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ON A NEW CLASS OF MULTIVALUED WEAKLY PICARD OPERATORS ON COMPLETE METRIC SPACES

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Abstract. In the present paper, the concept of nonlinear F-contraction for multivalued mappings in metric spaces is introduced and considering the new proof technique, which was used for single valued maps by Wardowski [D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl. 2012, 2012:94, 6 pp.], we demonstrate that multivalued nonlinear F-contractions of Ćirić type are weakly Picard operators on complete metric spaces. Finally, we give a nontrivial example to guarantee that our result is veritable generalization of recent result of Ćirić [Lj. B. Ćirić, Multi-valued nonlinear contraction mappings, *Nonlinear Analysis*, **71** (2009), 2716-2723]. Also, we show that many fixed point results in the literature can not be applied to this example.

1. Introduction and Preliminaries

Let (X, d) be a metric space. $P(X)$ denotes the family of all nonempty subsets of X, $C(X)$ denotes the family of all nonempty, closed subsets of X, $CB(X)$ denotes the family of all nonempty, closed and bounded subsets of X and $K(X)$ denotes the family of all nonempty compact subsets of X. It is clear that $K(X) \subseteq CB(X) \subseteq C(X) \subseteq P(X)$. For $A, B \in C(X)$, let

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\},$$

where $d(x, B) = \inf\{d(x, y) : y \in B\}$. Then H is called generalized Pompeiu-Hausdorff distance on $C(X)$. It is well known that H is a metric on $CB(X)$, which is called Pompeiu-Hausdorff metric induced by d. We can find detailed information about the Pompeiu-Hausdorff metric in [1, 5, 8]. An element $x \in X$ is said to be fixed point of a multivalued mapping $T : X \to P(X)$ if $x \in Tx$.

Following the Banach contraction principle, Nadler [11] first initiated the study of fixed point theorems for multivalued linear contraction mappings.

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Theorem 1. (Nadler [11]). *Let (X, d) be a complete metric space and $T : X \to CB(X)$. Assume that there there exists $L \in [0, 1)$ such that*

$$H(Tx, Ty) \leq Ld(x, y)$$

for all $x, y \in X$. Then T has a fixed point.

Then many fixed point theorists studied on fixed points of multivalued contractive maps as follows:

Theorem 2. (Reich [13]). *Let (X, d) be a complete metric space and $T : X \to K(X)$. Assume that there exists a map $\varphi : (0, \infty) \to [0, 1)$ such that*

$$\limsup_{t \to s^+} \varphi(t) < 1, \ \forall s > 0;$$

and

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y).$$

for all $x, y \in X$ with $x \neq y$. Then T has a fixed point.

In [14, 15], Reich asked the question as if the above theorem is also true for the map $T : X \to CB(X)$. The partial affirmative answer was given by Mizoguchi and Takahashi [10]. They proved the following theorem.

Theorem 3. (Mizoguchi-Takahashi [10]). *Let (X, d) be a complete metric space and $T : X \to CB(X)$. Assume that there exists a map $\varphi : (0, \infty) \to [0, 1)$ such that*

$$\limsup_{t \to s^+} \varphi(t) < 1, \ \forall s \geq 0;$$

and

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y).$$

for all $x, y \in X$ with $x \neq y$. Then T has a fixed point.

In [16] Suzuki gave a simple proof of Mizoguchi Takahashi fixed point theorem and also an example to show that it is a real generalization of Nadler's. On the other hand, Feng and Liu [7] obtained some interesting fixed point results for multivalued mappings without using the Pompeiu-Hausdorff metric. They proved the following theorem.

Theorem 4. (Feng-Liu [7]). *Let (X, d) be a complete metric space and $T : X \to C(X)$. Assume that the following conditions hold:*

(i) *the map $x \to d(x, Tx)$ is lower semi-continuous;*

(ii) *there exist $b, c \in (0, 1)$ with $c < b$ such that for any $x \in X$ there is $y \in Tx$ satisfying*

$$bd(x,y) \leq d(x,Tx)$$

and

$$d(y,Ty) \leq cd(x,y).$$

Then T has a fixed point.

