

## NEW RESULTS ON SYSTEMS OF GENERALIZED VECTOR QUASI-EQUILIBRIUM PROBLEMS

Monica Patriche

**Abstract.** This article aims to demonstrate the existence of new solutions for the vector quasi-equilibrium problems. Firstly we prove the existence of the equilibrium for the generalized abstract economy model under upper semicontinuity assumptions. By using these results, we solve the announced problem in case of multivalued trifunctions. Secondly, we consider the generalized strong vector quasi-equilibrium problem and prove the existence of its solutions in case of correspondences being weakly naturally quasi-concave or weakly biconvex and also in case of weak-continuity assumptions. In all situations, our theoretical analysis is based on fixed-point theorems. Our study indicates that the refinement of the hypotheses concerning the equilibrium problems plays an important role in the developing of this theory and it improves, by its novelty, the existent results obtained so far in literature.

### 1. INTRODUCTION

Research studies on the vector equilibrium problem are gaining an increasingly greater attention, since it is a unified model of other several problems, for instance, vector variational inequalities, vector optimization problems or Debreu-type equilibrium problems. For further relevant information on this topic, the reader is referred to the following publications available in our bibliography: [1, 2, 4, 9, 11, 12, 15, 16, 17, 19, 20, 22, 23, 24, 25, 27, 31].

The current paper focuses on the solving of two types of vector equilibrium problems. The first one is defined by the existence of the multivalued trifunctions, the second one is known as generalized strong vector quasi-equilibrium problem. In the first instance, our research establishes the existence equilibrium theorems for the generalized abstract economy model which can be proved under upper continuity assumptions. This methodology follows an open direction in literature, but it still proves to

---

Received November 26, 2013, accepted June 11, 2014.

Communicated by Boris S. Mordukhovich.

2010 *Mathematics Subject Classification*: 49Y53, 54C60, 90C33.

*Key words and phrases*: Generalized abstract economy, Vector quasi-equilibrium problem, Existence of solutions, Ky-Fan fixed point theorem, Weakly naturally quasi-concave correspondence.

bring fruitful results related to the solvability of this particular problem. It is important to mention some milestones in the development of the theory, for a better understanding of how we deduced our new solutions. The generalized abstract economy model was introduced by Kim and Tan [18] and generalizes the previous models of abstract economies defined by Debreu [6], Shafer and Sonnenshine [31] or Borglin and Keiding [5]. Kim and Tan based their work on the fact that any preference of the real agents of an economy could be made unstable by the fuzziness of consumers' behaviour and of other market situations.

In the second instance, we report new results concerning the existence of the solutions for the generalized strong vector quasi-equilibrium problems with correspondences having new properties of generalized convexity or fulfilling weak-continuity assumptions. In all situations, our theoretical analysis is based on fixed-point theorems. Our study indicates that the refinement of the hypotheses concerning the equilibrium problems plays an important role in the developing of this theory and it improves, by its novelty, the existent results obtained so far in other publications.

The rest of the paper is organized as follows. Section 2 contains preliminaries and notations. In Sections 3 the equilibrium existence of the generalized abstract economy model is obtained. Section 4 studies the existence of solutions for systems of generalized vector quasi-equilibrium problems. Section 5 presents types of convexity conditions which are sufficient in order to guarantee that the generalized vector quasi-equilibrium problems can be solved. The case of weak-continuity assumptions is approached in the end.

## 2. PRELIMINARIES AND NOTATION

For the reader's convenience, we present several properties of the correspondences which are used in our proofs.

Let  $X$  be a subset of a topological vector space  $E$ . The set  $X$  is said to *have the property (K)* if, for every compact subset  $B$  of  $X$ , the convex hull  $\text{co}B$  is relatively compact in  $E$ . It is clear that each compact convex set in a Hausdorff (resp., locally) topological vector space always has property (K). A normal topological space in which each open set is an  $F_\sigma$  is called *perfectly normal*.

Let  $X, Y$  be topological spaces and  $T : X \rightarrow 2^Y$  be a correspondence.  $T$  is said to be *upper semicontinuous* if for each  $x \in X$  and each open set  $V$  in  $Y$  with  $T(x) \subset V$ , there exists an open neighborhood  $U$  of  $x$  in  $X$  such that  $T(y) \subset V$  for each  $y \in U$ .  $T$  is said to be *lower semicontinuous* if for each  $x \in X$  and each open set  $V$  in  $Y$  with  $T(x) \cap V \neq \emptyset$ , there exists an open neighborhood  $U$  of  $x$  in  $X$  such that  $T(y) \cap V \neq \emptyset$  for each  $y \in U$ .  $T$  is said to have *open lower sections* if  $T^{-1}(y) := \{x \in X : y \in T(x)\}$  is open in  $X$  for each  $y \in Y$ .  $T$  is said to be *compact* if, for any  $x \in X$ , there exists an open neighborhood  $V(x)$  such that  $T(N(x)) = \cup_{y \in N(x)} T(y)$  is relatively compact in  $Y$ .

