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DIFFERENTIABILITY PROPERTIES OF ℓ-STABLE VECTOR FUNCTIONS IN INFINITE-DIMENSIONAL NORMED SPACES

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Abstract. The aim of this paper is to continue the study of properties of an ℓ-stable at a point vector function. We show that any ℓ-stable at a point function from arbitrary normed linear space is strictly differentiable at the considered point.

1. Introduction

The class of $C^{1,1}$ fuctions, i.e. the smooth functions with locally Lipschitz derivative, was intensively studied during last 30 years because, among others, these functions appear in several problems of applied mathematics including variational inequalities, the penalty function method and the proximal point method, see e.g. [3, 10, 11, 14, 15, 16, 17, 18, 20, 21, 26, 28].

In [4] it was introduced the notion of an ℓ-stable at a point scalar function and certain unconstrained optimality conditions were extended from $C^{1,1}$ to ℓ-stable functions. A function $f : X \to \mathbb{R}$, where X is a normed linear space, is ℓ-stable at $x \in X$ if there exist a neighborhood $\mathcal{U}$ of x and $K > 0$ such that

$$|f^\ell(y; h) - f^\ell(x; h)| \leq K\|y - x\|, \quad \forall y \in \mathcal{U}, \forall h \in S_X,$$

where S_X denotes the unit sphere of X, i.e. the set $\{z \in X; \|z\| = 1\}$ and

$$f^\ell(y; h) = \liminf_{t\downarrow 0} \frac{f(y + th) - f(y)}{t}.$$

The class of ℓ-stable functions was further studied in [5, 6, 7, 8, 9, 13, 22]. In the paper [4] there was presented an example of an ℓ-stable function which is not in the class $C^{1,1}$. Among others, the notion of an ℓ-stable scalar function was extended to a

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vector function because it seems that the class of ℓ-stable at a point functions is useful in some vector optimization problems. In fact, the functions from $\mathbb{R}^N$ to Y, where Y is a Banach space, were considered [9], but we can easily extend the definition of an ℓ-stable at a point function for the functions from X to Y, where X and Y are general normed linear spaces. In the present paper we will do it.

During the text of the paper the symbol $C \subset Y$ will denote a cone which we assume to be convex, closed and pointed (for definitions see for instance [19] or [27]). Its dual cone is defined by

$$C^* := \{\xi \in Y^*; \langle \xi, x\rangle \geq 0,\ \forall x \in C\}$$

where Y^* stands for the topological dual space of Y. We will suppose through the text that C^* has nonempty interior. The symbol Γ will denote the set $C^* \cap S_{Y^*}$.

Definition 1.1. Let X and Y be normed linear spaces, $f : X \to Y$ be a mapping and $x \in X$. We say that f is ℓ-stable at x provided that there are a neighborhood $\mathcal{U}$ of x and a constant $K > 0$ such that

$$|f^\ell(y; h)(\gamma) - f^\ell(x; h)(\gamma)| \leq K\|y - x\|,$$

for every $y \in \mathcal{U}$, for every $h \in S_X$ and for every $\gamma \in \Gamma$.

The symbol $f^\ell(x; h)(\gamma)$ denotes the lower Dini directional derivative of f at x in the direction $h \in X$ with respect to linear functional $\gamma \in \Gamma$. It is defined by the formula:

$$f^\ell(x; h)(\gamma) := \liminf_{t\downarrow 0} \frac{\langle \gamma, f(x + th) - f(x)\rangle}{t}.$$

Of course, $f^\ell(x; h) = f^\ell(x; h)(1)$ for scalar functions.

The main aim of this paper is to continue the solving of a problem whether or not an ℓ-stable at a point function is strictly differentiable at the considered point.

We say that $f : X \to Y$, where X and Y are normed linear spaces, is strictly differentiable at $x \in X$ if there is $f'(x) \in \mathcal{L}(X, Y)$ (i.e., $f'(x)$ is an element of the set of all continuous linear mappings from X to Y) such that

$$f'(x)h = \lim_{y\to x, t\downarrow 0} \frac{f(y + th) - f(y)}{t}, \quad \forall h \in S_X,$$

and the limit is uniform with respect to $h \in S_X$. It is easy to show that the strict differentiability implies the Fréchet differentiability.

