

GAP FUNCTIONS AND GLOBAL ERROR BOUNDS FOR SET-VALUED MIXED VARIATIONAL INEQUALITIES

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Abstract. In this paper, we introduce some gap functions for set-valued mixed variational inequalities under suitable conditions. We further use these gap functions to study global error bounds for the solutions of set-valued mixed variational inequalities in Hilbert spaces. The results presented in this paper generalize and improve some corresponding known results in literatures.

1. INTRODUCTION

The concept of a gap function was introduced for the study of a convex optimization problem and subsequently applied to variational inequalities. As is well known, gap functions play a crucial role in transforming a variational inequality into an optimization problem [6-8, 13, 15-20, 22, 24, 25, 28]. Thus, powerful optimization solution methods and algorithms can be applied for finding solutions of variational inequalities. On the other hand, gap functions have turned out to be very useful in deriving the error bounds, which provide a measure of the distance between solution set and an arbitrary point. Error bounds have played an important role not only in sensitivity analysis but also in convergence analysis of iterative algorithms for solving variational inequalities. It is therefore of interest to investigate error bounds for gap functions associated with various variational inequalities (see [6, 7, 17, 19, 22, 23, 28]).

Throughout this paper, let \mathbb{H} be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let $\Phi : \mathbb{H} \rightarrow (-\infty, +\infty]$ be a lower semicontinuous, proper and convex function and $F : \mathbb{H} \rightrightarrows \mathbb{H}$ be an upper semicontinuous set-valued mapping

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with nonempty compact convex values. We consider the following set-valued mixed variational inequality, denoted by $\text{SMVI}(F, \Phi)$, which consists in finding $x^* \in \text{dom } \Phi$ such that

$$(1.1) \quad \exists u^* \in F(x^*) : \langle u^*, y - x^* \rangle + \Phi(y) - \Phi(x^*) \geq 0, \quad \forall y \in \text{dom } \Phi,$$

It is well known that $\text{SMVI}(F, \Phi)$ is encountered in many applications, in particular, in mechanical problems and equilibrium problems (see [3, 4, 9, 10, 26, 27, 30]). Another problem closely related to $\text{SMVI}(F, \Phi)$ is the so-called weak set-valued mixed variational inequality, denoted by $\text{WSMVI}(F, \Phi)$, which consists in finding $x^* \in \text{dom } \Phi$ such that

$$(1.2) \quad \forall y \in \text{dom } \Phi, \exists u^* \in F(x^*) : \langle u^*, y - x^* \rangle + \Phi(y) - \Phi(x^*) \geq 0.$$

For the sake of convenience, the solution sets of $\text{SMVI}(F, \Phi)$ and $\text{WSMVI}(F, \Phi)$ are denoted by $\text{SOL}(F, \Phi)$ and $\text{SOL}_w(F, \Phi)$, respectively. It is easy to see that $\text{SOL}(F, \Phi) \subseteq \text{SOL}_w(F, \Phi)$.

Let K be a nonempty closed convex subset of \mathbb{H} . If $\Phi(\cdot)$ is the indicator function $\delta_K(\cdot)$ over the subset K , i.e., $\delta_K(x) = 0$, if $x \in K$ and $\delta_K(x) = +\infty$, if $x \notin K$, then $\text{SMVI}(F, \Phi)$ reduces to set-valued variational inequality (for short $\text{SVI}(F, K)$): Find $x^* \in K$ such that

$$(1.3) \quad \exists u^* \in F(x^*) : \langle u^*, y - x^* \rangle \geq 0, \quad \forall y \in K,$$

which has been investigated by Fan and Wang [7], Daniilidis and Hadjisavvas [5].

If F is single-valued, then both $\text{SMVI}(F, \Phi)$ and $\text{WSMVI}(F, \Phi)$ reduce to the mixed variational inequality (for short $\text{MVI}(F, \Phi)$): Find $x^* \in \text{dom } \Phi$ such that

$$(1.4) \quad \langle F(x^*), y - x^* \rangle + \Phi(y) - \Phi(x^*) \geq 0, \quad \forall y \in \text{dom } \Phi,$$

which has been considered by Solodov [22], Han and Reddy [10] and He [11].

If $\Phi(\cdot)$ is the indicator function $\delta_K(\cdot)$ over the subset K and F is single-valued, then both $\text{SMVI}(F, \Phi)$ and $\text{WSMVI}(F, \Phi)$ reduce to Stampacchia variational inequality (for short, $\text{VI}(F, K)$): Find $x^* \in K$ such that

$$(1.5) \quad \langle F(x^*), y - x^* \rangle \geq 0, \quad \forall y \in K.$$

For $\text{VI}(F, K)$, constrained differentiable optimization formulations have been proposed [8, 15, 25] and unconstrained differentiable optimization formulations have been studied [28]. Very recently, followed the ideas due to Yamashita and Fukushima [28], Fan and Wang [7] constructed new gap functions for $\text{SVI}(F, K)$ through the Moreau-Yosida regularization of some gap functions. The proposed gap functions constitute unconstrained optimization problems equivalent to problem (1.3) under suitable assumptions. Moreover, they derived global error bounds for the solution of problem

(1.3) by using the proposed gap functions. Li and Mastroeni [17] introduced several kinds of strong and weak scalar variational inequalities for studying strong and weak vector variational inequalities with set-valued mappings and suggested their gap functions and obtained the error bounds for gap functions. On the other hand, Huang et al. [13] introduced and studied a gap function for a system of vector equilibrium problems and proved some existence results of solutions for the problem. Some related work, we refer readers to [16, 12].

Inspired and motivated by the research works above, in this paper, we present some gap functions for problem (1.1) and give some error bounds based on them. The gap functions presented in this paper have the following desirable properties:

- (i) They are finite valued everywhere. Thus, problem (1.1) is equivalent to unconstrained optimization problems.
- (ii) They are differentiable even if without the differentiability of F and Φ (see Theorems 3.2 and 3.3).
- (iii) They provide global error bounds for problem (1.1) without the Lipschitz continuity of F (see Theorems 4.1 and 4.2).

The results presented in this paper generalize the corresponding known results for problem (1.3) in [28] from set-valued variational inequality to set-valued mixed variational inequality and from finite dimensional spaces to infinite dimensional Hilbert spaces.

The rest of paper is organized as follows. In the next section, we give some notations used in this paper and present some preliminary results. In particular, we provide a gap function induced by natural residual for problem (1.1). In the first subsection of Section 3, we introduce two regularized gap functions for problem (1.1), and denote them by $f_\alpha(\cdot)$ and $h_\beta(\cdot)$, respectively. Based on Moreau-Yosida regularization of $f_\alpha(\cdot)$ and $h_\beta(\cdot)$, in the latter subsection of Section 3, we give two desirable gap functions for problem (1.1) and study their differentiable properties. In Section 4, we present error bounds based on the gap functions mentioned above for problem (1.1).

