TAIWANESE JOURNAL OF MATHEMATICS
Vol. 19, No. 5, pp. 1317-1323, October 2015
DOI: 10.11650/tjm.19.2015.5285
This paper is available online at http://journal.taiwanmathsoc.org.tw

A CONJECTURE ON ALGEBRAIC CONNECTIVITY OF GRAPHS

Kinkar Ch. Das

Abstract. Let $G = (V,\ E)$ be a simple graph with vertex set $V(G) = \{v_1,\ v_2, \ldots,\ v_n\}$ and edge set $E(G)$. Let $A(G)$ be the adjacency matrix of graph G and also let $D(G)$ be the diagonal matrix with degrees of the vertices on the main diagonal. The Laplacian matrix of G is $L(G) = D(G) - A(G)$. Among all eigenvalues of the Laplacian matrix $L(G)$ of a graph G, the most studied is the second smallest, called the algebraic connectivity $(a(G))$ of a graph G [9]. Let $\alpha(G)$ be the independence number of graph G. Recently, it was conjectured that (see, [1]):

$$a(G) + \alpha(G)$$

is minimum for $\overline{K_{p,\,q}\backslash\{e\}}$, where e is any edge in $K_{p,\,q}$ and $p = \left\lfloor \frac{n}{2} \right\rfloor$, $q = \left\lceil \frac{n}{2} \right\rceil$ ($K_{p,\,q}$ is a complete bipartite graph). The aim of this paper is to show that this conjecture is true.

1. Introduction

All graphs considered in this paper are finite and simple. Let $G = (V, E)$ be a graph on vertex set $V(G) = \{v_1,\ v_2, \ldots,\ v_n\}$ and edge set $E = E(G)$, where $|V(G)| = n$ and $|E(G)| = m$. Also let d_i be the degree of vertex v_i for $i = 1,\ 2, \ldots,\ n$. The maximum vertex degree is denoted by $\Delta = \Delta(G)$. The diameter of a graph is the maximum distance between any two vertices of G. Let $d(G)$ be the diameter of graph G. Also let N_i be the neighbor set of the vertex $v_i \in V(G)$. Denote by $\overline{G}$ the complement graph of G. If vertices v_i and v_j are adjacent, we denote that by $v_iv_j \in E(G)$. Let $A(G)$ be the adjacency matrix of graph G and also let $D(G)$ be the diagonal matrix with degrees of the vertices on the main diagonal. Then the Laplacian matrix of G is $L(G) = D(G) - A(G)$. Let $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0$ denote the eigenvalues of $L(G)$. They are usually called the Laplacian eigenvalues of

Received September 13, 2014, accepted March 2, 2015.
Communicated by Sen-Peng Eu.
2010 *Mathematics Subject Classification*: 05C50.
Key words and phrases: Graph, Laplacian matrix, Laplacian spectral radius, Algebraic connectivity, Independence number.

G. Among all eigenvalues of the Laplacian of a graph, the most studied is the second smallest, called the algebraic connectivity of a graph [9]. It is well known that a graph is connected if and only if $a(G) = \mu_{n-1}(G) > 0$. Besides the algebraic connectivity, $\mu_1(G)$ is the invariant that interested the graph theorists.

Given a graph G, a subset S of $V(G)$ is called an independent set of G if $G[S]$, an induced subgraph by S, is a graph with $|S|$ isolated vertices. The independence number of G is denoted by $\alpha(G)$ and is defined to be the number of vertices in the largest independent set of G. The characteristic polynomial of a square matrix B is denoted by $\Phi(B,\,\mu) = \det(\mu\, I - B)$. In particular, if $B = L(G)$, we write $\Phi(L(G),\,\mu)$ by $\Phi(G,\,\mu)$ (the Laplacian characteristic polynomial of G) for convenience. As usual, we denote by $K_{p,\,q}$ the complete bipartite of order n $(q \geq p\,,\; p+q = n)$, $K_{1,\,n-1}$ the star of order n and K_n the complete graph of order n. A tree is called a double star $DS_{p,\,q}$ if it is obtained from $K_{1,\,p-1}$ and $K_{1,\,q-1}$ by connecting the center of $K_{1,\,p-1}$ with that of $K_{1,\,q-1}$ via an edge.

