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WEIGHTED NORM INEQUALITIES FOR FLAG SINGULAR INTEGRALS ON HOMOGENEOUS GROUPS

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Abstract. Let G be a homogeneous nilpotent Lie group. In this paper, we introduce a new class of multiparameter weights $A_p^{\mathcal{F}}$ associated with a flag $\mathcal{F}$ on G and show that such class of weights can be characterized via two type of flag maximal operators. We then prove that singular integrals with flag kernels are bounded on $L^p_w(G), 1 < p < \infty$, when $w \in A_p^{\mathcal{F}}(G)$, which extends a recent result of Nagel-Ricci-Stein-Wainger in [13]. As an application, we get weighted norm inequalities for the multiparameter Marcinkiewicz multipliers on Heisenberg groups introduced in [11].

1. INTRODUCTION

Flag singular integral operators were comprehensively studied in recent years and many applications were found in analysis on Heisenberg groups, theory of function spaces, several complex variables and *etc*. Such class of operators were introduced by Müller, Ricci and Stein [11] when they studied the Marcinkiewicz multiplier on the Heisenberg groups $\mathbb{H}^n$. They obtained the surprising result that certain Marcinkiewicz multipliers, invariant under a two-parameter group of dilations on $\mathbb{C}^n \times \mathbb{R}$, are bounded on $L^p(\mathbb{H}^n)$, despite the absence of a two-parameter automorphic group of dilations on $\mathbb{H}^n$. To study the $\Box_b$-complex on certain CR submanifolds of $\mathbb{C}^n$, Nagel, Ricci and Stein [12] studied further a class of product singular integrals with flag kernels. Applying the theory of flag kernels, Nagel and Stein [14] obtained remarkable results on the optimal estimates for solutions of the Kohn-Laplacian for certain classes of model domains in several complex variables. More recently, using Littlewood-Paley theory, Nagel, Ricci, Stein and Wainger [13] extended the above results to a more general setting, namely,

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homogeneous group. We would like to point out that Głowacki [5, 6] independently obtained similar results by using Melin calculus on homogeneous groups developed in [7]. The multiparameter Hardy spaces associated with flag kernels were developed by Han and Lu [8] in the Euclidean setting and by Han, Lu and Sawyer [9] in the setting of Heisenberg groups and atomic decomposition characterizations for the Hardy spaces were established in [16]. For other results about flag kernels, we refer the reader to [3, 17, 18]. While the theory of flag kernels are satisfactorily established, it is still open how to develop a theory of multiparameter weightes associated with flag kernels, even in the setting of Heisenberg groups.

The purpose of this paper is to address this question. More precisely, we shall introduce a new class of flag weights $A_p^{\mathcal{F}}$ (associated to a flag $\mathcal{F}$) on a homogeneous group G and provide characterizations of $A_p^{\mathcal{F}}$ via two kinds of maximal operators. We then prove that singular integrals with flag kernels are bounded on $L^p_w(G), 1 < p < \infty$, when $w \in A_p^{\mathcal{F}}(G)$, which extends a recent result of Nagel-Ricci-Stein-Wainger in [13]. As an application, we also get weighted norm inequalities for the multiparameter Marcinkiewicz multipliers introduced in [11] on Heisenberg groups.

To state our main results more precisely, we begin with recalling some basic definitions and notations on homogeneous groups. Let G be a homogeneous nilpotent Lie group with Lie algebra $\mathfrak{g}$. A Lie group G is homogeneous means that there is a one-parameter group of automorphisms $\delta_r : G \to G$ for $r > 0$, with $\delta_1 = Id$. As a manifold, G is an N-dimension real vector space, and we assume that with an appropriate choice of coordinates, $G = \mathbb{R}^N$ and the automorphisms are given by

$$\delta_r[x] = r \cdot x = (r^{d_1} x_1, \ldots, r^{d_N} x_N)$$

with $1 \le d_1 \le d_2 \le \cdots \le d_N$. We identify G with $\mathbb{R}^N$ as above. The bi-invariant Haar measure on G is Lebesgue measure $dy = dy_1 \cdots dy_N$. The convolution of functions $f, g \in L^1(G)$ is given by

$$f * g(x) = \int_G f(xy^{-1})g(y)dy = \int_G f(y)g(y^{-1}x)dy,$$

and the integral converges absolutely for almost all $x \in G$. For more details about homogeneous groups, we refer the reader to [4, 10]

A *standard flag* $\mathcal{F}$ associated to the partition $N = a_1 + \cdots + a_n$ $(a_i > 0)$ is a collection of increasing subspaces

$$(1.1) \quad (0) \subset \mathbb{R}^{a_n} \subset \mathbb{R}^{a_{n-1}} \oplus \mathbb{R}^{a_n} \subset \cdots \subset \mathbb{R}^{a_2} \oplus \cdots \oplus \mathbb{R}^{a_n} \subset \mathbb{R}^{a_1} \oplus \cdots \oplus \mathbb{R}^{a_n} = \mathbb{R}^N.$$