2. PRELIMINARIES

Definition 2.1. The mapping F is said to be

- (i) strongly monotone iff, there is $\beta > 0$ such that for all $(x, x^*), (y, y^*)$ in the graph F ,

$$\langle y^* - x^*, y - x \rangle \geq \beta \|y - x\|^2;$$

- (ii) monotone iff, for all $(x, x^*), (y, y^*)$ in the graph F ,

$$\langle y^* - x^*, y - x \rangle \geq 0;$$

- (iii) pseudomonotone iff, for all $(x, x^*), (y, y^*)$ in the graph F ,

$$\langle x^*, y - x \rangle \geq 0 \Rightarrow \langle y^*, y - x \rangle \geq 0;$$

- (iv) strongly pseudomonotone iff, there is $\beta > 0$ such that for all $(x, x^*), (y, y^*)$ in the graph F ,

$$\langle x^*, y - x \rangle \geq 0 \Rightarrow \langle y^*, y - x \rangle \geq \beta \|y - x\|^2;$$

- (v) Φ -pseudomonotone iff, for all $(x, x^*), (y, y^*)$ in the graph F ,

$$\langle x^*, y - x \rangle + \Phi(y) - \Phi(x) \geq 0 \Rightarrow \langle y^*, y - x \rangle + \Phi(y) - \Phi(x) \geq 0;$$

- (vi) Φ -strongly pseudomonotone iff, there is $\beta > 0$ such that for all $(x, x^*), (y, y^*)$ in the graph F ,

$$\langle x^*, y - x \rangle + \Phi(y) - \Phi(x) \geq 0 \Rightarrow \langle y^*, y - x \rangle + \Phi(y) - \Phi(x) \geq \beta \|y - x\|^2;$$

- (vii) Φ -strongly pseudomonotone with respect to \bar{x} with modulus $\beta > 0$ iff, for any $y \in \text{dom } \Phi$ and $y^* \in F(y)$, we have

$$\langle y^*, y - \bar{x} \rangle + \Phi(y) - \Phi(\bar{x}) \geq \beta \|y - \bar{x}\|^2;$$

- (viii) Lipschitz continuous on a subset B of $\text{dom } \Phi$ iff, there exists $L > 0$ such that

$$H(T(x), T(y)) \leq L \|x - y\|, \quad \forall x, y \in B,$$

where $H(\cdot, \cdot)$ is the Hausdorff metric on a nonempty bounded closed subset of H , i.e.,

$$H(T(x), T(y)) = \max\left\{ \sup_{r \in T(x)} \inf_{s \in T(y)} \|r - s\|, \sup_{s \in T(y)} \inf_{r \in T(x)} \|r - s\| \right\}, \quad \forall x, y \in B.$$

Remark 2.1. (i) We illustrate below the relationships between monotonicity and some generalized monotonicity:

strong pseudomonotonicity \Leftarrow strong monotonicity $\Rightarrow \Phi$ - strong pseudomonotonicity



pseudomonotonicity \Leftarrow monotonicity $\Rightarrow \Phi$ - pseudomonotonicity.

- (ii) It is easily seen that if \bar{x} is a solution of problem (1.1) and F is Φ -strongly pseudomonotone with modulus $\beta > 0$, then F is Φ -strongly pseudomonotone with respect to \bar{x} with modulus $\beta > 0$.
- (iii) If $\Phi \equiv \text{constant}$, then a Φ -pseudomonotone, Φ -strongly pseudomonotone mapping reduces to a pseudomonotone, strongly pseudomonotone mapping, respectively.
- (iv) We would like to point out the Φ -pseudomonotone mapping was used to study the F -complementarity problems in Banach spaces by Yin et al. [29], the stability for Minty mixed variational inequality in Banach spaces by Zhong and Huang [30], and to construct the algorithm for solving the mixed variational inequalities in finite dimensional spaces by He [11], respectively. The relationship between pseudomonotonicity and Φ -pseudomonotonicity was discussed by Zhong and Huang [30].

Definition 2.2. For a convex function $\Phi : \mathbb{H} \rightarrow (-\infty, +\infty]$, $\text{dom } \Phi = \{x \in \mathbb{H} : \Phi(x) < +\infty\}$ denotes its effective domain. For any given $x \in \text{dom } \Phi$

$$\partial\Phi(x) = \{p \in \mathbb{H} : \Phi(y) - \Phi(x) \geq \langle p, y - x \rangle, \forall y \in \mathbb{H}\}$$

denotes the subdifferential of Φ at x and a point $p \in \partial\Phi(x)$ is called a subgradient of Φ at x .

Definition 2.3. A function $M : \mathbb{H} \rightarrow (-\infty, +\infty]$ is called a gap function for the set-valued mixed variational inequality problem (1.1) if and only if

- (i) $M(x) \geq 0, \quad \forall x \in D \supset \text{dom } \Phi;$
- (ii) $M(x) = 0$ if and only if $x \in \text{dom } \Phi$ solves the problem (1.1),

where the set D is usually either the whole space or the set $\text{dom } \Phi$ itself.

One interesting application of gap functions is in deriving the so-called error bounds, i.e., upper estimates on the distance to the solution set $S(F, \Phi)$ of problem (1.1):

$$\text{dist}(x, S(F, \Phi)) \leq \gamma M(x)^\lambda, \quad \forall x \in D,$$

where $\gamma, \lambda > 0$ are independent of x .

Definition 2.4. The set-valued mapping $F : \mathbb{H} \rightrightarrows \mathbb{H}$ is said to be upper semicontinuous, if for each $x \in \mathbb{H}$ and each neighborhood $V \subset \mathbb{H}$ of $F(x)$, there exists a neighborhood U of x such that $F(z) \subset V$ for each $z \in U$.

Lemma 2.1. ([21] Theorem 4.2). *Let M be compact, N any space, f a function on $M \times N$ that is concave-convexlike. If $f(x, y)$ is upper semicontinuous in x for each y , then*

$$\sup_{x \in M} \inf_{y \in N} f(x, y) = \inf_{y \in N} \sup_{x \in M} f(x, y).$$

Lemma 2.2. *If $F : \mathbb{H} \rightrightarrows \mathbb{H}$ is a set-valued mapping with nonempty compact convex values, then the solution set of (1.1) coincides with the one of (1.2), i.e., $SOL(F, \Phi) = SOL_w(F, \Phi)$.*

Proof. Let

$$f(u, y) = \langle u, y - x \rangle + \Phi(y) - \Phi(x), \quad \forall (u, y) \in F(x) \times \text{dom } \Phi.$$

Since Φ is convex, it is easy to see that f is concave-convexlike on $F(x) \times \text{dom } \Phi$. Moreover, $F(x)$ is compact and $f(u, y)$ is continuous in u for each y . The conclusion is a direct application of Lemma 2.1. This completes the proof.

