsilon)$ is a lower barrier solution of (1.1), (1.2).

The boundary constraints $\alpha(0, \epsilon) \leq A_0$, $\alpha(1, \epsilon) - \delta\alpha(\eta, \epsilon) \leq B_0$ are still valid because the form (4.1) is the same as (3.10). If $t_* = \eta$, the function α is not differentiable at $t = t_* = \eta$ since

$$D^-\alpha(\eta, \epsilon) = u_0'(t_*^-) - w_L' - w_R' < u_0'(t_*^+) - w_L' - w_R' - k'(t_*^+) = D_+\alpha(\eta, \epsilon).$$

If $t_* \neq \eta$, the function α are not differentiable at t_* and η because of

$$D^-\alpha(t_*,\epsilon) = u_0'(t_*^-) - w_L' - w_R' - \Gamma_\epsilon'(t_*) < u_0'(t_*^+) - w_L' - w_R' - \Gamma_\epsilon'(t_*) = D_+\alpha(t_*,\epsilon)$$

and

$$D^-\alpha(\eta,\epsilon) = u_0'(\eta) - w_L' - w_R' < u_0'(\eta) - w_L' - w_R' - k'(\eta^+) = D_+\alpha(\eta,\epsilon).$$

The rest discussions of differential inequality $\epsilon\alpha'' \geq f(t,\alpha)$ on $(0,1)\backslash\{t_*,\eta\}$ are similar to the corresponding part in the proof of Theorem 3.1.

Claim 2. $\beta(t,\epsilon)$ is a upper barrier solution of (1.1), (1.2).

We first consider the boundary constraints when $q = 0$ or $q \geq 1$ respectively. For $q = 0$, it is obvious that $\beta(0,\epsilon) \geq A_0$ and

$$\begin{aligned}\beta(1,\epsilon) - \delta\beta(\eta,\epsilon) &= u_0(1) - \delta u_0(\eta) + |B_0 - u_0(1) - \delta u_0(\eta)| + \epsilon\big(k(1) - \delta k(\eta)\big)\\ &\quad + \frac{1}{2}\sqrt{\frac{\epsilon}{m}}|u_0'(t_*^+) - u_0'(t_*^-)|\big(e^{-\sqrt{\frac{m}{\epsilon}}|1-t_*|} - \delta e^{-\sqrt{\frac{m}{\epsilon}}|\eta-t_*|}\big)\\ &= u_0(1) - \delta u_0(\eta) + |B_0 - u_0(1) - \delta u_0(\eta)| + O(\epsilon) + o(\epsilon) \geq B_0.\end{aligned}$$

For $q \geq 1$, we have $\beta(0,\epsilon) \geq A_0$ easily and

$$\begin{aligned}&\beta(1,\epsilon) - \delta\beta(\eta,\epsilon)\\ &= \big(u_0(1) - \delta u_0(\eta)\big) + \big(\Gamma_\epsilon(1) - \delta\Gamma_\epsilon(\eta)\big) + \big(v_I(1,\epsilon) - \delta v_I(\eta,\epsilon)\big)\\ &\quad + \big(w_L(1,\epsilon) - \delta w_L(\eta,\epsilon)\big) + \big(w_R(1,\epsilon) - \delta w_R(\eta,\epsilon)\big)\\ &= \big(u_0(1) - \delta u_0(\eta)\big) + \epsilon^{\frac{1}{2q+1}}\big(k(1) - \delta k(\eta)\big)\\ &\quad + \tau_1[\big(1 + q\sqrt{\frac{m}{\epsilon(2q+1)!}}\tau_1^q|1-t_*|\big)^{-\frac{1}{q}} - \delta\big(1 + q\sqrt{\frac{m}{\epsilon(2q+1)!}}\tau_1^q|\eta-t_*|\big)^{-\frac{1}{q}}]\\ &\quad + |A_0 - u_0(0)|[\big(1 + \frac{\sigma_1|A_0 - u_0(0)|^q}{\sqrt{\epsilon}}\big)^{-\frac{1}{q}} - \delta\big(1 + \frac{\sigma_1|A_0 - u_0(0)|^q}{\sqrt{\epsilon}}\eta\big)^{-\frac{1}{q}}]\\ &\quad + |B_0 - u_0(1) + \delta u_0(\eta)|[1 - \delta\big(1 + \frac{\sigma_1|B_0 - u_0(1) + \delta u(\eta)|^q}{\sqrt{\epsilon}}(1-\eta)\big)^{-\frac{1}{q}}]\\ &= \big(u_0(1) - \delta u_0(\eta)\big) + |B_0 - u_0(1) + \delta u_0(\eta)| + O(\epsilon^{\frac{1}{2q+1}}) + o(\epsilon^{\frac{1}{2q+1}}) \leq B_0.\end{aligned}$$