The correspondence  $\bar{T}$  is defined by  $\bar{T}(x) := \{y \in Y : (x, y) \in \text{cl}_{X \times Y} \text{Gr } T\}$  (the set  $\text{cl}_{X \times Y} \text{Gr } (T)$  is called the adherence of the graph of  $T$ ). It is easy to see that  $\text{cl } T(x) \subset \bar{T}(x)$  for each  $x \in X$ .  $T$  is said to be *quasi-regular* if it has nonempty convex values, open lower sections and  $\bar{T}(x) = \text{cl}T(x)$  for each  $x \in X$ .  $T$  is said to be *regular* if it is quasi-regular and has an open graph.

If  $X$  and  $Y$  are topological vector spaces,  $K$  is a nonempty subset of  $X$ ,  $C$  is a nonempty closed convex cone and  $T : K \rightarrow 2^Y$  is a correspondence, then [26],  $T$  is called *upper  $C$ -continuous at  $x_0 \in K$*  if, for any neighborhood  $U$  of the origin in  $Y$ , there is a neighborhood  $V$  of  $x_0$  such that, for all  $x \in V$ ,  $T(x) \subset T(x_0) + U + C$ .  $T$  is called *lower  $C$ -continuous at  $x_0 \in K$*  if, for any neighborhood  $U$  of the origin in  $Y$ , there is a neighborhood  $V$  of  $x_0$  such that, for all  $x \in V$ ,  $T(x_0) \subset T(x) + U - C$ .

Now, we are presenting the approximation of upper semicontinuous correspondences as demonstrated by C. Ionescu-Tulcea.

Let  $X$  be a nonempty set, let  $Y$  be a nonempty subset of a topological vector space  $E$ , and let  $T : X \rightarrow 2^Y$ . A family  $(f_j)_{j \in J}$  of correspondences between  $X$  and  $Y$ , indexed by a nonempty filtering set  $J$  (denote by  $\leq$  the order relation in  $J$ ), is an *upper approximating family for  $F$*  [32] if (1)  $T(x) \subset f_j(x)$  for all  $x \in X$  and all  $j \in J$ ; (2) for each  $j \in J$  there is a  $j^* \in J$  such that, for each  $h \geq j^*$  and  $h \in J$ ,  $f_h(x) \subset f_j(x)$  for each  $x \in X$  and (3) for each  $x \in X$  and  $V \in \beta$ , where  $\beta$  is a base for the zero neighborhood in  $E$ , there is a  $j_{x,V} \in J$  such that  $f_h(x) \subset T(x) + V$  if  $h \in J$  and  $j_{x,V} \leq h$ . From (1)-(3), it is easy to deduce that for each  $x \in X$  and  $k \in J$ ,  $T(x) \subset \bigcap_{j \in J} f_j(x) = \bigcap_{k \leq j, k \in J} f_j(x) \subset \text{cl}T(x) \subset \bar{T}(x)$ .

Conditions for the existence of an approximating family for an upper semicontinuous correspondence are given in the following Lemma, which is obtained by observing Theorem 3 and the Remark of Tulcea [[32], p. 280 and pp 281-282].

**Lemma 1.** (see [7]). *Let  $(X_i)_{i \in I}$  be a family of paracompact spaces and let  $(Y_i)_{i \in I}$  be a family of nonempty closed convex subsets, each in a locally convex Hausdorff topological vector space and each has property  $(K)$ . For each  $i \in I$ , let  $T_i : X_i \rightarrow 2^{Y_i}$  be compact and upper semicontinuous with nonempty and convex values. Then, there is a common filtering set  $J$  (independent of  $i \in I$ ) such that, for each  $i \in I$ , there is a family  $(f_{ij})_{j \in J}$  of correspondences between  $X_i$  and  $Y_i$  with the following properties:*

- (i) for each  $j \in J$ ,  $(f_{ij})_{j \in J}$  is regular;
- (ii)  $(f_{ij})_{j \in J}$  and  $(\bar{f}_{ij})_{j \in J}$  are upper approximating families for  $F_i$ ;
- (iii) for each  $j \in J$ , the correspondence  $\bar{f}_{ij}$  is continuous if  $Y_i$  is compact.