Lemma 2.3. ([1] Theorem 1.4.16). *Let X, Y be metric spaces, a set-valued mapping $F : X \rightrightarrows Y$ and a function $f : \text{Graph}(F) \rightarrow \mathbb{R}$ be given. If f and F are upper semicontinuous and if the values of F are compact, then the function $g : X \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by*

$$g(x) = \sup_{y \in F(x)} f(x, y)$$

is upper semicontinuous.

Recall that the proximal map, $P_\alpha^\Phi : \mathbb{H} \rightarrow \text{dom } \Phi$, is given by

$$P_\alpha^\Phi(z) := \arg \min_{y \in \mathbb{H}} \left\{ \Phi(y) + \frac{1}{2\alpha} \|y - z\|^2 \right\}, \quad z \in \mathbb{H}, \alpha > 0.$$

Note that the objective function above is proper strongly convex. Since $\text{dom } \Phi$ is closed, $P_\alpha^\Phi(\cdot)$ is well defined and single-valued. It is not hard to see that $P_\alpha^\Phi(z) = (I + \alpha \partial \Phi)^{-1}(z)$ for any $z \in \mathbb{H}$. Define

$$R_\alpha(x, u) := x - P_\alpha^\Phi(x - \alpha u), \quad x \in \mathbb{H}, u \in F(x), \alpha > 0.$$

and

$$r_\alpha(x) := \inf_{u \in F(x)} \|R_\alpha(x, u)\|, \quad x \in \mathbb{H}, \alpha > 0.$$

It is easy to have the following result.

Proposition 2.1. *Let $\alpha > 0$ be arbitrary. Then the following statements are equivalent:*

- (i) *An element $x \in \mathbb{H}$ solves (1.1);*
- (ii) *There is some $u \in F(x)$ such that $R_\alpha(x, u) = 0$;*
- (iii) *$r_\alpha(x) = 0$.*

Proposition 2.2. *Let $\alpha > 0$ be arbitrary. Then*

- (i) *$r_\alpha(\cdot)$ is a gap function for (1.1);*
- (ii) *There is some $u \in F(x)$ such that $r_\alpha(x) = \|R_\alpha(x, u)\|$;*
- (iii) *$r_\alpha(\cdot)$ is lower semicontinuous.*

Proof.

- (i) Observe that nonnegativity of $r_\alpha(\cdot)$ and Proposition 2.1, it is easily seen that $r_\alpha(\cdot)$ is a gap function for (1.1).

- (ii) Since $F(x)$ is compact and $\|R_\alpha(x, u)\|$ is continuous in u , there is some $u \in F(x)$ such that $r_\alpha(x) = \|R_\alpha(x, u)\|$.
- (iii) Since $\|R_\alpha(x, u)\|$ is continuous and F is upper semicontinuous with nonempty compact convex values, from Lemma 2.3, we have

$$r_\alpha(x) = \inf_{u \in F(x)} \|R_\alpha(x, u)\| = - \sup_{u \in F(x)} (-\|R_\alpha(x, u)\|)$$

is lower semicontinuous. This completes the proof.

3. GAP FUNCTIONS

3.1. Regularized-gap functions

For any $\alpha > 0$, we define $g_\alpha : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R}$ by

$$(3.1) \quad g_\alpha(x; u) = \sup_{y \in \text{dom } \Phi} \{ \langle u, x - y \rangle + \Phi(x) - \Phi(y) - \frac{1}{2\alpha} \|x - y\|^2 \}, \quad \forall \alpha > 0,$$

and $f_\alpha : \mathbb{H} \rightarrow \mathbb{R}$ by

$$(3.2) \quad f_\alpha(x) = \inf_{u \in F(x)} g_\alpha(x; u).$$

Lemma 3.1. For any $\alpha > 0$,

$$(3.3) \quad g_\alpha(x, u) = \langle u, x - P_\alpha^\Phi(x - \alpha u) \rangle + \Phi(x) - \Phi(P_\alpha^\Phi(x - \alpha u)) - \frac{1}{2\alpha} \|x - P_\alpha^\Phi(x - \alpha u)\|^2, \quad \forall x \in \mathbb{H}.$$

Proof. If $x \notin \text{dom } \Phi$ then formula (3.3) is correct, because $\Phi(x) = +\infty$ while the other terms are all finite (recall that $P_\alpha^\Phi(z) \in \text{dom } \Phi$ for all $z \in \mathbb{H}$).

Consider now any $x \in \text{dom } \Phi$. Denote by $j(y)$ the function being maximized in (3.1). Let z be the (unique, by concavity of $j(y)$) element at which the maximum is realized in (3.1), equivalently, z is the argument at which the minimum of $-j(y)$ is obtained when $y \in \text{dom } \Phi$. Then z is uniquely characterized by the optimality condition

$$0 \in \partial(-j(z)) = u + \partial\Phi(z) + \frac{1}{\alpha}(z - x) = \partial\Phi(z) + \frac{1}{\alpha}[z - (x - \alpha u)].$$

It is to say that $z = \arg \min_{y \in \mathbb{H}} \{ \Phi(y) + \frac{1}{2\alpha} \|y - (x - \alpha u)\|^2 \} = P_\alpha^\Phi(x - \alpha u)$, where the second equation follows from the definition of proximal mapping $P_\alpha^\Phi(\cdot)$. This completes the proof.

Remark 3.1. If F is a single-valued mapping, then Lemma 3.1 coincides with Lemma 3 of [22] in the case $g = I$.

Lemma 3.2. *The following conclusions hold:*

- (i) For any $\alpha > 0$, $f_\alpha(\cdot)$ is nonnegative on $\text{dom } \Phi$;
- (ii) For any $x \in \mathbb{H}$, there exists some $u \in F(x)$ such that $f_\alpha(x) = g_\alpha(x; u)$;
- (iii) For any $\alpha > 0$, $f_\alpha(\cdot)$ is lower semicontinuous.

Proof.

- (i) Denote by $j(y)$ the function being maximized in (3.1), with $x \in \mathbb{H}$ and $u \in F(x)$ fixed. If $x \in \text{dom } \Phi$ then $j(x) = 0$ and so

$$0 = j(x) \leq \sup_{y \in \text{dom } \Phi} j(y) = g_\alpha(x; u).$$

If $x \notin \text{dom } \Phi$, then it holds that $j(y) = +\infty$ and so

$$+\infty = \sup_{y \in \text{dom } \Phi} j(y) = g_\alpha(x; u).$$

Thus, $f_\alpha(x) = \inf_{u \in F(x)} g_\alpha(x; u) \geq 0$.