Before focusing on differential inequality of β, we note that v_I is the solution of $\epsilon v'' = \frac{m}{(2q+1)!}v^{2q+1}$ in $(0,t_*)\cup(t_*,1)$ which satisfies

$$v_I(t_*^-,\epsilon) = v_I(t_*^+,\epsilon) = \tau_1,$$

and

$$v_I'(t_*^-,\epsilon) = -v_I'(t_*^+,\epsilon) = \frac{1}{2}|u_0'(t_*^+) - u_0'(t_*^-)|.$$

We also observe that if $t_* = \eta$, the function β is not differentiable at t_* since

$$D_-\beta(t_*, \epsilon) = \frac{1}{2}|u_0'(t_*^+) + u_0'(t_*^-)| + w_L' + w_R'$$
$$> \frac{1}{2}|u_0'(t_*^+) + u_0'(t_*^-)| + w_L' + w_R' + k(t_*^+) = D^+\beta(t_*, \epsilon).$$

If $t_* \neq \eta$, β is differentiable at t_*, indeed,

$$\beta'(t_*^-, \epsilon) = \beta'(t_*^+, \epsilon) = \frac{1}{2}|u_0'(t_*^+) + u_0'(t_*^-)| + w_L' + w_R' + \Gamma_\epsilon'(t_*)$$

and is not differentiable at η with $D_-\beta(\eta, \epsilon) > D^+\beta(\eta, \epsilon)$. One can show that $\epsilon\beta'' \leq f(t, \beta)$ on $(0,1)\backslash\{t_*, \eta\}$ by similar arguments in the proof of Theorem 3.1. α and β defined as (4.1) and (4.2) are respective lower and upper barrier solutions of (1.1), (1.2).

As $u_0'(t_*^-) > u_0'(t_*^+)$, we set, for $t \in [0,1]$,

$$\alpha(t, \epsilon) = u_0(t) - w_L(t, \epsilon) - w_R(t, \epsilon) - v_I(t, \epsilon) - \Gamma_\epsilon(t), \tag{4.3}$$

$$\beta(t, \epsilon) = u_0(t) + w_L(t, \epsilon) + w_R(t, \epsilon) + \Gamma_\epsilon(t), \tag{4.4}$$

and follow the same manner mentioned above. Thus, we complete this proof by applying Theorem 2.3. ■

Theorem 4.2. *Let* $n \geq 2$. *Assume that the reduced equation* (3.1) *has a solution* $u_0 = u_0(t)$ *of* $C^2[0,1]$, *except at* $t_* \in (0,1)$ *where* $u_0'(t_*^-) < u_0'(t_*^+)$ *and* $|u_0''(t_*^\pm)| < \infty$. *Assume also that* $A_0 \geq u_0(0)$, $B_0 \geq u_0(1) - \delta u_0(\eta)$, $u_0'' \geq 0$ *in* $(0, t_*) \cup (t_*, 1)$ *and there exists* $m > 0$ *such that* f *and* u_0 *satisfy* (3.14) *and* (3.15). *Then for* $\epsilon > 0$ *small enough, there exists a solution* u_ϵ *of* (1.1), (1.2) *such that*

$$0 \leq u_\epsilon(t) - u_0(t) \leq w_L(t, \epsilon) + w_R(t, \epsilon) + v_I(t, \epsilon) + \Delta_\epsilon(t).$$

Here w_L, w_R, Δ_ϵ *are as given in Theorem 3.2,*

$$v_I(t, \epsilon) = \tau_2\big(1 + \frac{n-1}{2}\sqrt{\frac{m}{\epsilon n!}}\tau^{\frac{n-1}{2}}|t - t_*|\big)^{-\frac{2}{n-1}},$$

where $\tau_2^{n+1} = \frac{\epsilon n!|u_0'(t_*^+) - u_0'(t_*^-)|^2}{4m}$.