Throughout this paper, we fix the partition and the flag on $G = \mathbb{R}^N$. In what follows, for $x \in \mathbb{R}^N$, we always write $x = (x_1, \ldots, x_n)$ with $x_l = (x_{p_l}, \ldots, x_{q_l}) \in \mathbb{R}^{a_l}$ so that $q_l = p_l + a_l - 1$. Denote by $J_l = \{p_l, \ldots, q_l\}$ the set of subscripts corresponding to the factor $\mathbb{R}^{a_l}$ so that $\{1, \ldots, N\}$ is the disjoint union $J_1 \cup \ldots \cup J_n$. With the family of dilations defined above, the action on the subspace $\mathbb{R}^{a_l}$ is given by

$$r \cdot x_l = (r^{d_{p_l}} x_{p_l}, \ldots, r^{d_{q_l}} x_{q_l}).$$

The homogeneous dimension of $\mathbb{R}^{a_l}$ is $Q_l = d_{p_l} + \cdots + d_{q_l} = \sum_{j \in J_l} d_j$. The function $N_l(x_l) = \sup_{p_l \le s \le q_l} |x_s|^{1/d_s}$ is a homogeneous norm on $\mathbb{R}^{a_l}$ so that $N_l(r \cdot x_l) = rN_l(x_l)$. If $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$, let $\bar{\alpha}_l = (\alpha_{p_l}, \ldots, \alpha_{q_l})$, and set

$$[\bar{\alpha}_l] = \alpha_{p_l} d_{p_l} + \cdots + \alpha_{q_l} d_{q_l} = \sum_{j \in J_l} \alpha_j d_j.$$

Let

$$\begin{aligned} G_k &= \{x = (x_1, \ldots, x_n) \in \mathbb{R}^N : x_1 = \cdots = x_{k-1} = 0\} \\ &= \{(0, \cdots, 0, x_k, \ldots, x_n) \in \mathbb{R}^N : x_j \in \mathbb{R}^{a_j}, k \le j \le n\}, \end{aligned}$$

and let $G_k^\perp$ denote the annihilator of G_k. We can identify G_k with $\mathbb{R}^{a_k} \oplus \cdots \oplus \mathbb{R}^{a_n}$ so that $G_k^\perp = \mathbb{R}^{a_1} \oplus \cdots \oplus \mathbb{R}^{a_{k-1}}$. For $x \in G$, we also write

$$x = (x_\perp^k, x^k) \in G_k^\perp \times G_k.$$

It follows from the formula for group multiplication that G_k is a subgroup of G. We let $m(E)$ denote the Lebesgue measure of a set $E \subset G = G_1$, and $m_k(F)$ denote the Lebesgue measure on G_k of subset $F \subset G_k$. For $s = (s_k, \ldots, s_n)$, let

$$R_s^{(k)} = R_s^{(k)}(0) = \{(x_k, \ldots, x_n) \in G_k : |x_k| \le s_k^{Q_k}, \ldots, |x_n| \le s_n^{Q_n}\}.$$

We say that the size of the rectangle R_s is *acceptable* if $s_k \le s_{k+1} \le \cdots \le s_n$.

Definition 1.1. The maximal function $M_{\mathcal{F}}$ on G is defined by

$$M_{\mathcal{F}}(f)(x) = \sup_{R_s} \frac{1}{|R_s|} \int_{R_s} |f(xy^{-1})| dy,$$

where the supremum is taken over all acceptable rectangles $R_s = R_s^{(1)} \subset G = G_1$.

We now introduce the Muckenhoupt weight $A_p^{\mathcal{F}}$ associated to the flag $\mathcal{F}$ on G. Define the *translated acceptable rectangles* $R_s(x) := x \cdot R_s(0) = \{x \cdot y : y \in B_s(0)\}$. Denote by $\mathcal{R}_{\mathcal{F}}$ the set of of translated acceptable rectangles.

Definition 1.2. Let w be a nonnegative measurable function on G. We say that w is a flag weight in $A_p^{\mathcal{F}}(G)$ if

$$\sup_{R \in \mathcal{R}_{\mathcal{F}}} \Big(\frac{1}{|R|} \int_R w(x) dx \Big) \Big(\frac{1}{|R|} \int_R w(x)^{-1/(p-1)} dx \Big)^{p-1} < \infty \quad \text{for } 1 < p < \infty,$$

$$M_{\mathcal{F}}(w)(x) \le Cw(x), \ a.e. \quad \text{for } p = 1.$$

We would like to point out that in the Euclidean setting when $G = \mathbb{R}^N$, the flag weights defined above are different from the product weights used in [2]. The standard maximal function M_k on the subgroup G_k is defined by

$$M_k(f)(x^k) = \sup_{\rho>0} \frac{1}{m(B(\rho))} \int_{B(\rho)} |f(x^k \cdot (y^k)^{-1})| dy^k,$$