The proofs of our results which use approximating methods are based on Lemma 1, which is a version of Lemma 1.1 in [33] (for  $D = Y$ , we obtain Lemma 1.1 in [33]).

**Lemma 2.** (see [29]). *Let  $X$  be a topological space,  $Y$  be a nonempty subset of a locally convex topological vector space  $E$  and  $T : X \rightarrow 2^Y$  be a correspondence. Let  $\beta$  be a basis of neighborhoods of 0 in  $E$  consisting of open absolutely convex symmetric sets. Let  $D$  be a compact subset of  $Y$ . If for each  $V \in \beta$ , the correspondence  $T^V : X \rightarrow 2^Y$  is defined by  $T^V(x) = (T(x) + V) \cap D$  for each  $x \in X$ , then  $\bigcap_{V \in \beta} \overline{T^V}(x) \subseteq \overline{T}(x)$  for every  $x \in X$ .*

Lemma 3 concerns the continuity of correspondences and it will be also crucial in our proofs.

**Lemma 3.** (see [33]). *Let  $X$  and  $Y$  be two topological spaces and let  $D$  be an open subset of  $X$ . Suppose  $T_1 : X \rightarrow 2^Y$ ,  $T_2 : X \rightarrow 2^Y$  are upper semicontinuous correspondences such that  $T_2(x) \subset T_1(x)$  for all  $x \in D$ . Then the correspondence  $T : X \rightarrow 2^Y$  defined by*

$$T(x) = \begin{cases} T_1(x), & \text{if } x \notin D, \\ T_2(x), & \text{if } x \in D \end{cases}$$

*is also upper semicontinuous.*

The property of properly  $C$ -quasiconvexity for correspondences is presented below.

Let  $X$  be a nonempty convex subset of a topological vector space  $E$ ,  $Z$  be a real topological vector space,  $Y$  be a subset of  $Z$  and  $C$  be a pointed closed convex cone in  $Z$  with its interior  $\text{int}C \neq \emptyset$ . Let  $T : X \rightarrow 2^Z$  be a correspondence with nonempty values.  $T$  is said to be *properly  $C$ -quasiconvex on  $X$* , if for any  $x_1, x_2 \in X$  and  $\lambda \in [0, 1]$ , either  $T(x_1) \subset T(\lambda x_1 + (1-\lambda)x_2) + C$  or  $T(x_2) \subset T(\lambda x_1 + (1-\lambda)x_2) + C$ .

### 3. EQUILIBRIUM EXISTENCE FOR GENERALIZED ABSTRACT ECONOMIES

Our research concerning the existence of the solutions for the vector quasi-equilibrium problems with multivalued trifunctions is mainly based on the existence equilibrium theorems for the generalized abstract economy, model which is defined below. It is characterized by the existence of the fuzzy constraint correspondences. Kim and Tan [18] asserts that their option for this types of correspondences is based on the fuzziness of consumers' behavior in a real market, which means that the preferences of the real economic agents are unstable. This model generalizes the previous ones defined by Debreu [6], Shafer and Sonnenshine [31] or Borglin and Keiding [5].

Before starting our exposure, we emphasize that the results from this section will be used in order to obtain the main theorems in Section 4, where we will consider systems of vector quasi-equilibrium problems under upper semicontinuity assumptions.

Let  $I$  be any set of agents (countable or uncountable). For each  $i \in I$ , let  $X_i$  be a nonempty set of actions available to the agent  $i$  in a topological vector space  $E_i$  and  $X = \prod_{i \in I} X_i$ .

**Definition 1.** [18]. A *generalized abstract economy*  $\Gamma = (X_i, A_i, F_i, P_i)_{i \in I}$  is defined as a family of ordered quadruples  $(X_i, A_i, F_i, P_i)$  where  $A_i : X \rightarrow 2^{X_i}$  is a constraint correspondence such that  $A_i(x)$  is the state attainable for the agent  $i$  at  $x$ ,  $F_i : X \rightarrow 2^{X_i}$  is a fuzzy constraint correspondence such that  $F_i(x)$  is the unstable state for the agent  $i$  and  $P_i : X \times X \rightarrow 2^{X_i}$  is a preference correspondence such that  $P_i(x, x)$  is the state preferred by the agent  $i$  at  $x$ .