- (ii) Since $F(x)$ is compact and $g_\alpha(x; u)$ is continuous in u , there is some $u \in F(x)$ such that $f_\alpha(x) = \inf_{u \in F(x)} g_\alpha(x; u)$.
- (iii) Let $\tilde{g}_\alpha((x, u), y) = \langle u, x - y \rangle + \Phi(x) - \Phi(y) - \frac{1}{2\alpha} \|x - y\|^2$. Since Φ is lower semicontinuous, it is easy to see that \tilde{g}_α is also lower semicontinuous in the argument (x, u) for each $y \in \text{dom } \Phi$. Therefore, $g_\alpha(x; u) = \sup_{y \in \text{dom } \Phi} \tilde{g}_\alpha((x, u), y)$ is lower semicontinuous, i.e., $-g_\alpha(x; u)$ is upper semicontinuous. Combining with the fact that F is upper semicontinuous with compact convex values, from Lemma 2.3, we obtain that the function $f_\alpha(\cdot)$ defined by

$$f_\alpha(x) = \inf_{u \in F(x)} g_\alpha(x; u) = - \sup_{u \in F(x)} [-g_\alpha(x; u)]$$

is lower semicontinuous. This completes the proof.

Remark 3.2. If $\mathbb{H} = \mathbb{R}$ and $\Phi(\cdot) = \delta_K(\cdot)$, where $\delta_K(\cdot)$ denotes the indicator function over the closed and convex set K , and we are needed to neglect the slight difference of coefficient α , then Lemma 3.2 reduces to Lemma 3.1 of [7].

Lemma 3.3. *If $\alpha > 0$, then it holds that*

$$(3.4) \quad f_\alpha(x) \geq \frac{1}{2\alpha} r_\alpha^2(x), \quad x \in \mathbb{H}.$$

In particular, $f_\alpha(x) = 0$ if and only if x is a solution of (1.1).

Proof. Fix any fixed $x \in \mathbb{H}$, and $\alpha > 0$. Observe that

$$x - \alpha u \in (I + \alpha \partial \Phi)(I + \alpha \partial \Phi)^{-1}(x - \alpha u) = (I + \alpha \partial \Phi)(P_\alpha^\Phi(x - \alpha u)),$$

which is equivalent to

$$-u + \frac{1}{\alpha}[x - P_\alpha^\Phi(x - \alpha u)] \in \partial \Phi(P_\alpha^\Phi(x - \alpha u)).$$

It follows from the definition of subdifferential that

$$(3.5) \quad \begin{aligned} & \langle u - \frac{1}{\alpha}[x - P_\alpha^\Phi(x - \alpha u)], y - P_\alpha^\Phi(x - \alpha u) \rangle \\ & + \Phi(y) - \Phi(P_\alpha^\Phi(x - \alpha u)) \geq 0, \quad \forall y \in \text{dom } \Phi. \end{aligned}$$

Taking $y = x$ in the inequality above, we have

$$\langle u - \frac{1}{\alpha}[x - P_\alpha^\Phi(x - \alpha u)], x - P_\alpha^\Phi(x - \alpha u) \rangle + \Phi(x) - \Phi(P_\alpha^\Phi(x - \alpha u)) \geq 0,$$

i.e.,

$$\Phi(x) - \Phi(P_\alpha^\Phi(x - \alpha u)) + \langle u, R_\alpha(x, u) \rangle \geq \frac{1}{\alpha} \|R_\alpha(x, u)\|^2.$$

Combining with (3.3), we obtain

$$(3.6) \quad \begin{aligned} g_\alpha(x, u) &= \langle u, R_\alpha(x, u) \rangle + \Phi(x) - \Phi(P_\alpha^\Phi(x - \alpha u)) - \frac{1}{2\alpha} \|R_\alpha(x, u)\|^2 \\ &\geq \frac{1}{\alpha} \|R_\alpha(x, u)\|^2 - \frac{1}{2\alpha} \|R_\alpha(x, u)\|^2 = \frac{1}{2\alpha} \|R_\alpha(x, u)\|^2. \end{aligned}$$

It follows from the property of infimum that

$$\inf_{u \in F(x)} g_\alpha(x, u) \geq \frac{1}{2\alpha} \left(\inf_{u \in F(x)} \|R_\alpha(x, u)\| \right)^2$$

and so $f_\alpha(x) \geq \frac{1}{2\alpha} r_\alpha^2(x)$.

To obtain the last assertion, from (3.4) and the nonnegativity of $r_\alpha(\cdot)$, we know that $f_\alpha(x) = 0$ if and only if $r_\alpha(x) = 0$. Hence, by Proposition 2.1, we deduce the conclusion immediately. This completes the proof.

Remark 3.3. (i) If F is single-valued, then Lemma 3.3 coincides with Theorem 4 of [22] in the case $g = I$;

(ii) If $\mathbb{H} = \mathbb{R}$ and $\Phi(\cdot) = \delta_K(\cdot)$, then the last conclusion of Lemma 3.3 reduces to Lemma 3.3 of [7];

(iii) If F is single-valued, $\mathbb{H} = \mathbb{R}$ and $\Phi(\cdot) = \delta_K(\cdot)$, then the last conclusion of Lemma 3.3 reduces to Lemma 2.1 of [28].

We define another function $h_\beta(\cdot) : \mathbb{H} \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$(3.7) \quad h_\beta(x) = \sup_{y \in \text{dom } \Phi, v \in F(y)} \{\langle v, x - y \rangle + \Phi(x) - \Phi(y) + \beta \|x - y\|^2\}$$

This function has been studied in [7] for the case $\Phi(\cdot) = \delta_K(\cdot)$, in [28] for the case $\Phi(\cdot) = \delta_K(\cdot)$ and F is single-valued, in [20] for the case $\Phi(\cdot) = \delta_K(\cdot)$, F is single-valued and $\beta = 0$.

Lemma 3.4. *For any $\beta \geq 0$, the function $h_\beta(\cdot)$ is lower semicontinuous convex function.*

Proof. For given $\beta \geq 0$ and $y \in \text{dom } \Phi$, so is $\langle v, \cdot - y \rangle + \Phi(\cdot) - \Phi(y) + \beta \|\cdot - y\|^2$. Therefore, it is easy to see that $h_\beta(\cdot)$ is lower semicontinuous. The convexity of $h_\beta(\cdot)$ follows from the definition (3.7) directly, since $\langle v, \cdot - y \rangle + \Phi(\cdot) - \Phi(y) + \beta \|\cdot - y\|^2$ is convex for every $y \in \text{dom } \Phi$ and $v \in F(y)$. This completes the proof.

Remark 3.4. If $\mathbb{H} = \mathbb{R}$ and $\Phi(\cdot) = \delta_K(\cdot)$, then Lemma 3.4 reduces to Lemma 3.2 of [7]. If, in addition, F is single-valued, then Lemma 3.4 reduces to Lemma 2.2 of [28].