Recently there has been vast research regarding conjectures, and a series of papers written on various graph invariants: average distance, independence number, largest eigenvalue of Laplacian and signless Laplacian matrix, Randić index and energy, etc. We continue this work and resolve some conjectures (see, [3-8, 11]). The following conjectures were proposed by Aouchiche and Hansen (see, [1]).

Conjecture 1. [1] Let G be a connected graph of order n with independence number $\alpha(G)$ and algebraic connectivity $a(G)$. Then $a(G) + \alpha(G)$ is minimum for $\overline{K_{p,\,q}\backslash\{e\}}$, where e is any edge in $K_{p,\,q}$ and $p = \left\lfloor \dfrac{n}{2} \right\rfloor,\, q = \left\lceil \dfrac{n}{2} \right\rceil$.

The aim of this paper is to confirm the validity of the above conjecture.

2. Preliminaries

In this section, we shall list some previously known results that will be needed in the next section.

Lemma 2.1. [12] *Let G be a simple graph on n vertices which has at least one edge. Then*

$$\mu_1(G) \geq \Delta + 1, \tag{1}$$

where Δ is the maximum degree in G. Moreover, if G is connected, then the equality holds in (1) if and only if $\Delta = n-1$.

Lemma 2.2. [2] *Let G be a connected graph with at least one edge. Then*

$$\mu_1(G) \leq \max_{v_i v_j \in E(G)} |N_i \cup N_j|\,, \tag{2}$$

where N_i is the neighbor set of vertex $v_i \in V(G)$. This upper bound for $\mu_1(G)$ does not exceed n.

The following result is related between the Laplacian eigenvalues of G and $\overline{G}$.

Lemma 2.3. [12] *Let G be a graph with Laplacian spectrum $\{0 = \mu_n, \mu_{n-1}, \ldots, \mu_2, \mu_1\}$. Then the Laplacian spectrum of $\overline{G}$ is $\{0, n-\mu_1, n-\mu_2, \ldots, n-\mu_{n-2}, n-\mu_{n-1}\}$, where $\overline{G}$ is the complement of the graph G.*

Lemma 2.4. [10] *Let G be a graph of n vertices and let H be a subgraph of G obtained by deleting an edge in G. Then*

$$\mu_1(G) \geq \mu_1(H) \geq \mu_2(G) \geq \mu_2(H) \geq \mu_3(G) \geq \cdots$$
$$\geq \mu_{n-1}(G) \geq \mu_{n-1}(H) \geq \mu_n(G) = \mu_n(H) = 0,$$

where $\mu_i(G)$ is the i-th largest Laplacian eigenvalue of G and $\mu_i(H)$ is the i-th largest Laplacian eigenvalue of H.

The following fact can be found in [13].

Lemma 2.5. [13] *Let G be a connected graph with a connected complement. If $d(G) = 3$, then $\overline{G}$ has a spanning subgraph which is a double star.*

The following result has been proved in Theorem 3.5 [8].

Lemma 2.6. [8] *Let G be a connected graph of diameter 2 with algebraic connectivity $a(G)$. Then $a(G) \geq 1$.*

3. Proof of Conjecture 1

In this section we prove Conjecture 1. For this we need the following result:

Lemma 3.1. *Let $G = K_{p,q}\backslash\{e\}$, where e is any edge in $K_{p,q}$ $(q \geq p\,,\, p+q = n)$. Then*

$$\mu_1(G) \leq \mu_1\left(K_{\left\lfloor \frac{n}{2}\right\rfloor,\,\left\lceil \frac{n}{2}\right\rceil}\backslash\{e\}\right) \tag{3}$$

with equality holding if and only if

$$G \cong K_{\left\lfloor \frac{n}{2}\right\rfloor,\,\left\lceil \frac{n}{2}\right\rceil}\backslash\{e\}\,.$$

Proof. If $p = \left\lfloor \frac{n}{2}\right\rfloor$ and $q = \left\lceil \frac{n}{2}\right\rceil$, then the equality holds in (3). Otherwise, $p < \left\lfloor \frac{n}{2}\right\rfloor$ and $q > \left\lceil \frac{n}{2}\right\rceil$. Thus we have $q \geq p+2$. The characteristic polynomial of $L(G)$ is

$$\begin{aligned}\phi(G,\mu) = \mu(\mu-p)^{q-2}(\mu-q)^{p-2}\Big[&\mu^3-(2p+2q-2)\mu^2+(3pq+p^2+q^2\\ &-3p-3q+2)\mu+2pq+p^2+q^2-pq^2-p^2q-p-q\Big],\end{aligned} \tag{4}$$