where $B(\rho) = B^{(k)}(\rho)$ is the automorphic one-parameter ball given by

$$B^{(k)}(\rho) = \{x^k = (x_k, \ldots, x_n) \in G_k : |x_k| \le \rho^{Q_k}, |x_{k+1}| \le \rho^{Q_{k+1}}, \ldots, |x_n| \le \rho^{Q_n}\},$$

and $\cdot$ denote the group multiplication on subgroup G_k. Let

$$B^{(k)}_{x^k}(\rho) := \{x^k \cdot y^k : \ y^k \in B^{(k)}(\rho)\}.$$

Define the maximal operator $\widetilde{M}_k$ on G by

$$\widetilde{M}_k(f)(x) \equiv (\delta_{G_k^\perp} \otimes M_k)(f)(x) = M_k(f(x_\perp^k, \cdot))(x^k),$$

where $\delta_{G_k^\perp}$ is the Dirac mass at $(0) \in G_k^\perp$. We then define another type of flag maximal operator by

$$\widetilde{M}_{\mathcal{F}} = \widetilde{M}_n \circ \widetilde{M}_{n-1} \circ \ldots \circ \widetilde{M}_1.$$

For $k = 1, \ldots, n$, the one-parameter weight classes $A_p^{(k)}$, relative to $\widetilde{M}_k$ on G, are defined as follows.

Definition 1.3. Let w be a nonnegative measurable function on G. We say that w is in $A_p^{(k)}(G), 1 < p < \infty$, if

$$\left(\frac{1}{|B^{(k)}|} \int_{B^{(k)}} w(x_\perp^k, x^k) dx^k\right) \left(\frac{1}{|B^{(k)}|} \int_{B^{(k)}} w(x_\perp^k, x^k)^{-1/(p-1)} dx^k\right)^{p-1} < C$$

for all $B^{(k)} \subset G_k$ and almost all $x_\perp^k \in G_k^\perp$. If $\widetilde{M}_k(w)(x) \le Cw(x), a.e.$, then we say $w \in A_1^{(k)}(G)$.

By definition, $A_p^{(k)}(G)$ consists of those weight functions that satisfy *uniform* A_p *property* in flag subvariables of $G_k, k = 1, \ldots, n$.

Our first main result is as follows.

Theorem 1.4. *Let* $1 < p < \infty$ *and* w *be a nonnegative measurable function on* G. *Then the following four conditions are equivalent:*

(1) $w \in A_p^{\mathcal{F}}(G)$;

(2) $w \in A_p^{(1)}(G) \cap A_p^{(2)}(G) \cap \cdots \cap A_p^{(n)}(G)$;

(3) $\widetilde{M}_{\mathcal{F}}$ *is bounded on* $L^p_w(G)$*;*

(4) $M_{\mathcal{F}}$ *is bounded on* $L^p_w(G)$.

Using Theorem 1.4, we get the following weighted Fefferman-Stein vector-valued inequality.

Theorem 1.5. *Let* $1 < p < \infty$ *and* w *be a nonnegative locally integrable function on* G*. Then the following weighted Fefferman-Stein vector-valued inequality holds*

$$\Big\| \Big(\int_{\mathbb{R}^n_+} |M_{\mathcal{F}}(f_t)|^2 \frac{dt}{[t]} \Big)^{1/2} \Big\|_{L^p_w(G)} \le C \Big\| \Big(\int_{\mathbb{R}^n_+} |f_t|^2 \frac{dt}{[t]} \Big)^{1/2} \Big\|_{L^p_w(G)}, \tag{1.2}$$

if and only if $w \in A^{\mathcal{F}}_p(G)$.

The following definition of flag kernels on G was introduced in [13].

Definition 1.6. A flag kernel adapted to the flag $\mathcal{F}$ is a distribution $\mathcal{K} \in \mathcal{S}'(\mathbb{R}^N)$ which satisfies the following differential inequalities (part (a)) and cancellation conditions (part (b))

(a) For test functions supported away from the subspace $x_1 = 0$, the distribution $\mathcal{K}$ is given by integration against a $\mathcal{C}^\infty$-function K. Moreover for every $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}^N$ there is a constant C_α so that if $\alpha_k = (\alpha_{p_k}, \ldots, \alpha_{q_k})$, then for $x_1 \ne 0$,

$$|\partial^\alpha K(x)| \le C_\alpha \prod_{k=1}^n [N_1(x_1) + \cdots + N_k(x_k)]^{-Q_k - [\bar{\alpha}_k]}.$$