In the paper we will use also the first-order directional derivative of $f : X \to Y$ at $x \in X$ in the direction $h \in X$ with respect to $\gamma \in \Gamma$ defined by

$$f'(x; h)(\gamma) := \lim_{t\downarrow 0} \frac{\langle \gamma, f(x + th) - f(x)\rangle}{t},$$

and upper Dini directional derivative of $f : X \to Y$ at $x \in X$ in the direction $h \in X$ with respect to $\gamma \in \Gamma$ defined by

$$f^u(x;h)(\gamma) := \limsup_{t \downarrow 0} \frac{\langle \gamma, f(x+th) - f(x) \rangle}{t}.$$

Again, $f'(x;h) = f'(x;h)(1)$ and $f^u(x;h) = f^u(x;h)(1)$ for scalar function.

2. Asplund Spaces and RNP

Let us recall that a Banach space X is said to be an Asplund space provided that every continuous convex function defined on a nonempty open convex subset D of X is Fréchet differentiable at each point of some dense G_δ subset of D. More information about Asplund spaces can be found in [24].

Theorem 2.1. [5, Theorem 3]. *Let X be an Asplund space and $f : X \to \mathbb{R}$ be a continuous function near $x \in X$ which is ℓ-stable at x. Then f is strictly differentiable at x.*

Remark 2.1. We note that in [5, Example 2] it was considered a classical Banach space ℓ_1 of real infinite sequences together with ℓ_1 norm $\|\cdot\|_1$ defined for $x = \{x_m\}_{m=1}^{\infty} \in \ell_1$ by the formula

$$f(x) = \|x\|_1 = \sum_{m=1}^{\infty} |x_m|.$$

It was shown in [24] that $\|\cdot\|_1$ is not Fréchet differentiable at any point $x \in \ell_1$ and thus ℓ_1 is not an Asplund space.

The authors of paper [5] asserted that $\|\cdot\|_1$ was ℓ-stable at arbitrary $x = \{x_m\}_{m=1}^{\infty} \in \ell_1$, where $x_m > 0$ for every $m \in \mathbb{N}$. In fact, the previous assertion is false. Indeed, it can be shown that $\|\cdot\|_1$ is Gâteaux differentiable at $x = \{x_m\}_{m=1}^{\infty} \in \ell_1$ iff for every $m \in \mathbb{N}$, $x_m \neq 0$. In this case, for every $h = \{h_m\}_{m=1}^{\infty} \in \ell_1$, we have $\|\cdot\|_1'(x;h) = \sum_{m \in \mathbb{N}} sgn(x_m) h_m$. Considering $x = \{\frac{1}{2^m}\}_{m=1}^{\infty}$, and sequences $\{y^n\}_{n=1}^{\infty} \subset \ell_1$, $\{h^n\}_{n=1}^{\infty} \subset \ell_1$ such that $y^n = \{y^n_m\}_{m=1}^{\infty}$ and $h^n = \{h^n_m\}_{m=1}^{\infty}$ satisfy, for every $n \in \mathbb{N}$, respectively,

$$y^n_m = \begin{cases} \dfrac{1}{2^m} & , \ \text{if} \ \ n \neq m, \\ -\dfrac{1}{2^m} & , \ \text{if} \ \ n = m. \end{cases}$$

and

$$h^n_m = \begin{cases} 0 & , \ \text{if} \ \ n \neq m, \\ 1 & , \ \text{if} \ \ n = m. \end{cases}$$

Then $y^n \to x$ and $h^n \in S_{\ell_1}$ for every $n \in \mathbb{N}$, but

$$|f^\ell(y^n; h^n) - f^\ell(x; h^n)| = 2,$$

and hence $\|\cdot\|_1$ is not ℓ-stable at x.

Thus, we can try to generalize Theorem 2.1 for arbitrary normed linear space instead of Asplund space.

Supposing that B is a subset of a Banach space Y, we recall that a set S is a slice of B if there exist $\varphi \in Y^*$ and $\lambda \in \mathbb{R}$ such that

$$S = B \cap \{y \in Y; \varphi(y) \leq \lambda\}.$$

Now, we recall that a Banach space Y is said to have the RNP if each bounded subset of Y has slices of arbitrarily small diameter. For details see [1, 2, 24]. We note only that a Banach space X is an Asplund space if and only if X^* has the RNP.