- Lemma 3.5.** (i) *If F is Φ -pseudomonotone and upper semicontinuous, then x^* is a solution of (1.1) if and only if $h_0(x^*) = 0$;*
(ii) *If F is upper semicontinuous, if there exists $\beta > 0$ with $h_\beta(x^*) = 0$, then x^* solves (1.1);*
(iii) *If x^* solves (1.1), F is Φ -strongly pseudomonotone with respect to x^* with modulus $\mu > 0$, and β is chosen to satisfy $0 \leq \beta \leq \mu$, then $h_\beta(x^*) = 0$;*
(iv) *If F is upper semicontinuous and Φ -strongly pseudomonotone with respect to x^* with modulus $\mu > 0$, and β is chosen to satisfy $0 \leq \beta \leq \mu$, then x^* solves (1.1) if and only if $h_\beta(x^*) = 0$.*

Proof. (i) If x^* solves (1.1), then there exists some $u^* \in F(x^*)$ such that

$$\langle u^*, y - x^* \rangle + \Phi(y) - \Phi(x^*) \geq 0, \quad \forall y \in \text{dom } \Phi.$$

Since F is Φ -pseudomonotone, then we have

$$\langle v, y - x^* \rangle + \Phi(y) - \Phi(x^*) \geq 0, \quad \forall y \in \text{dom } \Phi, v \in F(y),$$

which yields that

$$h_0(x^*) = \sup_{y \in \text{dom } \Phi, v \in F(y)} \{\langle v, x^* - y \rangle + \Phi(x^*) - \Phi(y)\} \leq 0.$$

Combining with nonnegativity of $h_\beta(\cdot)$, we have $h_0(x^*) = 0$.

Conversely, if $h_0(x^*) = 0$, then by the definition of h , we have

$$(3.8) \quad \langle v, x^* - y \rangle + \Phi(x^*) - \Phi(y) \leq 0, \quad \forall y \in \text{dom } \Phi, v \in F(y).$$

We will show that x^* solves (1.2). If not, there is some $y_0 \in \text{dom } \Phi$ such that for all $u^* \in F(x^*)$, it holds that

$$\langle u^*, y_0 - x^* \rangle + \Phi(y_0) - \Phi(x^*) < 0.$$

Since the set $A = \{u^* \in \mathbb{H} : \langle u^*, y_0 - x^* \rangle + \Phi(y_0) - \Phi(x^*) < 0\}$ is a neighborhood of $F(x^*)$ and F is upper semicontinuous, then setting $x_t = ty_0 + (1 - t)x^* \in \text{dom } \Phi$ and taking t close to zero, we obtain $F(x_t) \subset A$, i.e., for each $u_t \in F(x_t)$, it holds that

$$\langle u_t, y_0 - x^* \rangle + \Phi(y_0) - \Phi(x^*) < 0.$$

Thus, it follows from the convexity of Φ that

$$(3.9) \quad \begin{aligned} & \langle u_t, x_t - x^* \rangle + \Phi(x_t) - \Phi(x^*) \\ &= \langle u_t, t(y_0 - x^*) \rangle + \Phi(ty_0 + (1 - t)x^*) - \Phi(x^*) \\ &\leq t\langle u_t, y_0 - x^* \rangle + t\Phi(y_0) + (1 - t)\Phi(x^*) - \Phi(x^*) \\ &= t[\langle u_t, y_0 - x^* \rangle + \Phi(y_0) - \Phi(x^*)] < 0, \end{aligned}$$

which contradicts (3.8). So x^* is a solution of (1.2), thus, by Lemma 2.2, x^* solves (1.1).

(ii) If $\beta > 0$ and $h_\beta(x^*) = 0$, it is easy to see that

$$\langle v, x^* - y \rangle + \Phi(x^*) - \Phi(y) \leq 0, \quad \forall y \in \text{dom } \Phi, v \in F(y),$$

From the proof of (i), we know that x^* is a solution of (1.1).

(iii) Since F is Φ -strongly pseudomonotone with respect to x^* with modulus $\mu > 0$, for any $y \in \text{dom } \Phi, v \in F(y)$, we have

$$\langle v, y - x^* \rangle + \Phi(y) - \Phi(x^*) \geq \mu\|y - x^*\|^2.$$

This implies that

$$\langle v, x^* - y \rangle + \Phi(x^*) - \Phi(y) + \beta\|y - x^*\|^2 \leq (\beta - \mu)\|y - x^*\|^2 \leq 0,$$

where the second inequality follows from $0 \leq \beta \leq \mu$, which yields that $h_\beta(x^*) \leq 0$. Combining with the nonnegativity of $h_\beta(\cdot)$, we have $h_\beta(x^*) = 0$.

(iv) Since F is upper semicontinuous and Φ -strongly pseudomonotone with respect to x^* , the conclusion follows immediately from (ii) and (iii). This completes the proof.

Remark 3.5. Lemma 3.5 generalizes and improves Lemma 3.4 of [7] in the following aspects: (a) If $\mathbb{H} = \mathbb{R}$ and $\Phi(\cdot) = \delta_K(\cdot)$, then Lemma 3.5 reduces to Lemma 3.4 of [7]; (b) In item (ii) of Lemma 3.5, we removes the pseudomonotonicity of F of [7].

3.2. Gap functions based on Moreau-Yosida regularization of $f_\alpha(\cdot)$ and $h_\beta(\cdot)$

Next, we consider the following functions defined by

$$(3.10) \quad \varphi_{f,\alpha,\lambda}(x) = \inf_{z \in \text{dom } \Phi} \{f_\alpha(z) + \lambda\|x - z\|^2\}$$

and

$$(3.11) \quad \varphi_{h,\beta,\lambda}(x) = \inf_{z \in \text{dom } \Phi} \{h_\beta(z) + \lambda\|x - z\|^2\},$$

where λ is a positive constant, $f_\alpha(\cdot)$ and $h_\beta(\cdot)$ are defined by (3.2) and (3.7), respectively. In fact, combining with the definitions of $f_\alpha(\cdot)$ and $h_\beta(\cdot)$, $\varphi_{f,\alpha,\lambda}(\cdot)$ and $\varphi_{h,\beta,\lambda}(\cdot)$ can be rewritten as

$$(3.12) \quad \varphi_{f,\alpha,\lambda}(x) = \inf_{z \in \text{dom } \Phi, u \in F(z)} \left\{ \sup_{y \in \text{dom } \Phi} \left\{ \langle u, z - y \rangle + \Phi(z) - \Phi(y) - \frac{1}{2\alpha} \|z - y\|^2 \right\} + \lambda\|x - z\|^2 \right\}$$

and

$$(3.13) \quad \varphi_{h,\beta,\lambda}(x) = \inf_{z \in \text{dom } \Phi} \left\{ \sup_{y \in \text{dom } \Phi, v \in F(y)} \left\{ \langle v, z - y \rangle + \Phi(z) - \Phi(y) + \beta \|z - y\|^2 \right\} + \lambda\|x - z\|^2 \right\}$$

Some special cases of these functions have been studied in [7, 28].