(b) Let $\{1, \ldots, n\} = L \cup M$ with $L = \{l_1, \ldots, l_\alpha\}$, $M = \{m_1, \ldots, m_\beta\}$ and $L \cap M = \emptyset$ be any pair of complementary subsets. For any $\psi \in \mathcal{C}^\infty_0(\mathbb{R}^{N_b})$ and any positive real numbers $R_1, \ldots, R_\beta$, put $\psi_R(x_{m_1}, \ldots, x_{m_\beta}) = \psi(R_1 \cdot x_{m_1}, \ldots, R_\beta \cdot x_{m_\beta})$. Define a distribution $\mathcal{K}^\#_{\psi,R} \in \mathcal{S}'(\mathbb{R}^{a_{l_1} + \cdots + a_{l_r}})$ by setting

$$\langle \mathcal{K}^\#_{\psi,R}, \varphi \rangle = \langle \mathcal{K}, \psi_R \otimes \varphi \rangle$$

for any test function $\varphi \in \mathcal{S}(\mathbb{R}^{a_{l_1} + \cdots + a_{l_r}})$. Then the distribution $\mathcal{K}^\#_{\psi,R}$ satisfies the differential inequalities of part (a) for the decomposition $\mathbb{R}^{a_{l_1}} \oplus \cdots \oplus \mathbb{R}^{a_{l_r}}$. Moreover, the corresponding constants that appear in these differential inequalities are independent of the parameters $\{R_1, \ldots, R_s\}$, and depend only on the constants $\{C_\alpha\}$ from part (a) and the semi-norms of ψ.

The constants $\{C_\alpha\}$ in part and the implicit constant in part are called the flag kernel constants for the flag kernel $\mathcal{K}$.

The second main result of this paper is the following

Theorem 1.7. *Let $1 < p < \infty$ and $w \in A_p^{\mathcal{F}}(G)$. If $\mathcal{K}$ a flag kernel on G, then the convolution operator $T(f) = f * \mathcal{K}$ is bounded on $L_w^p(G)$.*

Finally, we give an application of Theorem 1.7 to Marcinkiewicz multipliers on Heisenberg group $\mathbb{H}^n$. Recall that the Heisenberg group $\mathbb{H}^n$ is a two-step homogeneous nilpotent group on $\mathbb{C}^n \times \mathbb{R}$ with the multiplication law

$$(z,t) \cdot (z',t') = (z + z', t + t' + 2\Im(z \cdot \bar{z'})).$$

Let $\mathcal{Z}$ be the center of $\mathbb{H}^n$ define by

$$\mathcal{Z} = \{(0,t) : 0 \in \mathbb{C}^n, t \in \mathbb{R}\}.$$

The flag $\widetilde{\mathcal{F}}$ on $\mathbb{H}^n$ is given by

$$(0) \subset \mathcal{Z} \subset \mathbb{H}^n.$$

In [11], Müller-Ricci-Stein studied a class of multiparameter Marcinkiewicz multipliers on $\mathbb{H}^n$. They showed that such class of Marcinkiewicz multipliers can be characterized by the flag singular integrals on $\mathbb{H}^n$. Applying Theorem 1.7, we then get the following weighted norm inequalities for the multiparameter Marcinkiewicz multipliers studied in [11] on $\mathbb{H}^n$.

Theorem 1.8. *Let $1 < p < \infty$ and $w \in A_p^{\widetilde{\mathcal{F}}}(\mathbb{H}^n)$. Suppose that $m(\xi, \eta)$ is a function on $\mathbb{R}^+ \times \mathbb{R}$ satisfying*

$$|(\xi\partial_\xi)^\alpha(\eta\partial_\eta)^\beta m(\xi,\eta)| \le C_{\alpha,\beta}$$

for all $\alpha, \beta \le N$, with N large enough. Then $m(\mathcal{L}, i\mathcal{T})$ is a bounded operator on $L_w^p(\mathbb{H}^n)$. Here $\mathcal{L}$ is the sub-Laplacian and $\mathcal{T}$ is the central element of the Heisenberg Lie algebra.

2. Proof of Theorem 1.4

We prove Theorem 1.4 by showing $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$.

We show first $(1) \Rightarrow (2)$. Note that all one-parameter balls $B^{(1)}(\rho)$ are acceptable, we thus have $A_p^{\mathcal{F}}(G) \subset A_p^{(1)}(G), 1 < p < \infty$. To see that $A_p^{\mathcal{F}}(G) \subset A_p^{(k)}(G), 1 < k \le n$, we assume that $w \in A_p^{\mathcal{F}}(G)$. Given any translated one-parameter ball $B^{(k)}(x^k, \rho)$ and $r \le \rho$, set

$$\begin{aligned} B_\perp^{(k)}(r) &= B_\perp^{(k)}(x_\perp^k, r) \\ &= \{x_\perp^k \cdot y : (y_1, \ldots, y_{k-1}) \in G_k^\perp : |y_1| \le r^{Q_1}, \cdots, |y_{k-1}| \le r^{Q_{k-1}}\}. \end{aligned}$$