Theorem 2.2. [9, Theorem 9]. *Let a Banach space Y have the RNP and let $f : \mathbb{R}^N \to Y$ be ℓ-stable at $x \in \mathbb{R}^N$. Then f is strictly differentiable at x.*

Before we make the joint generalization of Theorems 2.1 and 2.2 it seems to be useful to recall the following fact.

Proposition 2.1. [9, Theorem 7]. *Let Y be a normed linear space and $f : \mathbb{R}^N \to Y$ be an ℓ-stable mapping at $x \in \mathbb{R}^N$. Then f is continuous on a certain neighborhood of the point x.*

On the other hand, the previous result is not true for arbitrary ℓ-stable function. Indeed, it suffices to consider an arbitrary linear, and thus ℓ-stable at a point, functional from an infinite dimensional normed linear space to $\mathbb{R}$ which is not continuous at the considered point.

Finishing this section, we note that Theorem 2.1 was proved with help of the theorem of D. Preiss [25], i.e. any locally Lipschitz real-valued function on an Asplund space is Fréchet differentiable at the points of a dense set, and Theorem 2.2 was proved using theorem of P. Mankiewicz [23] which states that a Lipschitz mapping $f : \mathbb{R}^N \to Y$ is Gâteaux differentiable on a dense set for a Banach space Y having the RNP.

3. Strict Derivative

In this section, we present several auxiliary assertions at first. If $a, b \in X$, X is a normed linear space, (a, b) and $[a, b]$ denote an open and closed interval in X, respectively, i.e.

$$(a, b) = \{z \in X; z = ta + (1 - t)b, t \in (0, 1)\},$$

$$[a, b] = \{z \in X; z = ta + (1-t)b, t \in [0, 1]\}.$$

The following lemma is a straightforward consequence of the Diewert mean value theorem [12].

Lemma 3.1. [5, Lemma 1]. *Let X be a normed linear space, $f : X \to \mathbb{R}$ be a continuous function, and let $a, b \in X$. Then there exist $\xi_1, \xi_2 \in (a, b)$ such that*

$$f^{\ell}(\xi_1; b-a) \leq f(b) - f(a) \leq f^{\ell}(\xi_2; b-a).$$

An easy consequence is the following lemma. We state it without proof.

Lemma 3.2. *Let X and Y be normed linear spaces, $f : X \to Y$ be a continuous function, $\gamma \in Y^*$ and let $a, b \in X$. Then there are points $\xi_1, \xi_2 \in (a, b)$ such that*

$$f^{\ell}(\xi_1; b-a)(\gamma) \leq \langle \gamma, f(b) - f(a)\rangle \leq f^{\ell}(\xi_2; b-a)(\gamma).$$

Observe that using liminf and limsup calculus, we can prove an analogous assertion in terms of upper Dini directional derivative.

The following lemma was proved in [9, Lemma 5] for a Banach space but the proof can be used also for a normed linear space.

Lemma 3.3. *Let Y be a normed linear space and $C \subset Y$ be a cone. Then*

$$L := \inf_{c \in S_Y} \sup_{\gamma \in \Gamma} |\langle \gamma, c\rangle| > 0.$$

The proof of the following proposition repeated the proof of Theorem 8 from [9] which was stated for the case $X = \mathbb{R}^N$ but for the sake of completness we include it here.

Proposition 3.1. *Let X and Y be normed linear spaces and $f : X \to Y$ be a continuous function near $x \in X$. If f is an ℓ-stable function at x, then f is Lipschitz on a certain neighborhood of x.*

Proof.