- Theorem 3.1.** (i) For any $\alpha > 0$, $\beta \geq 0$ and $\lambda > 0$, the functions $\varphi_{f,\alpha,\lambda}(\cdot)$ and $\varphi_{h,\beta,\lambda}(\cdot)$ are nonnegative on \mathbb{H} .
- (ii) For any $\alpha > 0$ and $\lambda > 0$, x^* is a solution of (1.1) if and only if $\varphi_{f,\alpha,\lambda}(x^*) = 0$.
- (iii) If F is Φ -pseudomonotone, then, for any $\lambda > 0$, x^* is a solution of (1.1) if and only if $\varphi_{h,\beta,\lambda}(x^*) = 0$.
- (iv) Let x^* be a solution of (1.1). If F is Φ -strongly pseudomonotone with respect to x^* with modulus $\mu > 0$, β is chosen to satisfy $0 \leq \beta \leq \mu$, then for any $\lambda > 0$, $\varphi_{h,\beta,\lambda}(x^*) = 0$.
- (v) If F is Φ -strongly pseudomonotone with modulus $\mu > 0$, for any β, λ satisfying $0 \leq \beta \leq \mu$ and $\lambda > 0$, then x^* is a solution of (1.1) if and only if $\varphi_{h,\beta,\lambda}(x^*) = 0$.

Proof. (i) For any $\alpha > 0$, $\beta \geq 0$, $f_\alpha(\cdot)$ and $h_\beta(\cdot)$ are nonnegative on \mathbb{H} , we can easily deduce from the definitions of $\varphi_{f,\alpha,\lambda}(\cdot)$ and $\varphi_{h,\beta,\lambda}(\cdot)$ that they are nonnegative for all $x \in \mathbb{H}$.

(ii) Suppose that x^* is a solution of (1.1). Then, we have

$$\begin{aligned}
 \varphi_{f,\alpha,\lambda}(x^*) &= \inf_{z \in \text{dom } \Phi} \{f_\alpha(z) + \lambda\|x^* - z\|^2\} \\
 &\leq f_\alpha(x^*) + \lambda\|x^* - x^*\|^2 \\
 &= 0.
 \end{aligned}
 \tag{3.14}$$

where the last equality follows from $f_\alpha(x^*) = 0$ (by Lemma 3.3). Since $\varphi_{f,\alpha,\lambda}(x) \geq 0$ for all x as shown above, we obtain $\varphi_{f,\alpha,\lambda}(x^*) = 0$.

Conversely, suppose $\varphi_{f,\alpha,\lambda}(x^*) = 0$. Then since $f_\alpha(z) \geq 0$ for all $z \in \text{dom } \Phi$, it follows from the definition of $\varphi_{f,\alpha,\lambda}(\cdot)$ that there exists a minimizing sequence $\{z_n\}$ in $\text{dom } \Phi$ such that, for any positive integer n , we have

$$f_\alpha(z_n) + \lambda\|z_n - x^*\|^2 < \frac{1}{n},$$

i.e., there exists a sequence $\{z_n\}$ in $\text{dom } \Phi$ such that $f_\alpha(z_n) \rightarrow 0$ and $\|z_n - x^*\| \rightarrow 0$. Since the set $\text{dom } \Phi$ is closed (by the lower semicontinuity of Φ), $z_n \rightarrow x^*$ and $z_n \in \text{dom } \Phi$ imply that $x^* \in \text{dom } \Phi$. Since $f_\alpha(\cdot)$ is lower semicontinuous and nonnegative (by Lemma 3.2), we have

$$0 \leq f_\alpha(x^*) \leq \liminf_{n \rightarrow \infty} f_\alpha(z_n) = 0,$$

which yields that $f_\alpha(x^*) = 0$. Therefore from Lemma 3.3, we obtain that x^* is a solution of (1.1).

By using Lemmas 3.4 and 3.5, the proof of (iii)-(v) for the functions $\varphi_{h,\beta,\lambda}(\cdot)$ can be done analogously. This completes the proof.

Remark 3.6. Theorem 3.1 generalizes Theorem 3.1 of [7] from set-valued variational inequality (SVI(F, K)) to set-valued mixed variational inequality (SMVI(F, Φ)) and from finite dimensional spaces to infinite dimensional spaces.

Theorem 3.1 shows us the unconstrained minimization problems

$$\min_{x \in \mathbb{H}} \varphi_{f,\alpha,\lambda}(x) \quad \text{and} \quad \min_{x \in \mathbb{H}} \varphi_{h,\beta,\lambda}(x)$$

are equivalent to the problem (1.1) under certain assumptions of F and the associated parameters. Thus it is convenient to use unconstrained minimization methods to solve the problem (1.1) which satisfies the conditions in Theorem 3.1. In order for these minimization problems to be practically useful, it is desirable that the objective functions $\varphi_{f,\alpha,\lambda}(\cdot)$ and $\varphi_{h,\beta,\lambda}(\cdot)$ are everywhere differentiable. For the discussions to follow, for

any $\alpha > 0$, $\beta \geq 0$ and $\lambda > 0$, we define the functions $\Psi_{f,\alpha,\lambda}(\cdot, \cdot) : \mathbb{H} \times \text{dom } \Phi \rightarrow (-\infty, +\infty]$ and $\Psi_{h,\beta,\lambda}(\cdot, \cdot) : \mathbb{H} \times \text{dom } \Phi \rightarrow (-\infty, +\infty]$ by

$$\Psi_{f,\alpha,\lambda}(x, z) = f_\alpha(z) + \lambda\|x - z\|^2 \quad \text{and} \quad \Psi_{h,\beta,\lambda}(x, z) = h_\beta(z) + \lambda\|x - z\|^2,$$

respectively. By the definitions of $\varphi_{f,\alpha,\lambda}(\cdot)$ in (3.10) and $\varphi_{h,\beta,\lambda}(\cdot)$ in (3.11), we have

$$\varphi_{f,\alpha,\lambda}(x) = \inf_{z \in \text{dom } \Phi} \Psi_{f,\alpha,\lambda}(x, z) \quad \text{and} \quad \varphi_{h,\beta,\lambda}(x) = \inf_{z \in \text{dom } \Phi} \Psi_{h,\beta,\lambda}(x, z).$$

The following theorems show us that the differentiability of $\varphi_{f,\alpha,\lambda}(\cdot)$ and $\varphi_{h,\beta,\lambda}(\cdot)$ do not need to rely on the some differentiability property of set-valued mapping F .