Then $B_\perp^{(k)}(r)$ shrinks to $x_\perp^k \in G_k^\perp$ as r tends to zero. Moreover, $B_\perp^{(k)}(x_\perp^k, r) \times B^{(k)}(x^k, \rho) \in \mathcal{R}_{\mathcal{F}}$. Thus, $w \in A_p^{\mathcal{F}}(G)$ gives

$$\begin{aligned} &\Big(\frac{1}{|B^{(k)}(\rho)|}\int_{B^{(k)}(\rho)}\Big(\frac{1}{|B_\perp^{(k)}(r)|}\int_{B_\perp^{(k)}(r)} w(\bar{y}, y')d\bar{y}\Big)dy'\Big) \\ &\times \Big(\frac{1}{|B^{(k)}(\rho)|}\int_{B^{(k)}(\rho)}\Big(\frac{1}{|B_\perp^{(k)}(r)|}\int_{B_\perp^{(k)}(r)} w(\bar{y}, y')^{-1/(p-1)}d\bar{y}\Big)dy'\Big)^{p-1} < C. \end{aligned}$$

Letting $r \to 0$ and applying Lebesgue's differential theorem, we get

$$\Big(\frac{1}{|B^{(k)}(\rho)|}\int_{B^{(k)}(\rho)} w(x_\perp^k, y')dy'\Big)\Big(\frac{1}{|B^{(k)}(\rho)|}\int_{B^{(k)}(\rho)} w(x_\perp^k, y')^{-1/(p-1)}dy'\Big)^{p-1} < C$$

for all $B^{(k)}(\rho) \subset G_k$ and almost all $x_\perp^k \in G_k^\perp$. This verifies $w \in A_p^{(k)}(G)$ and thus the implication $(1) \Rightarrow (2)$ is proved.

The second implication can be proved as in the Euclidean case (see [15]) while the third follows immediately from the following Lemma (see [13, Lemma 9.3]).

Lemma 2.1. *There is a constant C so that for all $x \in G$,*

$$M_{\mathcal{F}}(f)(x) \le C\widetilde{M}_{\mathcal{F}}(f)(x).$$

To show the last implication $(4) \Rightarrow (1)$, we assume that $M_{\mathcal{F}}$ is bounded on $L^p_w(G)$ for a non-negative locally integrable function w. Apply this to the function $f\chi_R$ supported in an acceptable rectangle R and use that $1/|R|\int_R |f| \lesssim M_{\mathcal{F}}(f\chi_R)(x)$ for all $x \in R$ to obtain

$$\begin{aligned} w(R)\cdot\Big(\frac{1}{|R|}\int_R |f(x)|dx\Big)^p &\le C\int_R [M_{\mathcal{F}}(f\chi_B)(x)]^p w(x)dx \\ &\le C_p\int_R |f(x)|^p w(x)dx, \end{aligned}$$

where $w(R) = \int_R w(x)dx$. It follows that

$$\Big(\frac{1}{|R|}\int_R |f(x)|dx\Big)^p \le \frac{C_p}{w(R)}\int_R |f(x)|^p w(x)dx,$$

for all $R \in \mathcal{R}_{\mathcal{F}}$ and all functions f. Now we take $f = w^{-p'/p}$, which gives $f^p w = w^{-p'/p}$. We thus get that w should satisfy the inequality (1) under additional assumption that $\inf_R w > 0$ for all acceptable rectangles R. If $\inf_R w = 0$ for some acceptable rectangles R, we take $f = (w+\varepsilon)^{-p'/p}$. Repeating the similar argument, we can derive

$$\Big(\frac{1}{|R|}\int_R w(x)dx\Big)\Big(\frac{1}{|R|}\int_R (w(x)+\varepsilon)^{-\frac{p'}{p}}dx\Big)^{p-1} \le C_p,$$

from which we can still get the conclusion (1) via the Lebesgue monotone convergence theorem by letting $\varepsilon \to 0$. This ends the proof of the implication $(4) \Rightarrow (1)$ and hence Theorem 1.4 follows. ∎

3. Proof of Theorem 1.5

To prove the sufficient part of Theorem 1.5, we need the following one-parameter weighted Fefferman-Stein's inequality.

Lemma 3.1. *Let $1 \leq k \leq n$, $1 < p, q < \infty$ and let u be a $A_p(G_k)$ weight (i.e. the classical Muckenhoupt weight on G_k).*

$$\Big\| \Big(\sum_{j \in \mathbb{Z}} |M_k(f_j)|^2 \Big)^{1/2} \Big\|_{L^p_u(G_k)} \leq C \Big\| \Big(\sum_{j \in \mathbb{Z}} |f_j|^2 \Big)^{1/2} \Big\|_{L^p_u(G_k)}. \tag{3.1}$$

The key tool in the proof of the above inequality in Euclidean setting is the the Calderón-Zygmund decomposition. Such decomposition in the current setting was provided in [15]. Based on the Calderón-Zygmund decomposition, the proof of Lemma 3.1 is just a recreation of the Euclidean one in [1]. We omit the details here.