Step 1. At first we show that

$$\alpha := \sup\{|f^{\ell}(x; h)(\gamma)| : h \in S_X, \gamma \in \Gamma\} < +\infty.$$

Suppose for a contradiction that there are sequences $\{h_n\}_{n=1}^{+\infty} \subset S_X$ and $\{\gamma_n\}_{n=1}^{+\infty} \subset \Gamma$ such that

$$\lim_{n \to +\infty} |f^{\ell}(x; h_n)(\gamma_n)| = +\infty.$$

Without any loss of generality we can assume that either

$$\lim_{n \to +\infty} f^{\ell}(x; h_n)(\gamma_n) = -\infty.$$

or

$$\lim_{n\to+\infty} f^\ell(x;h_n)(\gamma_n) = +\infty.$$

We suppose that the first case occurs (the second case can be treated by an analogous way). Next we will assume that for a certain $\delta > 0$ the condition of ℓ-stability is fulfilled on $B(x;\delta) = \{z \in X; \|z-x\| < \delta\}$ and moreover f is continuous and bounded on $B(x;\delta)$.

Now, if we combine the property of ℓ-stability and Lemma 3.2, for each sufficiently large $n \in \mathbb{N}$ there exists $\xi_n \in (x, x+\delta h_n)$ such that

$$\begin{aligned}
&\langle f(x+\delta h_n), \gamma_n\rangle \le \langle f(x), \gamma_n\rangle + \delta f^\ell(\xi_n;h_n)(\gamma_n)\\
&= \langle f(x), \gamma_n\rangle + \delta(f^\ell(\xi_n;h_n)(\gamma_n) - f^\ell(x;h_n)(\gamma_n) + f^\ell(x;h_n)(\gamma_n))\\
&\le \langle f(x), \gamma_n\rangle + \delta K\|\xi_n - x\| + \delta f^\ell(x;h_n)(\gamma_n).
\end{aligned}$$

Since f is bounded on $B(x,\delta)$ and $\lim_{n\to+\infty} f^\ell(x;h_n)(\gamma_n) = -\infty$, the previous inequality does not hold for infinitely many $n \in \mathbb{N}$, a contradiction.

Step 2. Now we will show that f is Lipschitz on $B(x,\delta)$. We take arbitrary distinct $a, b \in B(x,\delta)$. We can suppose without loss of generality that $f(a) \neq f(b)$. It follows from Lemma 3.3 that for every $c \in S_Y$ there exists $\gamma \in \Gamma$ such that

$$0 < \frac{L}{2} \le |\langle \gamma, c\rangle|.$$

Thus, using the previous argument jointly with Lemma 3.2, and setting

$$c = \frac{f(b) - f(a)}{\|f(b) - f(a)\|},$$

we can find $\gamma \in \Gamma$ and $\xi \in (a,b)$ such that

$$\begin{aligned}
\|f(b) - f(a)\| &\le \frac{2}{L}|\langle \gamma, f(b) - f(a)\rangle| \le \frac{2}{L}|f^\ell(\xi; b-a)(\gamma)|\\
&\le \frac{2}{L}(|f^\ell(x;b-a)(\gamma)| + K\|x-\xi\|\|b-a\|)\\
&\le \frac{2}{L}(\alpha + K\delta)\|b-a\|,
\end{aligned}$$

where $\alpha < +\infty$ by STEP 1. ■

Now, we are able to prove the main result of our paper.

Theorem 3.1. *Let X be a normed linear spaces, Y a Banach space, and $f : X \to Y$ be a continuous function near $x \in X$. If f is an ℓ-stable function at x, then f is strictly differentiable at x.*

Proof. At first we show that for every $h \in S_X$ the following limit

$$\lim_{y \to x, t \downarrow 0} \frac{f(y+th) - f(y)}{t} \tag{1}$$

exists. Indeed, for some $h \in S_X$ we suppose on the contrary that there are $c > 0$ and sequences $\{y_n^1\}_{n=1}^{\infty} \subset X$, $\{y_n^2\}_{n=1}^{\infty} \subset X$, $\{t_n^1\}_{n=1}^{\infty} \subset \mathbb{R}$, $\{t_n^2\}_{n=1}^{\infty} \subset \mathbb{R}$ such that

$$\lim_{n \to +\infty} y_n^1 = \lim_{n \to +\infty} y_n^2 = x \tag{2}$$

$t_n^1 > 0$, $t_n^2 > 0$ for every $n \in \mathbb{N}$, and

$$\lim_{n \to +\infty} t_n^1 = \lim_{n \to +\infty} t_n^2 = 0, \tag{3}$$

satisfying

$$c \le \left\| \frac{f(y_n^1 + t_n^1 h) - f(y_n^1)}{t_n^1} - \frac{f(y_n^2 + t_n^2 h) - f(y_n^2)}{t_n^2} \right\|, \quad \forall n \in \mathbb{N}. \tag{4}$$