Theorem 3.2. *Let $\alpha > 0$ and $\lambda > 0$. If the function $\Psi_{f,\alpha,\lambda}(x, \cdot)$ attains its unique minimum $z_{f,\alpha,\lambda}(x)$ on $\text{dom } \Phi$ for each $x \in \mathbb{H}$ and $z_{f,\alpha,\lambda}(x)$ is continuous, then $\varphi_{f,\alpha,\lambda}(\cdot)$ is differentiable on \mathbb{H} and*

$$\nabla \varphi_{f,\alpha,\lambda}(x) = 2\lambda(x - z_{f,\alpha,\lambda}(x)).$$

Proof. From the definitions of $\varphi_{f,\alpha,\lambda}(\cdot)$, $\Psi_{f,\alpha,\lambda}(\cdot, \cdot)$ and $z_{f,\alpha,\lambda}(\cdot)$, for each $d \in \mathbb{H}$ and $\mu > 0$, we have

$$\begin{aligned} & \varphi_{f,\alpha,\lambda}(x + \mu d) - \varphi_{f,\alpha,\lambda}(x) \\ (3.15) \quad & \leq \Psi_{f,\alpha,\lambda}(x + \mu d, z_{f,\alpha,\lambda}(x)) - \Psi_{f,\alpha,\lambda}(x, z_{f,\alpha,\lambda}(x)) \\ & = \lambda[\|x + \mu d - z_{f,\alpha,\lambda}(x)\|^2 - \|x - z_{f,\alpha,\lambda}(x)\|^2] \\ & = \lambda[2\mu\langle x - z_{f,\alpha,\lambda}(x), d \rangle + \mu^2\|d\|^2]. \end{aligned}$$

By dividing μ in the leftmost and rightmost sides of the inequality above and tends $\mu \rightarrow 0$, we get

$$(3.16) \quad \limsup_{\mu \rightarrow 0} \frac{\varphi_{f,\alpha,\lambda}(x + \mu d) - \varphi_{f,\alpha,\lambda}(x)}{\mu} \leq 2\lambda\langle x - z_{f,\alpha,\lambda}(x), d \rangle.$$

On the other hand, for each $d \in \mathbb{H}$ and $\mu > 0$, let $x_\mu = x + \mu d$. It follows from the definitions of $\varphi_{f,\alpha,\lambda}(\cdot)$, $\Psi_{f,\alpha,\lambda}(\cdot, \cdot)$ and $z_{f,\alpha,\lambda}(\cdot)$ again that

$$\begin{aligned} & \varphi_{f,\alpha,\lambda}(x + \mu d) - \varphi_{f,\alpha,\lambda}(x) \\ (3.17) \quad & = \varphi_{f,\alpha,\lambda}(x_\mu) - \varphi_{f,\alpha,\lambda}(x) \\ & \geq \Psi_{f,\alpha,\lambda}(x_\mu, z_{f,\alpha,\lambda}(x_\mu)) - \Psi_{f,\alpha,\lambda}(x, z_{f,\alpha,\lambda}(x_\mu)) \\ & = \lambda[\|x + \mu d - z_{f,\alpha,\lambda}(x_\mu)\|^2 - \|x - z_{f,\alpha,\lambda}(x_\mu)\|^2] \\ & = \lambda[2\mu\langle x - z_{f,\alpha,\lambda}(x_\mu), d \rangle + \mu^2\|d\|^2]. \end{aligned}$$

By dividing μ in the leftmost and rightmost sides of the inequality above and tends $\mu \rightarrow 0$, observing the continuity of $z_{f,\alpha,\lambda}(\cdot)$, we have

$$(3.18) \quad \liminf_{\mu \rightarrow 0} \frac{\varphi_{f,\alpha,\lambda}(x + \mu d) - \varphi_{f,\alpha,\lambda}(x)}{\mu} \geq 2\lambda \langle x - z_{f,\alpha,\lambda}(x), d \rangle.$$

It follows from (3.16) and (3.18) that for each $d \in \mathbb{H}$

$$\nabla \varphi_{f,\alpha,\lambda}(x; d) = \lim_{\mu \rightarrow 0} \frac{\varphi_{f,\alpha,\lambda}(x + \mu d) - \varphi_{f,\alpha,\lambda}(x)}{\mu} = 2\lambda \langle x - z_{f,\alpha,\lambda}(x), d \rangle.$$

In other words, $\nabla \varphi_{f,\alpha,\lambda}(x) = 2\lambda(x - z_{f,\alpha,\lambda}(x))$.

Theorem 3.3. *If $\beta \geq 0$ and $\lambda > 0$, then the function $\Psi_{h,\beta,\lambda}(x, \cdot)$ attains its unique minimum $z_{h,\beta,\lambda}(x)$ on $\text{dom } \Phi$ for each $x \in \mathbb{H}$. Moreover, $\varphi_{h,\beta,\lambda}(\cdot)$ is a differentiable convex function on \mathbb{H} and*

$$\nabla \varphi_{h,\beta,\lambda}(x) = 2\lambda(x - z_{h,\beta,\lambda}(x)).$$

Proof. By Lemma 3.4, $h_\beta(\cdot)$ is a closed convex function. By the strict convexity of the function $\|\cdot - x\|^2$, we know that $\Psi_{h,\beta,\lambda}(x, \cdot)$ is strict convex, thus $\Psi_{h,\beta,\lambda}(x, \cdot)$ attains its minimum on $\text{dom } \Phi$ uniquely. Observing that $z_{h,\beta,\lambda}(\cdot)$ is actually the proximal mapping with respect to the convex function $h_\beta(\cdot)$, it is well known that $z_{h,\beta,\lambda}(\cdot)$ is firmly nonexpansive (see, for example, Section 1 of [14]), thus $z_{h,\beta,\lambda}(\cdot)$ is continuous. In the sequel, the proof follows the pattern of the proof of Theorem 3.2 with f and α replaced by h and β , respectively. We obtain that $\varphi_{h,\beta,\lambda}(\cdot)$ is differentiable and its gradient is represented as indicated in the theorem. The convexity of $\varphi_{h,\beta,\lambda}(\cdot)$ follows from the convexity of $h_\beta(\cdot)$ (see the proof of Proposition 4.1 in [2]).

Theorem 3.4. *Assume that the problem (1.1) has a solution. Let $\lambda > 0$. If F is Φ -pseudomonotone on $\text{dom } \Phi$, then any stationary point of $\varphi_{h,0,\lambda}(\cdot)$ is a solution of the problem (1.1). Moreover, if F is Φ -strongly pseudomonotone with modulus $\mu > 0$ and β is chosen to satisfy $0 \leq \beta \leq \mu$, then any stationary point of $\varphi_{h,\beta,\lambda}(\cdot)$ is a solution of the problem (1.1).*

Proof. It follows from Theorem 3.3 that for each $\beta \geq 0$ and $\lambda > 0$, $\varphi_{h,\beta,\lambda}(\cdot)$ is a differentiable convex function. Thus, $\nabla \varphi_{h,\beta,\lambda}(x) = 0$ if and only if $\varphi_{h,\beta,\lambda}(\cdot)$ attains its global minimum at x . The conclusion then follows from item (iii) and (v) of Theorem 3.1. This completes the proof.

Remark 3.7. If $\mathbb{H} = \mathbb{R}$ and $\Phi(\cdot) = \delta_K(\cdot)$, then Theorems 3.2, 3.3 and 3.4 reduce to Propositions 3.1, 3.2 and Theorem 3.2 of [7], respectively. If, in addition, F is single-valued, then Theorems 3.2, 3.3 and 3.4 collapse to Propositions 2.5, 2.6 and Theorem 2.9 of [28], respectively.