We now assume that $w \in A_p^{\mathcal{F}}(G)$. By Theorem 1.4, $w \in A_p^{(k)}, k = 1, \ldots, n$. Thus for each $x_\perp^k \in G_k^\perp$, $w(x_\perp^k, \cdot)$ is in $A_p(G_k)$ uniformly for $x_\perp^k$. Thus the the weighted Fefferman-Stein'sinequality in Lemma 3.1 hold for $u = w(x_\perp^k, \cdot)$. Lifting to G yields

$$\Big\| \Big(\sum_{j \in \mathbb{Z}} |\widetilde{M}_k(f_j)|^2 \Big)^{1/2} \Big\|_{L^p_w(G)} \leq C \Big\| \Big(\sum_{j \in \mathbb{Z}} |f_j|^2 \Big)^{1/2} \Big\|_{L^p_w(G)}, \qquad 1 \leq k \leq n.$$

By iteration,

$$\Big\| \Big(\sum_{j \in \mathbb{Z}} |\widetilde{M}_{\mathcal{F}}(f_j)|^2 \Big)^{1/2} \Big\|_{L^p_w(G)} \leq C \Big\| \Big(\sum_{j \in \mathbb{Z}} |f_j|^2 \Big)^{1/2} \Big\|_{L^p_w(G)}.$$

This together with Lemma 2.1 yields

$$\Big\| \Big(\sum_{j \in \mathbb{Z}} |M_{\mathcal{F}}(f_j)|^2 \Big)^{1/2} \Big\|_{L^p_w(G)} \leq C \Big\| \Big(\sum_{j \in \mathbb{Z}} |f_j|^2 \Big)^{1/2} \Big\|_{L^p_w(G)}. \tag{3.2}$$

To pass to the continuous one in Theorem 1.5, we assume first that $f_t(x)$ is jointly continuous and has compact support. For each $\epsilon > 0$, apply the conclusion (3.2) to the case where $\{f_j(x)\}$ are enumeration of the $\epsilon^{n/2} F_{\epsilon j_1, \epsilon j_2, \ldots, \epsilon j_n}(x)$, for $(j_1, j_2, \ldots, j_n)$ ranging over $(\mathbb{Z}^+)^n$, and then let $\epsilon \to 0$, obtaining the desired result in this case. For the general f_t, assuming that is finite, find a sequence $f_t^{(k)}(x)$ of continuous functions of compact support, with $f_t^{(k)}(x) \to f_t(x)$ almost everywhere, so that

$$\Big\| \Big(\int_{\mathbb{R}^n_+} |f_t^{(k)}|^2 dt \Big)^{1/2} \Big\|_{L^p_w(G)} \to \Big\| \Big(\int_{\mathbb{R}^n_+} |f_t|^2 dt \Big)^{1/2} \Big\|_{L^p_w(G)}.$$

Applying the previous case and Fatou's lemma, we obtain the desired estimate.

Concerning the necessity of $w \in A_p^{\mathcal{F}}(G)$, take $f_t = f$ for $t \in [1,2]^n$ and $f_t = 0$ otherwise. Then we see that $w \in A_p^{\mathcal{F}}(G)$ is necessary by the scaler-valued result in Theorem 1.4. This concludes the proof of Theorem 1.5. ∎

4. Proof of Theorem 1.7

The proof is similar to the unweighted case in [13] and we provide it for the sake of completeness. We first recall the square functions introduced in [13]. Since each subgroup G_k is a homogeneous group with family of dilations δ_r, there exists a finite-dimensional inner-product space V_k and a pair $\varphi^{(k)}$, $\psi^{(k)}$ of V_k-valued functions, with $\varphi^{(k)} \in \mathcal{C}_0^\infty(G_k)$ supported in the unit ball, and $\psi^{(k)} \in \mathcal{S}(G_k)$ a Schwartz function, so that

$$\int_{G_k} \varphi^{(k)}(x)dx = \int_{G_k} \psi^{(k)}(x)dx = 0$$

and

$$\int_0^\infty \psi_a^{(k)}(xy^{-1}) \cdot \varphi_a^{(k)}(y)\frac{da}{a} = \delta_0. \tag{4.1}$$

Here

$$\varphi_a^{(k)} = a^{-Q_k - Q_{k+1} \cdots - Q_n} \varphi^{(k)}(\delta_{a^{-1}}(x))$$

with a similar definition for $\psi_a^{(k)}(x)$ and $\cdot$ denotes the inner product in V_k. See [4, Theorem 1.61].