where

$$G = K_{p,q} \backslash \{e\}\,.$$

Similarly, the characteristic polynomial of $L(G^*)$ is

$$\phi(G^*,\,\mu) = \mu(\mu-p)^{q-3}(\mu-q)^{p-1}\Big[\mu^3-(2p+2q-2)\mu^2 \\ +(3pq+p^2+q^2-4p-2q+1)\mu+2pq+2p^2-pq^2-p^2q\Big], \tag{5}$$

where

$$G^* = K_{p+1\,,\,q-1} \backslash \{e\}.$$

Now,

$$\phi(G\,,p+q-1) = -(p+q-1)(p-1)^{p-2}(q-1)^{q-2}$$

and

$$\phi(G\,,p+q) = q^{q-2}p^{p-2}(p+q)(q-p).$$

If $p=1$, then $\mu_1(G)=p+q-1=n-1$. Otherwise, $q>p\geq 2$ and from the above, we get $n=p+q>\mu_1(G)>p+q-1=n-1$. Since $\mu_1(G)$ is the largest Laplacian eigenvalue of $L(G)$, we have

$$\mu_1^3(G)-(2p+2q-2)\mu_1^2(G)+(3pq+p^2+q^2-3p-3q+2)\mu_1(G) \\ +2pq+p^2+q^2-pq^2-p^2q-p-q=0\,,\;\; q>p\geq 2\,.$$

Using the above result in (5), we get

$$\begin{aligned}\phi(G^*,\,\mu_1(G)) &= \mu_1(G)\Big(\mu_1(G)-p\Big)^{q-3}\Big(\mu_1(G)-q\Big)^{p-1}\Big[\mu_1^3(G)-(2p+2q-2)\mu_1^2(G) \\ &\qquad +(3pq+p^2+q^2-4p-2q+1)\mu_1(G)+2pq+2p^2-pq^2-p^2q\Big] \\ &= \mu_1(G)\Big(\mu_1(G)-p\Big)^{q-3}\Big(\mu_1(G)-q\Big)^{p-1}(q-p-1)\Big(\mu_1(G)-p-q\Big) \\ &< 0\end{aligned}$$

as $q\geq p+2$ and $\mu_1(G)<p+q$.

Since $\phi(G^*,\,\mu)\to+\infty$ as $\mu\to\infty$. Using the above result, we get $\mu_1(G^*)>\mu_1(G)$, that is, $\mu_1(K_{p,q}\backslash\{e\})<\mu_1\left(K_{p+1\,,\,q-1}\backslash\{e\}\right)$. Repeating the procedure sufficient number times and we conclude that

$$\mu_1(K_{p,q}\backslash\{e\})<\mu_1(K_{p+1\,,\,q-1}\backslash\{e\})<\cdots<\mu_1\left(K_{\left\lfloor\frac{n}{2}\right\rfloor-1\,,\,\left\lceil\frac{n}{2}\right\rceil-1}\backslash\{e\}\right) \\ <\mu_1\left(K_{\left\lfloor\frac{n}{2}\right\rfloor,\,\left\lceil\frac{n}{2}\right\rceil}\backslash\{e\}\right).$$

This completes the proof. ■

Let H be a graph of order n $(n \geq 4)$ such that $\overline{H} = K_{\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil} \backslash \{e\}$, where e is any edge in $K_{\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil}$ (see, Fig. 1). Then $\alpha(H) = 2$.

$H \left(\text{Here } \; p = \left\lfloor \frac{n}{2} \right\rfloor, q = \left\lceil \frac{n}{2} \right\rceil\right)$

Fig. 1. Graph H.

Lemma 3.2. *Let H be the graph of order n (≥ 4) defined above. Then $a(H) < 1$.*

Proof. From Lemma 2.1, we get $\mu_1(H) > \left\lceil \frac{n}{2} \right\rceil + 1$. Let H^* be a disconnected graph of order n such that

$$H^* = K_{\left\lfloor \frac{n}{2} \right\rfloor} \cup K_{\left\lceil \frac{n}{2} \right\rceil}.$$

Thus we have

$$\sum_{i=1}^{n-1} \Big(\mu_i(H) - \mu_i(H^*)\Big) = 2.$$

Since

$$\mu_1(H^*) = \left\lceil \frac{n}{2} \right\rceil \quad \text{and} \quad a(H^*) = 0,$$

using the above results with Lemma 2.4, we must have $a(H) < 1$. ■

Now we are ready to give a proof of Conjecture 1.