4. ERROR BOUNDS

In this section, we present error bounds based on the gap functions $f_\alpha(\cdot)$, $h_\beta(\cdot)$, $\varphi_{f,\alpha,\lambda}(\cdot)$ and $\varphi_{h,\beta,\lambda}(\cdot)$ for the set-valued mixed variational inequality (1.1). To begin with, we discuss how the gap functions $f_\alpha(\cdot)$, $h_\beta(\cdot)$ provide error bounds for the problem (1.1).

Lemma 4.1. *Suppose that F is Φ -strongly pseudomonotone with modulus $\mu > 0$ with respect to solution x^* of (1.1). If α is chosen to satisfy $\alpha > \frac{1}{2\mu}$, then we have*

$$(4.1) \quad f_\alpha(x) \geq \left(\mu - \frac{1}{2\alpha}\right)\|x - x^*\|^2, \quad \forall x \in \text{dom } \Phi.$$

Proof. By Lemma 3.2, for any $x \in \text{dom } \Phi$, there exists $u_x \in F(x)$ such that $f_\alpha(x) = g_\alpha(x; u_x)$. Since x^* is a solution of (1.1), i.e., $\langle u^*, x - x^* \rangle + \Phi(x) - \Phi(x^*) \geq 0$ with $u^* \in F(x^*)$, and F is Φ -strongly pseudomonotone with modulus $\mu > 0$ with respect to solution x^* , it holds that $\langle u_x, x - x^* \rangle + \Phi(x) - \Phi(x^*) \geq \mu\|x - x^*\|^2$. Thus, we have

$$(4.2) \quad \begin{aligned} f_\alpha(x) &= g_\alpha(x; u_x) \\ &= \sup_{y \in \text{dom } \Phi} \left\{ \langle u_x, x - y \rangle + \Phi(x) - \Phi(y) - \frac{1}{2\alpha}\|x - y\|^2 \right\} \\ &\geq \langle u_x, x - x^* \rangle + \Phi(x) - \Phi(x^*) - \frac{1}{2\alpha}\|x - x^*\|^2 \\ &\geq \mu\|x - x^*\|^2 - \frac{1}{2\alpha}\|x - x^*\|^2 \\ &= \left(\mu - \frac{1}{2\alpha}\right)\|x - x^*\|^2. \end{aligned}$$

This completes the proof.

Remark 4.1. If F is single-valued, then Lemma 4.1 reduces to Theorem 5 of [22] in the case $g = I$. This theorem also generalizes Lemma 4.1 of [7], Lemma 4.1 of [28] and Theorem 2 of [23].

Lemma 4.2. *If $\beta > 0$ and x^* is a solution of (1.1), then*

$$h_\beta(x) \geq \beta\|x - x^*\|^2, \quad \forall x \in \text{dom } \Phi.$$

Proof. Let $x^* \in \text{dom } \Phi$ be arbitrary. It follows from $x^* \in \text{SOL}(F, \Phi)$ that there exists $u^* \in F(x^*)$ such that

$$\langle u^*, x - x^* \rangle + \Phi(x) - \Phi(x^*) \geq 0.$$

Then we have

$$\begin{aligned}
 h_\beta(x) &= \sup_{y \in \text{dom } \Phi, v \in F(y)} \{ \langle v, x - y \rangle + \Phi(x) - \Phi(y) + \beta \|x - y\|^2 \} \\
 (4.3) \quad &\geq \langle u^*, x - x^* \rangle + \Phi(x) - \Phi(x^*) + \beta \|x - x^*\|^2 \\
 &\geq \beta \|x - x^*\|^2.
 \end{aligned}$$

This completes the proof.

Using the results above, we prove below that $\varphi_{f,\alpha,\lambda}(\cdot)$ and $\varphi_{h,\beta,\lambda}(\cdot)$ provide global error bounds for the set-valued mixed variational inequality (1.1) on the whole space \mathbb{H} .

Theorem 4.1. *Suppose that F is Φ -strongly pseudomonotone with modulus $\mu > 0$ with respect to solution x^* of (1.1). If α is chosen to satisfy $\alpha > \frac{1}{2\mu}$. Then for any $\lambda > 0$, we have*

$$(4.4) \quad \frac{1}{2} \min\{\mu - \frac{1}{2\alpha}, \lambda\} \|x - x^*\|^2 \leq \varphi_{f,\alpha,\lambda}(x) \leq \lambda \|x - x^*\|^2, \quad \forall x \in \mathbb{H}.$$

Proof. First we consider the right-hand inequality. Since x^* is a solution of (1.1), from item (ii) of Theorem 3.1, we have $\varphi_{f,\alpha,\lambda}(x^*) = 0$. Thus

$$\begin{aligned}
 \varphi_{f,\alpha,\lambda}(x) &= \inf_{z \in \text{dom } \Phi} \{ f_\alpha(z) + \lambda \|x - z\|^2 \} \\
 (4.5) \quad &\leq f_\alpha(x^*) + \lambda \|x - x^*\|^2 \\
 &= \lambda \|x - x^*\|^2,
 \end{aligned}$$

Next, we prove the left-hand inequality. It follows from Theorem 4.1 that

$$\begin{aligned}
 \varphi_{f,\alpha,\lambda}(x) &= \inf_{z \in \text{dom } \Phi} \{ f_\alpha(z) + \lambda \|x - z\|^2 \} \\
 &\geq \inf_{z \in \text{dom } \Phi} \{ (\mu - \frac{1}{2\alpha}) \|z - x^*\|^2 + \lambda \|x - z\|^2 \} \\
 (4.6) \quad &\geq \min\{\mu - \frac{1}{2\alpha}, \lambda\} \inf_{z \in \text{dom } \Phi} \{ \|z - x^*\|^2 + \|x - z\|^2 \} \\
 &\geq \frac{1}{2} \min\{\mu - \frac{1}{2\alpha}, \lambda\} \|x - x^*\|^2,
 \end{aligned}$$

where the last inequality follows from the inequality

$$\|a\|^2 + \|b\|^2 \geq \frac{\|a - b\|^2}{2}, \quad \forall a, b \in \mathbb{H}.$$

This completes the proof.

Theorem 4.2. *Suppose that F is Φ -strongly pseudomonotone with modulus $\mu > 0$ with respect to solution x^* of (1.1). If β is chosen to satisfy $0 < \beta \leq \mu$. Then for any $\lambda > 0$, we have*

$$(4.7) \quad \frac{1}{2} \min\{\beta, \lambda\} \|x - x^*\|^2 \leq \varphi_{h,\beta,\lambda}(x) \leq \lambda \|x - x^*\|^2, \quad \forall x \in \mathbb{H}.$$

Proof. Observing that $x^* \in \text{SOL}(F, \Phi)$ and $h_\beta(x^*)$, the right-hand inequality can be proved in a way similar to the first part of the proof of Theorem 4.1. Moreover, by using Lemma 4.2, we can prove the left-hand inequality analogously to the last part of the proof of Theorem 4.1. This completes the proof.

Remark 4.2. Lemma 4.2, Theorems 4.1 and 4.2 generalize the corresponding results of [7, 28].

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