The operators $P_a^{(k)}$ and $Q_a^{(k)}$, acting on functions on G_k, are defined by $P_a^{(k)}(f) = f * \varphi_a^{(k)}$ and $Q_a^{(k)}(f) = f * \psi_a^{(k)}$. Note that (4.1) implies that

$$\int_0^\infty P_a^{(k)} \cdot Q_a^{(k)} \frac{da}{a} = Id. \tag{4.2}$$

Next, define the square functions S_k and $S_k^\#$ by setting

$$S_k(f)(x) = \Big(\int_0^\infty |P_a^{(k)}(f)(x)|^2 \frac{da}{a} \Big)^{1/2}, \quad S_k^\#(f)(x) = \Big(\int_0^\infty |Q_a^{(k)}(f)(x)|^2 \frac{da}{a} \Big)^{1/2}.$$

Since $w \in A_p^{\mathcal{F}}(G)$, by Theorem 1.4, $w(x_\perp^k, \cdot)$ is in $A_p(G_k)$, uniformly in $x_\perp^k \in G_k^\perp$. Then the classical weighted Littlewood-Paley theory gives

$$\|f\|_{L^p_{w(x_\perp^k, \cdot)}(G_k)} \sim \|S_k(f)\|_{L^p_{w(x_\perp^k, \cdot)}(G_k)} \sim \|S_k^\#(f)\|_{L^p_{w(x_\perp^k, \cdot)}(G_k)} \tag{4.3}$$

for $1 < p < \infty$. Now, we transfer these inequalities to the whole group G. Let

$$\begin{aligned}
\widetilde{P}_a^{(k)}(f) &= f * (\delta_{x_1 \cdots x_{k-1}} \otimes \varphi_a^{(k)}), \\
\widetilde{Q}_a^{(k)}(f) &= f * (\delta_{x_1 \cdots x_{k-1}} \otimes \psi_a^{(k)}), \\
\widetilde{S}_k(f) &= \Big(\int_0^\infty |\widetilde{P}_a^{(k)}(f)|^2 \frac{da}{a} \Big)^{1/2}, \\
\widetilde{S}_k^\#(f) &= \Big(\int_0^\infty |\widetilde{Q}_a^{(k)}(f)|^2 \frac{da}{a} \Big)^{1/2}.
\end{aligned}$$

Then by (4.3) we have

$$\|f\|_{L^p_w(G)} \sim \|\widetilde{S}_k(f)\|_{L^p_w(G)} \sim \|\widetilde{S}^{\#}_k(f)\|_{L^p_w(G)}.$$

Moreover, these inequalities also hold for Hilbert-valued functions.

For each $t = (t_1, \ldots, t_n) \in (\mathbb{R}^+)^n$, set

$$P_t = \widetilde{P}^{(n)}_{t_n} \cdot \widetilde{P}^{(n-1)}_{t_{n-1}} \cdot \cdots \cdot \widetilde{P}^{(1)}_{t_1}.$$

That is, $P_t(f) = f * \Phi_t$, where $\Phi_t = \widetilde{\varphi}^{(1)}_{t_1} * \widetilde{\varphi}^{(2)}_{t_2} * \cdots * \widetilde{\varphi}^{(n)}_{t_n}$, and $\widetilde{\varphi}^{(k)}_{t_k} = \delta_{x_1 \cdots x_{k-1}} \otimes \varphi^{(k)}_{t_k}$. Similarly, define

$$\begin{aligned} P^*_t &= \widetilde{P}^{(1)}_{t_1} \cdot \widetilde{P}^{(2)}_{t_2} \cdot \cdots \cdot \widetilde{P}^{(n)}_{t_n}, \\ Q_t &= \widetilde{Q}^{(n)}_{t_n} \cdot \widetilde{Q}^{(n-1)}_{t_{n-1}} \cdot \cdots \cdot \widetilde{Q}^{(1)}_{t_1}, \\ Q^*_t &= \widetilde{Q}^{(1)}_{t_1} \cdot \widetilde{Q}^{(2)}_{t_2} \cdot \cdots \cdot \widetilde{Q}^{(n)}_{t_n}. \end{aligned}$$

Note that $Q_t(f) = f * \bar{\psi}_t$, with $\bar{\psi}_t = \widetilde{\psi}^{(1)}_{t_1} * \cdots * \widetilde{\psi}^{(n)}_{t_n}$ and $\bar{\psi}_t$ is also V-valued. Finally, set

$$S(f)(x) = \Big(\int_{(\mathbb{R}^+)^n} |P_t(f)(x)|^2 \frac{t}{[t]} \Big)^{1/2},$$

$$S^{\#}(f)(x) = \Big(\int_{(\mathbb{R}^+)^n} |Q_t(f)(x)|^2 \frac{t}{[t]} \Big)^{1/2},$$

and

$$\mathfrak{S}(f)(x) = \Big(\int_{(\mathbb{R}^+)^n} |(M_{\mathcal{F}} \circ M_{\mathcal{F}} \circ Q_t)(f)(x)|^2 \frac{t}{[t]} \Big)^{1/2},$$

where $[t] = t_1 \cdot t_2 \cdot \cdots \cdot t_n$. Then by (4.2), we get the following reproducing formula

$$\int_{(\mathbb{R}^+)^n} P^*_t Q_t \frac{dt}{[t]} = Id.$$

To prove Theorem 1.7, we need the following

Lemma 4.1. *Let* $1 < p < \infty$ *and* $w \in A^{\mathcal{F}}_p(G)$. *We have*

(a) $\|S(f)\|_{L^p_w(G)} \sim \|S^{\#}(f)\|_{L^p_w(G)} \sim \|f\|_{L^p_w(G)}$;

(b) $\|\mathfrak{S}(f)\|_{L^p_w(G)} \leq C\|f\|_{L^p_w(G)}$.