Then Klim and Wardowski [9] generalized Theorem 4 as follows:

Theorem 5. (Klim-Wardowski [9]). *Let (X,d) be a complete metric space and $T : X \to C(X)$. Assume that the following conditions hold:*

(i) *the map $x \to d(x,Tx)$ is lower semi-continuous;*

(ii) *there exists $b \in (0,1)$ and a function $\varphi : [0,\infty) \to [0,b)$ satisfying*

$$\limsup_{t \to s^+} \varphi(t) < b, \ \forall s \geq 0;$$

(iii) *for any $x \in X$, there is $y \in Tx$ satisfying*

$$bd(x,y) \leq d(x,Tx)$$

and

$$d(y,Ty) \leq \varphi(d(x,y))d(x,y).$$

Then T has a fixed point.

Considering the same direction, in 2009, Ćirić [6] introduced new multivalued nonlinear contractions and established a few nice fixed point theorems for such mappings, one of them is as follows:

Theorem 6. *Let (X,d) be a complete metric space and $T : X \to C(X)$. Assume that the following conditions hold:*

(i) the map $x \to d(x,Tx)$ is lower semi-continuous;

(ii) there exists a function $\varphi : [0,\infty) \to [a,1)$, $0 < a < 1$, satisfying

$$\limsup_{t \to s^+} \varphi(t) < 1, \ \forall s \geq 0;$$

(iii) for any $x \in X$, there is $y \in Tx$ satisfying

$$\sqrt{\varphi(d(x,Tx))}d(x,y) \leq d(x,Tx)$$

and

$$d(y,Ty) \leq \varphi(d(x,Tx))d(x,y).$$

Then T has a fixed point.

On the other hand, Berinde and Berinde [4] introduced a general class of multivalued contractions and proved the following fixed point theorems:

Theorem 7. (Berinde-Berinde [4], Theorem 3). *Let* (X, d) *be a complete metric space and* $T : X \to CB(X)$ *be a multivalued almost contraction, that is, there exist two constants* $\delta \in (0, 1)$ *and* $L \geq 0$ *such that*

$$H(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx) \tag{1.1}$$

for all $x, y \in X$. *Then* T *has a fixed point.*

Theorem 8. (Berinde-Berinde [4], Theorem 4). *Let* (X, d) *be a complete metric space and* $T : X \to CB(X)$. *If there exist a constant* $L \geq 0$ *and a function* $\varphi : [0, \infty) \to [0, 1)$ *satisfying*

$$\limsup_{t \to s^+} \varphi(t) < 1, \ \forall s \geq 0,$$

such that

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y) + Ld(y, Tx) \tag{1.2}$$

for all $x, y \in X$. *Then* T *has a fixed point.*

Analyzing the proofs of above all theorems, we can observe that the mentioned maps on complete metric spaces are weakly Picard operators. We know that, a multivalued map T on a metric space is weakly Picard operator if there exists a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ for any initial point x_0, converges to a fixed point of T.

In the present paper, by introducing a new and different class of multivalued mappings in metric spaces, we give some multivalued weakly Picard operators in complete metric spaces. Our results are extend and generalize many fixed point theorems including Theorem 6 and they are based on a new approach of contraction mapping, which is called F-contraction. The concept of F-contraction for single valued mappings on complete metric space was introduced by Wardowski [17]. Now, we recall this new concept and some related results.

Let $F : (0, \infty) \to \mathbb{R}$ be a function. For the sake of completeness, we will consider the following conditions:

(F1) F is strictly increasing, i.e., for all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$,

(F2) For each sequence $\{\alpha_n\}$ of positive numbers

$$\lim_{n \to \infty} \alpha_n = 0 \text{ if and only if } \lim_{n \to \infty} F(\alpha_n) = -\infty,$$

(F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$,

(F4) $F(\inf A) = \inf F(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$.

We denote by $\mathcal{F}$ and $\mathcal{F}_*$ be the set of all functions F satisfying (F1)-(F3) and (F1)-(F4), respectively. It is clear that $\mathcal{F}_* \subset \mathcal{F}$ and some examples of the functions belonging $\mathcal{F}_*$ are $F_1(\alpha) = \ln \alpha$, $F_2(\alpha) = \alpha + \ln \alpha$, $F_3(\alpha) = -\frac{1}{\sqrt{\alpha}}$ and $F_4(\alpha) = \ln\left(\alpha^2 + \alpha\right)$. If we define $F_5(\alpha) = \ln \alpha$ for $\alpha \leq 1$ and $F_5(\alpha) = 2\alpha$ for $\alpha > 1$, then $F_5 \in \mathcal{F}\backslash\mathcal{F}_*$. If F satisfies (F1), then it satisfies (F4) if and only if it is right continuous.

Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then, Wardowski [17] say that T is an F-contraction if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y)) \tag{1.3}$$

for all $x, y \in X$ with $d(Tx, Ty) > 0$. Also, Wardowski [17] proved that every F-contractions on complete metric spaces has a unique fixed point. We can find some detailed information for F-contractions in [17].

By combining the ideas of Wardowski's and Nadler's, Altun et al [2] introduced the concept of multivalued F-contractions and obtained some fixed point results for mappings of this type on complete metric space.

Definition 1. ([2]). Let (X, d) be a metric space and $T : X \to CB(X)$ be a mapping. Then we say that T is a multivalued F-contraction if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that

$$\tau + F(H(Tx, Ty)) \leq F(d(x, y))$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$.

By the considering $F(\alpha) = \ln \alpha$, then every multivalued contraction in the sense of Nadler is also multivalued F-contraction.

Theorem 9. ([2]). *Let (X, d) be a complete metric space and $T : X \to K(X)$ be a multivalued F-contraction, then T has a fixed point in X.*

Here, the following question may come to mind: Can we take $CB(X)$ instead of $K(X)$ in Theorem 9? The answer is negative as shown in Example 1 in [3]. Nevertheless, by adding the condition (F4) on F, we can we take $CB(X)$ instead of $K(X)$.

Theorem 10. ([2]). *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multivalued F-contraction with $F \in \mathcal{F}_*$, then T has a fixed point in X.*

On the other hand Olgun et al [12] proved the following theorems, one of them is a generalization of famous Mizoguchi-Takahashi fixed point theorem for multivalued contractive mappings. These results are also nonlinear case of Theorem 9 and Theorem 10, respectively.

Theorem 11. ([12]). *Let (X, d) be a complete metric space and $T : X \to K(X)$. If there exist $F \in \mathcal{F}$ and $\tau : (0, \infty) \to (0, \infty)$ such that*

$$\liminf_{t \to s^+} \tau(t) > 0, \ \forall s \geq 0,$$

satisfying

$$\tau(d(x, y)) + F(H(Tx, Ty)) \leq F(d(x, y))$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$. Then T has a fixed point in X.

Theorem 12. ([12]). *Let (X, d) be a complete metric space and $T : X \to CB(X)$. If there exist $F \in \mathcal{F}_*$ and $\tau : (0, \infty) \to (0, \infty)$ such that*

$$\liminf_{t \to s^+} \tau(t) > 0, \ \forall s \geq 0, \tag{1.4}$$

satisfying

$$\tau(d(x, y)) + F(H(Tx, Ty)) \leq F(d(x, y))$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$. Then T has a fixed point in X.

If we examine the proofs of Theorem 9, 10, 11 and 12, we can see that the mentioned maps on complete metric spaces are weakly Picard operators.

2. Main Results

Now, we shall prove a theorem which extends and generalizes Theorem 6.

Theorem 13. *Let (X, d) be a complete metric space, $T : X \to C(X)$ and $F \in \mathcal{F}_*$. Assume that the following conditions hold:*

(i) the map $x \to d(x, Tx)$ is lower semi-continuous;

(ii) there exists a function $\tau : (0, \infty) \to (0, \sigma]$, $\sigma > 0$ such that

$$\liminf_{t \to s^+} \tau(t) > 0, \ \forall s \geq 0; \tag{2.1}$$

(iii) for any $x \in X$ with $d(x, Tx) > 0$, there is $y \in Tx$ satisfying

$$F(d(x, y)) \leq F(d(x, Tx)) + \frac{\tau(d(x, Tx))}{2} \tag{2.2}$$

and

$$\tau(d(x, Tx)) + F(d(y, Ty)) \leq F(d(x, y)). \tag{2.3}$$

Then T is a weakly Picard operator.