Using Lemma 3.3, Lemma 3.2, and ℓ-stability, we can find $L > 0$, $K > 0$, and, for every $n \in \mathbb{N}$, $\gamma_n \in \Gamma$, $\alpha_n^1 \in (0,1)$, $\alpha_n^2 \in (0,1)$ such that

$$\begin{aligned} &\left\| \frac{f(y_n^1 + t_n^1 h) - f(y_n^1)}{t_n^1} - \frac{f(y_n^2 + t_n^2 h) - f(y_n^2)}{t_n^2} \right\| \\ \le\ & \frac{1}{L} \left| \left\langle \gamma_n, \frac{f(y_n^1 + t_n^1 h) - f(y_n^1)}{t_n^1} - \frac{f(y_n^2 + t_n^2 h) - f(y_n^2)}{t_n^2} \right\rangle \right| \\ \le\ & \frac{1}{L} | f^\ell(y_n^1 + \alpha_n^1 t_n^1 h; h)(\gamma_n) - f^\ell(x;h)(\gamma_n) \\ & + f^\ell(x;h)(\gamma_n) - f^\ell(y_n^2 + \alpha_n^2 t_n^2 h)(\gamma_n) | \\ \le\ & \frac{K}{L} (\|y_n^1 - x + \alpha_n^1 t_n^1\| + \|y_n^2 - x + \alpha_n^2 t_n^2\|), \quad \forall n \in \mathbb{N}. \end{aligned} \tag{5}$$

On the base of inequalities (4) and (5), we obtain

$$c \le \frac{K}{L} (\|y_n^1 - x + \alpha_n^1 t_n^1\| + \|y_n^2 - x + \alpha_n^2 t_n^2\|), \quad \forall n \in \mathbb{N},$$

but it is a contradiction with formulas (2) and (3).

Now it is easy to show that the mapping $T : X \to Y$,

$$T(h) = \lim_{y \to x, t \downarrow 0} \frac{f(y+th) - f(y)}{t}, \quad h \in X,$$

is linear. We note that the mapping T is continuous because f is Lipschitz near x due to Proposition 3.1.

Finishing our proof, it suffices to show that the limit (1) is uniform for $h \in S_X$. Lemmas 3.2, 3.3, and ℓ-stability imply that for every y sufficiently close to x, for every $t > 0$ sufficiently small, and for every $h \in S_X$, there are $\gamma \in \Gamma$ and $\xi \in (y, y + th)$ such that it holds

$$\begin{aligned} &\left\| \frac{f(y+th) - f(y)}{t} - T(h) \right\| \\ &\leq \frac{1}{L} \left| \left\langle \gamma, \frac{f(y+th) - f(y)}{t} - T(h) \right\rangle \right| \\ &= \frac{1}{L} \left| \left\langle \gamma, \frac{f(y+th) - f(y)}{t} \right\rangle - f^{\ell}(x;h)(\gamma) \right| \\ &\leq \frac{1}{L} |f^{\ell}(\xi;h)(\gamma) - f^{\ell}(x;h)(\gamma)| \leq \frac{K}{L} \|\xi - x\|. \end{aligned}$$

Summarizing the previous considerations, the mapping T is a strict derivative of f at x, i.e. $T = f'(x)$. ■

We can use Theorem 3.1 for a characterization of ℓ-stability at a point by means of u-stability at a point. For a mapping $f : X \to Y$, where X and Y are normed linear spaces, we say that it is an u-stable at $x \in X$ if there are a neighborhood $\mathcal{U}$ of x and a constant $K > 0$ such that

$$|f^u(y;h)(\xi) - f^u(x;h)(\xi)| \leq K\|y - x\|,$$

for every $y \in \mathcal{U}$, for every $h \in S_X$ and for every $\xi \in \Gamma$.