Proof. We first use induction on n to prove the $\lesssim$ part of (a). The case $n = 1$ follows from (4.3). Assume that the assertion holds for $n = k - 1$. The Hilbert-valued version of inequality (4.3) together with the inductive hypothesis yields

$$\begin{aligned} &\|S(f)\|_{L^p_w(G)} \\ &\lesssim \Big\| \Big(\int_{[0,\infty)^{k-1}} |\widetilde{P}^{(k-1)}_{t_{k-1}} \cdot \cdots \cdot \widetilde{P}^{(1)}_{t_1}(f)|^2 \frac{dt_1 \cdots dt_{k-1}}{t_1 \cdots t_{k-1}} \Big)^{1/2} \Big\|_{L^p_w(G)} \lesssim \|f\|_{L^p_w(G)}. \end{aligned} \tag{4.4}$$

This gives the desired conclusion for $n = k$.

The converse inequality follows by duality argument. Indeed, by Cauchy-Schwarz's inequality,

$$
\begin{aligned}
&\|f\|_{L^p_w} \\
&= \sup_{g\in L^{p'}_{w^{1-p'}}} \Big|\int_G f(x)\overline{g(x)}\,dx\Big| = \sup_{g\in L^{p'}_{w^{1-p'}}} \Big|\int_G \Big(\int_{\mathbb{R}^n_+} P_t^* Q_t(f)(x)\frac{dt}{[t]}\Big)\overline{g(x)}\,dx\Big| \\
&= \sup_{g\in L^{p'}_{w^{1-p'}}} \Big|\int_G \int_{\mathbb{R}^n_+} P_t(\bar g)(x) Q_t(f)(x)\frac{dt}{[t]}\,dx\Big| \\
&\le \sup_{g\in L^{p'}_{w^{1-p'}}} \int_G \Big(\int_{\mathbb{R}^n_+} |P_t(\bar g)(x)|^2\frac{dt}{[t]}\Big)^{1/2}\Big(\int_{\mathbb{R}^n_+} |Q_t(f)(x)|^2\frac{dt}{[t]}\Big)^{1/2}\,dx \\
&\le \sup_{g\in L^{p'}_{w^{1-p'}}} \Big\|\Big(\int_{\mathbb{R}^n_+} |P_t(\bar g)(x)|^2\frac{dt}{[t]}\Big)^{1/2}\Big\|_{L^{p'}_{w^{1-p'}}} \Big\|\Big(\int_{\mathbb{R}^n_+} |Q_t(f)(x)|^2\frac{dt}{[t]}\Big)^{1/2}\Big\|_{L^p_w} \\
&\lesssim \Big\|\Big(\int_{\mathbb{R}^n_+} |Q_t(f)(x)|^2\frac{dt}{[t]}\Big)^{1/2}\Big\|_{L^p_w}.
\end{aligned}
$$

Similarly, we can show that the assertion (a) continues to hold if the operator S is replaced by $S^{\#}$.

To prove the assertion (b), we use the weighted Fefferman-Stein's inequality in Corollary 1.5 and a similar inequality to (4.4) with S replaced by $S^{\#}$ to get

$$
\begin{aligned}
\|\mathfrak{S}(f)\|_{L^p_w(G)} &= \Big\|\Big(\int_{(\mathbb{R}^+)^n} |(M_{\mathcal{F}}\circ M_{\mathcal{F}}\circ Q_t)(f)|^2\frac{t}{[t]}\Big)^{1/2}\Big\|_{L^p_w(G)} \\
&\lesssim \|S^{\#}(f)\|_{L^p_w(G)} \lesssim \|f\|_{L^p_w(G)}.
\end{aligned}
$$

This proves assertion (b) and hence Lemma 4.1 follows. ∎

We also need the following lemma, whose proof can be found in [13].

Lemma 4.2. *Suppose $\mathcal{K}$ is a flag kernel and $T(f) = f * \mathcal{K}$. Then*

$$S[T(f)](x) \lesssim \mathfrak{S}(f)(x).$$

Now, we are ready to give

Proof of Theorem 1.7. Applying Lemmas 4.1 and 4.2 yields

$$\|T(f)\|_{L^p_w(G)} \lesssim \|S[T(f)]\|_{L^p_w(G)} \lesssim \|\mathfrak{S}(f)\|_{L^p_w(G)} \lesssim \|f\|_{L^p_w(G)}.$$

Hence Theorem 1.7 is proved. ∎

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