Proof. This conclusion follows from similar steps of proof of Theorem 4.1 to show that

$$\alpha(t, \epsilon) = u_0(t)$$

and

$$\beta(t, \epsilon) = u_0(t) + w_L(t, \epsilon) + w_R(t, \epsilon) + v_I(t, \epsilon) + \Delta_\epsilon(t)$$

are corresponding lower and upper barriers of (1.1) and (1.2). We also note that the function $\Delta_\epsilon(t)$ as (3.18) is well-defined because $u_0 = u_0(t) \in C^2[0,1]$, except $t_* \in (0,1)$ with $|u_0''(t_*^\pm)| < \infty$ will implies $||u_0''||_\infty < \infty$. ■

Theorem 4.3. *Let* $n \geq 2$. *Assume that the reduced equation* (3.1) *has a solution* $u_0 = u_0(t)$ *of* $C^2[0,1]$, *except at* $t_* \in (0,1)$ *where* $u_0'(t_*^-) > u_0'(t_*^+)$ *and* $|u_0''(t_*^{\pm})| < \infty$. *Assume also that* $A_0 \leq u_0(0)$, $B_0 \leq u_0(1) - \delta u_0(\eta)$, $u_0'' \leq 0$ *in* $(0,t_*) \cup (t_*,1)$ *and there exists* $m > 0$ *such that* f *and* u_0 *satisfy* (3.19) *and* (3.20). *Then for* $\epsilon > 0$ *small enough, there exists a solution* u_ϵ *of* (1.1), (1.2) *such that*

$$-w_L(t,\epsilon) - w_R(t,\epsilon) - v_I(t,\epsilon) - \Delta_\epsilon(t) \leq u_\epsilon(t) - u_0(t) \leq 0.$$

Here w_L, w_R, v_I *and* Δ_ϵ *are as given in Theorem 4.2.*

Proof. This conclusion follows from similar steps of proof of Theorem 4.1 to show that

$$\alpha(t,\epsilon) = u_0(t) - w_L(t,\epsilon) - w_R(t,\epsilon) - v_I(t,\epsilon) - \Delta_\epsilon(t)$$

and

$$\beta(t,\epsilon) = u_0(t)$$

are corresponding lower and upper barriers of (1.1) and (1.2). ■

Remark. (i) In Theorem 4.1, 4.2 and 4.3, the solution $u_\epsilon(t)$ of (1.1), (1.2) tends to $u_0(t)$ as $\epsilon \to 0$, uniformly on every compact subset of $(0,1)$. In this case an interior layer in the derivative u_ϵ' takes place at points t_* where the derivative u_0' is discontinuous. (ii) These results(Theorem 4.1-4.3) can be extended to the case of finitely many points of nondifferentiability of the reduced solution u_0.

References

1. E. E. Bukzhalev, On the construction of upper and lower solutions by the Nagumo method, *Diff. Equ.*, **40(6)** (2004), 771-779.
2. R. Vrabel, Three point boundary value problem for singularly perturbed semilinear differential equations, *E. J. Qualit. Th. Diff. Eqns.*, **70** (2009), 1-4.
3. R. Vrabel, V. Liska and I. Mankova, Boundary layer analysis for nonlinear singularly perturbed differential equations, *Electron. J. Qual. Theory Differ. Equ.*, **2011(32)** (2011), 1-11.
4. R. Vrabel, Boundary layer phenomenon for three-point boundary value problem for the nonlinear singularly perturbed systens, *Kybernetika*, **47(4)** (2011), 644-652.
5. X. J. Lin and W. B. Liu, Singular perturbation of a second-order three-point boundary value problem for nonlinear systems, *J. Shanghai Univ.* (*English ed.*), **13(3)** (2009), 16-19.
6. W. Zhao, Singular perturbations of boundary value problems for a class of third-order nonlinear ordinary differential equations, *J. Diff. Eqn.*, **88** (1990), 265-278.
7. Z. Du, C. Xue, W. Ge and M. Zhou, Singular perturbations for third-order nonlinear multi-point boundary value problem, *J. Diff. Eqn.*, **218** (2005), 69-90.