Proof. First, we show that T has fixed point in X. Assume the contrary, then $d(x, Tx) > 0$ for all $x \in X$. Therefore, since $\tau(t) > 0$ for all $t > 0$ and $F \in \mathcal{F}_*$, then for any $x \in X$ there exists $y \in Tx$ such that (2.2) holds. Let $x_0 \in X$ be an initial point. Then by assumptions (2.2) and (2.3) we can choose $x_1 \in Tx_0$ such that

$$F(d(x_0, x_1)) \leq F(d(x_0, Tx_0)) + \frac{\tau(d(x_0, Tx_0))}{2} \tag{2.4}$$

and

$$\tau(d(x_0, Tx_0)) + F(d(x_1, Tx_1)) \leq F(d(x_0, x_1)). \tag{2.5}$$

From (2.4) and (2.5), we get

$$\frac{\tau(d(x_0, Tx_0))}{2} + F(d(x_1, Tx_1)) \leq F(d(x_0, Tx_0)). \tag{2.6}$$

Now we choose $x_2 \in Tx_1$ such that

$$F(d(x_1, x_2)) \leq F(d(x_1, Tx_1)) + \frac{\tau(d(x_1, Tx_1))}{2}$$

and

$$\tau(d(x_1, Tx_1)) + F(d(x_2, Tx_2)) \leq F(d(x_1, x_2)).$$

Hence we get

$$\frac{\tau(d(x_1, Tx_1))}{2} + F(d(x_2, Tx_2)) \leq F(d(x_1, Tx_1)).$$

Continuing this process we can choose a sequence $\{x_n\}$ such that $x_{n+1} \in Tx_n$ satisfying

$$F(d(x_n, x_{n+1})) \leq F(d(x_n, Tx_n)) + \frac{\tau(d(x_n, Tx_n))}{2} \tag{2.7}$$

and

$$\frac{\tau(d(x_n, Tx_n))}{2} + F(d(x_{n+1}, Tx_{n+1})) \leq F(d(x_n, Tx_n)) \tag{2.8}$$

for all $n \geq 0$.

Now, we shall show that $d(x_n, Tx_n) \to 0$ as $n \to \infty$. From (2.8), we conclude that $\{d(x_n, Tx_n)\}$ is a strictly decreasing sequence of positive real numbers. Therefore, there exists $\delta \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, Tx_n) = \delta.$$

Suppose $\delta > 0$. Then, since F is right continuous, taking the limit infimum on both sides of (2.8) and having in mind the assumption (2.1), we have

$$\liminf_{d(x_n, Tx_n)\to\delta^+} \frac{\tau(d(x_n, Tx_n))}{2} + F(\delta) \leq F(\delta),$$

which is a contradiction. Thus $\delta = 0$, that is,

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{2.9}$$

Now we shall show that $\{x_n\}$ is a Cauchy sequence in X. Let

$$\alpha = \liminf_{d(x_n,Tx_n)\to\delta^+} \frac{\tau(d(x_n,Tx_n))}{2} > 0.$$

and $0 < q < \alpha$. Then there exists $n_0 \in \mathbb{N}$ such that $\frac{\tau(d(x_n,Tx_n))}{2} > q$ for all $n \geq n_0$. Thus from (2.8),

$$q + F(d(x_{n+1}, Tx_{n+1})) \leq F(d(x_n, Tx_n))$$

for each $n \geq n_0$. Hence, by induction, for all $n \geq n_0$

$$\begin{aligned} F(d(x_{n+1}, Tx_{n+1})) &\leq F(d(x_n, Tx_n)) - q \\ &\;\vdots \\ &\leq F(d(x_{n_0}, Tx_{n_0})) - (n+1-n_0)q. \end{aligned} \tag{2.10}$$

Since $0 < \tau(t) \leq \sigma$ for all $t > 0$. From (2.7), we get

$$F(d(x_n, x_{n+1})) \leq F(d(x_n, Tx_n)) + \sigma.$$

Thus, by (2.10), for all $n \geq n_0$

$$\begin{aligned} F(d(x_n, x_{n+1})) &\leq F(d(x_n, Tx_n)) + \sigma \\ &\leq F(d(x_{n_0}, Tx_{n_0})) - (n-n_0)q + \sigma. \end{aligned} \tag{2.11}$$

From (2.11), we get $\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty$. Thus from (F2) we have $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Therefore, from (F3) there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} [d(x_n, x_{n+1})]^k F(d(x_n, x_{n+1})) = 0.$$

By (2.11), for all $n \geq n_0$

$$\begin{aligned} &[d(x_n, x_{n+1})]^k F(d(x_n, x_{n+1})) - [d(x_n, x_{n+1})]^k F(d(x_{n_0}, Tx_{n_0})) \\ &\leq -[d(x_n, x_{n+1})]^k [(n-n_0)q + \sigma] \leq 0. \end{aligned} \tag{2.12}$$