Theorem 3.2. *Let X and Y be normed linear spaces and let $f : X \to Y$ be a continuous function near $x \in X$. Then f is ℓ-stable at x if and only if f is u-stable at x.*

Proof. We suppose that f is ℓ-stable at x. In order to prove that f is u-stable at x, we will assume on the contrary that there are sequences $\{z_n\}_{n=1}^{\infty} \subset X$, $\{h_n\}_{n=1}^{\infty} \subset S_X$ and $\{\gamma_n\}_{n=1}^{\infty} \subset \Gamma$ such that for each $n \in \mathbb{N}$, $z_n \neq x$, $z_n \to x$ as $n \to +\infty$, and

$$|f^u(z_n;h_n)(\gamma_n) - f'(x;h_n)(\gamma_n)| \geq n\|z_n - x\|, \quad \forall n \in \mathbb{N}. \tag{6}$$

In the previous formula we notice that for every $n \in \mathbb{N}$ we have

$$f^u(x;h_n)(\gamma_n) = f^{\ell}(x;h_n)(\gamma_n) = f'(x;h_n)(\gamma_n)$$

by Theorem 3.1.

Note that then it holds for almost any $n \in \mathbb{N}$:

$$f^u(z_n;h_n)(\gamma_n) - f'(x;h_n)(\gamma_n) \geq 0.$$

Indeed, otherwise we would have for infinitely many $n \in \mathbb{N}$:

$$f^{\ell}(z_n; h_n)(\gamma_n) - f'(x; h_n)(\gamma_n) \leq f^{u}(z_n; h_n)(\gamma_n) - f'(x; h_n)(\gamma_n) < 0.$$

From this it follows due to formula (6) for infinitely many $n \in \mathbb{N}$ that

$$n\|z_n - x\| \leq |f^{u}(z_n; h_n)(\gamma_n) - f'(x; h_n)(\gamma_n)| \leq |f^{\ell}(z_n; h_n)(\gamma_n) - f'(x; h_n)(\gamma_n)|,$$

and this contradicts the ℓ-stability of f at x. Next (6) implies for almost any $n \in \mathbb{N}$:

$$\begin{aligned} n\|z_n - x\| &\leq f^{u}(z_n; h_n)(\gamma_n) - f'(x; h_n)(\gamma_n) \\ &= \inf_{\delta>0}\big(\sup_{t\in(0,\delta)} \frac{\langle \gamma_n, f(z_n + th_n) - f(z_n)\rangle}{t}\big) - f'(x; h_n)(\gamma_n). \end{aligned}$$

Thus for almost any $n \in \mathbb{N}$ there exists $t_n > 0$ such that $t_n \leq \frac{\|z_n - x\|}{2}$ and

$$(n-1)\|z_n - x\| < \frac{\langle \gamma_n, f(z_n + t_n h_n) - f(z_n)\rangle}{t_n} - f'(x; h_n)(\gamma_n).$$

Using Lemma 3.2 and ℓ-stability, for almost any $n \in \mathbb{N}$ we can find $\xi_n \in (z_n, z_n + t_n h_n)$ such that

$$\begin{aligned} (n-1)\|z_n - x\| &< f^{\ell}(\xi_n; h_n)(\gamma_n) - f'(x; h_n)(\gamma_n) \\ &\leq K\left(\|\xi_n - z_n\| + \|z_n - x\|\right) \\ &\leq Kt_n + K\left\|z_n - x\right\| \\ &\leq K\frac{\|z_n - x\|}{2} + K\left\|z_n - x\right\| \\ &= \frac{3K}{2}\left\|z_n - x\right\|, \end{aligned}$$

but it is a contradiction.

The reverse implication follows from what we have already proved and from easily verifiable fact that

$$f^{u}(x; h)(\gamma) = -(-f)^{\ell}(x; h)(\gamma). \qquad \blacksquare$$

We note that the previous result generalizes Corollary 1 in [5], where $Y = \mathbb{R}$, and also Theorem 2.6 in [13] stated for the case that X and Y are finite dimensional.

Remark 3.1. Strict differentiability of ℓ-stable functions plays an important role in some finite-dimensional problems of vector optimization. Thus, we hope that the results obtained in this paper will be useful in some infinite-dimensional problems of vector optimization.

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