8. E. E. Bukzhalev, A singularly perturbed equation with a boundary-layer solution whose expanded variables depend on various powers of a perturbation parameter, *Comput. Math. Math. Phys.*, **43(12)** (2003), 1707-1717.
9. E. E. Bukzhalev, Contrast steplike structures with stretched variables depend on various powers of a perturbation parameter, *Comput. Math. Math. Phys.*, **44(4)** (2004), 626-639.
10. A. B. Vasil'eva and E. E. Bukzhalev, A singularly perturbed boundary value problem for a second-order differential equation whose right-hand side is a quadratic function of the derivative of the unknown function, *Diff. Eqns.*, **38(6)** (2002), 760-771.
11. R. Vrabel, *Nonlocal Four-Point Boundary Value Problem for the Singularly Perturbed Semilinear Differential Equations*, Bound. Value Probl., Vol. 2011, Article ID 570493, 2011, 9 pages.
12. Z. Du and L. Kong, Asymptotic solutions of singularly perturbed second-order differential equations and application to multi-point boundary value problems, *Appl. Math. Lett.*, **23** (2010), 980-983.
13. Z. Weili, Singular perturbations of boundary value problems for a class of third order nonlinear ordinary differential equations, *J. Diff. Equ.*, **88** (1990), 265-278.
14. M. Fe̎can, Singularly perturbed higher order boundary value problems, *J. Diff. Equ.*, **111** (1994), 79-102.
15. G. Akram and N. Amin, Solution of a fourth order singularly perturbed boundary value problem using quintic spline, *Int. Math. Forum*, **7(44)** (2012), 2179-2190.
16. S. Valarmathi and N. Ramanujam, An asymptotic numerical method for singularly perturbed third-order ordinary differential equations of convection-diffusion type, *Comput. Math. Appl.*, **44** (2002), 693-710.
17. F. A. Howes, The asymptotic solution of a class of third-order boundary value problem arising in the theory of thin film flow, *SIAM J. Appl. Math.*, **43** (1983), 993-1004.
18. Z. Du, C. Xue and W. Ge, Multiple solutions for three-point boundary value problem with nonlinear terms depending on the first order derivative, *Arch. Math.*, **84** (2005), 341-349.
19. L. Kong and Q. Kong, Multi-point boundary value problems of second-oder differential equations (I), *Nonl. Anal.*, **58** (2004), 909-931.
20. R. A. Khan and J. R. L. Webb, Existence of at least three solutions of a second-order three-point boundary value problem, *Nonlinear Anal.*, **64** (2006), 1356-1366.
21. P. Minghe and S. K. Chang, The generalized quasilinearization method for second-order three-point boundary value problems, *Nonlinear Analysis*, **68(9)** (2008), 2779-2790.
22. S. P. Wang, On three-point boundary value problems with nonlinear source terms containing first order derivatives, *J. Pure and Applied Mathemaitcs: Advanced and Applications*, **2(2)** (2009), 209-226.
23. Y. Guo, W. Shan and W. Ge, Positive solutions for second-order m-point boundary value problems, *J. Comput. Appl. Math.*, **151** (2003), 415-424.

24. C. D. Coster and P. Habets, *Two-Point Boundary Value Problems: Lower and Upper Solutions*, Elsevier, 2006.

25. R. Figueroa, Second-order functional differential equations with past, present and future dependence, *Appl. Math. Comput.*, **217** (2011), 7448-7454.

26. J. R. Graef, L. Kong, F. M. Minhós and J. Fialho, On the lower and upper solution mehtod for higher order functional boundary value problems, *Appl. Anal. Discrete Math.*, **5** (2011), 133-146.

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