Letting $n \to \infty$ in (2.12), we obtain that

$$\lim_{n\to\infty} [d(x_n, x_{n+1})]^k [(n-n_0)q + \sigma] = 0. \tag{2.13}$$

From (2.13), there exits $n_1 \in \mathbb{N}$ such that $[d(x_n, x_{n+1})]^k [(n-n_0)q+\sigma] \leq 1$ for all $n \geq n_1$. We can take $n_1 > n_0$. So, we have, for all $n \geq n_1$

$$d(x_n, x_{n+1}) \leq \frac{1}{[(n-n_0)q+\sigma]^{\frac{1}{k}}}. \tag{2.14}$$

In order to show that $\{x_n\}$ is a Cauchy sequence consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the triangular inequality for the metric and from (2.14), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} d(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{[(i-n_0)q+\sigma]^{1/k}} \end{aligned}$$

By the convergence of the series $\sum\limits_{i>n_0-\frac{\sigma}{q}} \frac{1}{[(i-n_0)q+\sigma]^{1/k}}$, passing to limit $n, m \to \infty$, we get $d(x_n, x_m) \to 0$. This yields that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, the sequence $\{x_n\}$ converges to some point $z \in X$, that is, $\lim_{n\to\infty} x_n = z$. Since $x \to d(x, Tx)$ is lower semi-continuous, from (2.9) we have

$$0 \leq d(z, Tz) \leq \liminf_{n\to\infty} d(x_n, Tx_n) = 0.$$

Hence $d(z, Tz) = 0$, which is a contradiction. Therefore, T has a fixed point in X.

It can be seen that, we can construct a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ for any initial point x_0, converges to a fixed point of T. That is, T is a weakly Picard operator. ∎

Remark 1. Let A be a compact subset of a metric space (X, d) and $x \in X$, then there exists $a \in A$ such that $d(x, a) = d(x, A)$.

Remark 2. If we take $K(X)$ instead of $CB(X)$ in Theorem 13, we can remove the condition (F4) on F. Therefore, by taking into account Remark 1 the proof of the following theorem is obvious.

Theorem 14. *Let (X, d) be a complete metric space, $T : X \to K(X)$ and $F \in \mathcal{F}$. Assume that the following conditions hold:*

(i) *the map $x \to d(x, Tx)$ is lower semi-continuous;*

(ii) *there exists a function $\tau : (0, \infty) \to (0, \sigma]$, $\sigma > 0$ such that*

$$\liminf_{t\to s^+} \tau(t) > 0, \ \forall s \geq 0;$$

(iii) *for any $x \in X$ with $d(x, Tx) > 0$, there is $y \in Tx$ satisfying*

$$F(d(x,y)) \leq F(d(x,Tx)) + \frac{\tau(d(x,Tx))}{2}$$

and

$$\tau(d(x,Tx)) + F(d(y,Ty)) \leq F(d(x,y)).$$

Then T is a weakly Picard operator.

Taking into account our results, T is a weakly Picard operator in the following nontrivial example. We also show that Theorems 1, 3, 4, 5, 6, 8, 10, 12 can not be applied to this example.

Example 1. Let $X = \{\frac{1}{n^2} : n \in \mathbb{N}\} \cup \{0\}$ and $d(x,y) = |x-y|$, then (X,d) is complete metric space. Let $T : X \to CB(X)$ be defined by

$$Tx = \begin{cases} \left\{0, \dfrac{1}{(n+1)^2}\right\}, & x = \dfrac{1}{n^2} \\ \{x\}, & x \in \{0,1\} \end{cases}.$$

It is easy to see that

$$d(x,Tx) = \begin{cases} 0, & x \in \{0,1\} \\ \dfrac{2n+1}{n^2(n+1)^2}, & x = \dfrac{1}{n^2},\ n \geq 2 \end{cases}$$

and it is lower semi-continuous.

Let $\tau(t) = \ln 2$ and $\sigma = 4$, then the condition (ii) of Theorem 13 is satisfied.