Theorem 3.3. *Let G be a connected graph of order n (≥ 4) with independence number $\alpha(G)$ and algebraic connectivity $a(G)$. Then*

$$a(G) + \alpha(G) \geq a(H) + \alpha(H) \tag{6}$$

with equality holding if and only if $G \cong H$.

Proof. Since G is connected graph, $a(G) > 0$. We have $\alpha(H) = 2$ and by Lemma 3.2, $a(H) < 1$. Thus we have $a(H) + \alpha(H) < 3$. For $\alpha(G) = 1$, $G \cong K_n$ and hence $a(G) + \alpha(G) = n + 1 \geq 3 > a(H) + \alpha(H)$. For $\alpha(G) \geq 3$, one can see easily that $a(G) + \alpha(G) > 3 > a(H) + \alpha(H)$ as $a(G) > 0$. Otherwise, $\alpha(G) = 2$.

Let $d(G)$ be the diameter of G. Since $\alpha(G) = 2$, we have $d(G) \geq 2$. For $d(G) = 2$, by Lemma 2.6, $a(G) \geq 1$ and hence $a(G) + \alpha(G) \geq 3 > a(H) + \alpha(H)$. For $d(G) \geq 4$, we must have a path P_5: $v_1v_2v_3v_4v_5$, a subgraph of the diametral path $P_{d(G)+1}$ in G. Hence, we have $\alpha(G) \geq 3$, a contradiction. Otherwise, we have to prove the result (6) for $d(G) = 3$ and $\alpha(G) = 2$. If $\overline{G}$ is a disconnected graph, then by Lemma 2.2, $\mu_1(\overline{G}) \leq n - 1$ and hence $a(G) \geq 1$, by Lemma 2.3. As above we have $a(G) + \alpha(G) \geq 3 > a(H) + \alpha(H)$. Otherwise, $\overline{G}$ is a connected graph. By Lemma 2.5, $\overline{G}$ has a spanning subgraph which is a double star. Hence $\overline{G}$ has diameter at most 3.

First we assume that $\overline{G}$ is bipartite graph. Since G is connected, $\overline{G} \ncong K_{p,q}$, $n = p + q$. Since $\overline{G}$ is connected bipartite graph, by Lemma 2.4, we get

$$\mu_1(\overline{G}) \leq \mu_1(K_{p,q} \backslash \{e\}),$$

where e is any edge in $K_{p,q}$ and $p + q = n$. Using the above result with Lemma 3.1, we get

$$\mu_1(\overline{G}) \leq \mu_1 \left(K_{\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil} \backslash \{e\} \right) = \mu_1(\overline{H})$$

with equality holding if and only if $\overline{G} \cong K_{\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil} \backslash \{e\}$, that is, $G \cong H$. By Lemma 2.3, from the above, we get

$$a(G) \geq a(H)$$

with equality holding if and only if $G \cong H$, that is,

$$a(G) + \alpha(G) \geq a(H) + \alpha(H) \ \ (\text{ as } \ \alpha(G) = \alpha(H) = 2)$$

with equality holding if and only if $G \cong H$.

Next we assume that $\overline{G}$ is non-bipartite graph. Then there is at least one odd cycle in $\overline{G}$. Since $d(\overline{G}) \leq 3$, graph $\overline{G}$ contains induced subgraph, cycle C_3 or C_5 or C_7. Since $\alpha(G) = 2$, cycle of length three (C_3) is not in $\overline{G}$. If $\overline{G}$ contains any cycles of length five or seven $(C_5$ or $C_7)$, then by Lemma 2.2, one can see easily that

$$\mu_1(\overline{G}) \leq n - 1.$$

From the above, again we have $a(G) \geq 1$ and hence $a(G) + \alpha(G) \geq 3 > a(H) + \alpha(H)$ as $\alpha(G) = 2$. This completes the proof. ■

ACKNOWLEDGMENT

The author is much grateful to two referees for valuable comments on our paper, which have improved the presentation of this paper. This work is supported by the National Research Foundation funded by the Korean government with Grant no. 2013R1A1A2009341.

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Kinkar Ch. Das
Department of Mathematics
Sungkyunkwan University
Suwon 440-746
Republic of Korea
E-mail: kinkardas2003@googlemail.com