Now we show that the condition (iii) of Theorem 13 is satisfied with

$$F(\alpha) = \begin{cases} \dfrac{\ln\alpha}{\sqrt{\alpha}}, & 0 < \alpha < e^2 \\ \alpha - e^2 + \frac{2}{e}, & \alpha \geq e^2 \end{cases}.$$

We can see that $F \in \mathcal{F}_*$. Note that $\sup_{x,y\in X} d(x,y) = 1 < e^2$. If $d(x,Tx) > 0$, then $x = \frac{1}{n^2}$, $n \geq 2$. Then we choose $y = \frac{1}{(n+1)^2} \in Tx = \left\{0, \frac{1}{(n+1)^2}\right\}$. Therefore, we have

$$d(x,y) = d(x,Tx) = \frac{2n+1}{n^2(n+1)^2}$$

and

$$d(y,Ty) = \frac{2n+3}{(n+1)^2(n+2)^2}.$$

Since $d(x,y) = d(x,Tx)$, (2.2) is clearly satisfied. To see (2.3), we must show that

$$\ln 2 + F(d(y,Ty)) \le F(d(x,y))$$

or equivalently

$$|y-Ty|^{\frac{1}{\sqrt{|y-Ty|}}} |x-y|^{-\frac{1}{\sqrt{|x-y|}}} \le \frac{1}{2}. \tag{2.15}$$

Now, for $x = \frac{1}{n^2}$ and $y = \frac{1}{(n+1)^2}$, we obtain

$$\begin{aligned}
&|y-Ty|^{\frac{1}{\sqrt{|y-Ty|}}} |x-y|^{-\frac{1}{\sqrt{|x-y|}}} \\
&= \left(\frac{2n+3}{(n+1)^2(n+2)^2}\right)^{\frac{(n+1)(n+2)}{\sqrt{2n+3}}} \left(\frac{2n+1}{n^2(n+1)^2}\right)^{-\frac{n(n+1)}{\sqrt{2n+1}}} \\
&= \left(\frac{2n+3}{(n+1)^2(n+2)^2}\right)^{\frac{(n+1)(n+2)}{\sqrt{2n+3}}} \left(\frac{2n+1}{n^2(n+1)^2}\frac{2n+3}{2n+3}\frac{(n+1)^2(n+2)^2}{(n+1)^2(n+2)^2}\right)^{-\frac{n(n+1)}{\sqrt{2n+1}}} \\
&= \left(\frac{2n+3}{(n+1)^2(n+2)^2}\right)^{\frac{(n+1)(n+2)}{\sqrt{2n+3}}-\frac{n(n+1)}{\sqrt{2n+1}}} \left(\frac{(2n+3)n^2(n+1)^2}{(2n+1)(n+1)^2(n+2)^2}\right)^{\frac{n(n+1)}{\sqrt{2n+1}}}.
\end{aligned}$$

On the other hand, for all $n \ge 2$, since

$$\frac{2n+3}{(n+1)^2(n+2)^2} \le \frac{1}{2},$$

$$\frac{(n+1)(2n+1)}{\sqrt{2n+3}} - \frac{n(n+1)}{\sqrt{2n+1}} \ge 1$$

and

$$\frac{(2n+3)n^2(n+1)^2}{(2n+1)(n+1)^2(n+2)^2} < 1,$$

then we have

$$|y-Ty|^{\frac{1}{\sqrt{|y-Ty|}}} |x-y|^{-\frac{1}{\sqrt{|x-y|}}} \le \frac{1}{2}.$$

Therefore (2.15) is satisfied. Thus all conditions of Theorem 13 are satisfied and so T has a fixed point in X.

Now we show that mentioned fixed point theorems can not be applied to this example.

Berinde-Berinde, Mizoguchi-Takahashi, Nadler [4, 10, 11].

Suppose that there exist a constant $L \ge 0$ and a function $\varphi : [0,\infty) \to [0,1)$ satisfying the assumptions in Theorem 8. Therefore, for $x = \frac{1}{n^2}$ and $y = \frac{1}{(n+1)^2}$, then $d(y,Tx) = 0$,

$$H(Tx,Ty) = \frac{2n+3}{(n+1)^2(n+2)^2} \text{ and } d(x,y) = \frac{2n+1}{n^2(n+1)^2}.$$

Thus

$$\begin{aligned} H(Tx,Ty) &\leq \varphi(d(x,y))d(x,y) + Ld(y,Tx) \\ \Leftrightarrow \frac{2n+3}{(n+1)^2(n+2)^2} &\leq \varphi(\frac{2n+1}{n^2(n+1)^2})\frac{2n+1}{n^2(n+1)^2} \\ \Leftrightarrow \frac{(2n+3)n^2}{(2n+1)(n+2)^2} &\leq \varphi(\frac{2n+1}{n^2(n+1)^2}). \end{aligned}$$

Taking limit supremum as $n \to \infty$, we have

$$1 \leq \limsup_{n\to\infty} \varphi(\frac{2n+1}{n^2(n+1)^2}) \leq \limsup_{t\to 0^+} \varphi(t) < 1,$$

which is a contradiction. Therefore Theorem 8 can not be applied to this example. Also, T is not multivalued almost contraction. Since Theorem 8 is a generalized version of Mizoguchi-Takahashi and Nadler fixed point theorems, these theorems can not be also applied to this example.

Klim-Wardowski, Feng-Liu [9, 7].

Suppose that there exist a constant $b \in (0,1)$ and a function $\varphi : [0,\infty) \to [0,1)$ satisfying the assumptions in Theorem 5. Take $x = \frac{1}{n^2}$, then $Tx = \{0, \frac{1}{(n+1)^2}\}$. If $y = 0$, then

$$bd(x,y) \leq d(x,Tx) \Leftrightarrow b \leq \frac{2n+1}{(n+1)^2}.$$

Taking limit $n \to \infty$, we have $b = 0$, which is a contradiction. If $y = \frac{1}{(n+1)^2}$, then

$$\begin{aligned} d(y,Ty) &\leq \varphi(d(x,y))d(x,y) \\ \Leftrightarrow \frac{2n+3}{(n+1)^2(n+2)^2} &\leq \varphi(\frac{2n+1}{n^2(n+1)^2})\frac{2n+1}{n^2(n+1)^2} \\ \Leftrightarrow \frac{(2n+3)n^2}{(2n+1)(n+2)^2} &\leq \varphi(\frac{2n+1}{n^2(n+1)^2}). \end{aligned}$$

Taking limit supremum as $n \to \infty$, we have

$$1 \leq \limsup_{n\to\infty} \varphi(\frac{2n+1}{n^2(n+1)^2}) \leq \limsup_{t\to 0^+} \varphi(t) < b,$$

which is a contradiction. Therefore Theorem 5 as well as Theorem 4 can not be applied to this example.

Ćirić [6].

Suppose that there exist a constant $a \in (0,1)$ and a function $\varphi : [0,\infty) \to [a,1)$ satisfying the assumptions in Theorem 6. Take $x = \frac{1}{n^2}$, then $Tx = \{0, \frac{1}{(n+1)^2}\}$. If

$y = 0$, then

$$\sqrt{\varphi(d(x,Tx))}d(x,y) \leq d(x,Tx) \Leftrightarrow \varphi(\frac{2n+1}{n^2(n+1)^2}) \leq \frac{(2n+1)^2}{(n+1)^4}.$$

Taking limit as $n \to \infty$, we have

$$0 < a \leq \lim_{n\to\infty} \varphi(\frac{2n+1}{n^2(n+1)^2}) \leq 0,$$

which is a contradiction. If $y = \frac{1}{(n+1)^2}$, then

$$\begin{aligned} d(y,Ty) &\leq \varphi(d(x,Tx))d(x,y) \\ \Leftrightarrow \frac{2n+3}{(n+1)^2(n+2)^2} &\leq \varphi(\frac{2n+1}{n^2(n+1)^2})\frac{2n+1}{n^2(n+1)^2} \\ \Leftrightarrow \frac{(2n+3)n^2}{(2n+1)(n+2)^2} &\leq \varphi(\frac{2n+1}{n^2(n+1)^2}). \end{aligned}$$

Taking limit supremum as $n \to \infty$, we have

$$1 \leq \limsup_{n\to\infty} \varphi(\frac{2n+1}{n^2(n+1)^2}) \leq \limsup_{t\to 0^+} \varphi(t) < 1,$$

which is a contradiction. Therefore Theorem 6 can not be applied to this example.

Olgun et al, Altun et al. [12, 2].

Since $H(T0,T1) = 1 = d(0,1)$, then for all $F \in \mathcal{F}_*$ and $\tau : (0,\infty) \to (0,\infty)$ satisfying inequality (1.4), we have

$$\tau(d(0,1)) + F(H(T0,T1)) > F(d(0,1)).$$

Therefore Theorem 12 can not be applied to this example. Also, T is not multivalued F-contraction.

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