**Definition 2.** An *equilibrium* for  $\Gamma$  is a point  $(x^*, y^*) \in X \times X$  such that for each  $i \in I$ ,  $x_i^* \in A_i(x^*)$ ,  $y_i^* \in F_i(x^*)$  and  $P_i(x^*, y^*) \cap A_i(x^*) = \emptyset$ .

If for each  $i \in I$  and each  $x \in X$ ,  $F_i(x) = X_i$  and the preference correspondence  $P_i$  satisfies  $P_i(x, y) = P_i(x, y')$  for each  $x, y, y' \in X$ , the definition of a generalized abstract economy and an equilibrium coincides with the usual definition of an abstract economy and an equilibrium established by Shafer and Sonnenschein [31].

The following theorem is the compact version of Theorem 5.1 of Lin et al. [21]. It states the existence of the equilibrium points for the generalized abstract economy with the set  $X$  being compact and the correspondences  $A_i$ ,  $F_i$  and  $P_i$  having open lower sections. Theorem 1 will be used to prove the existence of the equilibrium for the same model, but with correspondences being upper semicontinuous.

**Theorem 1.** For each  $i \in I$  ( $I$  finite), let  $X_i$  be a nonempty compact convex subset of a topological vectorspace  $E_i$ ,  $X = \prod_{i \in I} X_i$ ,  $A_i : X \rightarrow 2^{X_i}$  a constraint correspondence,  $P_i : X \times X \rightarrow 2^{X_i}$  a preference correspondence and  $F_i : X \rightarrow 2^{X_i}$  a fuzzy constraint correspondence. Assume that the following conditions are fulfilled:

- (i) For all  $x \in X$ ,  $A_i(x)$  and  $F_i(x)$  are nonempty and convex;
- (ii) For all  $y_i \in X_i$ ,  $A_i^{-1}(y_i)$ ,  $F_i^{-1}(y_i)$  and  $P_i^{-1}(y_i)$  are open sets (the correspondences  $A_i$ ,  $F_i$  and  $P_i$  have open lower sections);
- (iii) For all  $(x, y) \in X \times X$ ,  $x_i \notin \text{co}P_i(x, y)$ ;
- (iv) The set  $W_i := \{(x, y) \in X \times X : x_i \in A_i(x) \text{ and } y_i \in F_i(x)\}$  is closed in  $X \times X$ .

Then, there exists  $(x^*, y^*) \in X \times X$  such that for each  $i \in I$ ,  $x_i^* \in A_i(x^*)$ ,  $y_i^* \in F_i(x^*)$  and  $A_i(x^*) \cap P_i(x^*, y^*) = \emptyset$ .

We establish the following result which is an equilibrium existence theorem for a generalized abstract economy with upper semicontinuous correspondences. We use a method of approximation of upper semi-continuous correspondences developed by C. I. Tulcea [32]. Theorem 1 is also crucial for the demonstration.

**Theorem 2.** For each  $i \in I$  ( $I$  finite), let  $X_i$  be a nonempty compact convex subset with property (K) of a topological vector space  $E_i$ ,  $X = \prod_{i \in I} X_i$  be perfectly

normal,  $A_i : X \rightarrow 2^{X_i}$  a constraint correspondence,  $P_i : X \times X \rightarrow 2^{X_i}$  a preference correspondence and  $F_i : X \rightarrow 2^{X_i}$  a fuzzy constraint correspondence. Assume that the following conditions are fulfilled:

- (i) For all  $x \in X$ ,  $A_i(x)$  and  $F_i(x)$  are nonempty and convex;
- (ii) For all  $y_i \in X_i$ ,  $P_i^{-1}(y_i)$  are open sets and the correspondences  $F_i$ ,  $A_i$  are upper semicontinuous, compact, with nonempty convex closed values;
- (iii) For all  $(x, y) \in X \times X$ ,  $x_i \notin \text{co}P_i(x, y)$ ;
- (iv) The set  $U_i := \{(x, y) \in X \times X : P_i(x, y) \cap A_i(x) \neq \emptyset\}$  is open.

Then, there exists  $(x^*, y^*) \in X \times X$  such that for each  $i \in I$ ,  $x_i^* \in \overline{A_i}(x^*)$ ,  $y_i^* \in \overline{F_i}(x^*)$  and  $A_i(x^*) \cap P_i(x^*, y^*) = \emptyset$ .

*Proof.* Our approach and the upper approximation method require the application of Lemma 1. Hence, there is a common filtering set  $J$  such that, for every  $i \in I$ , there exists a family  $(A_{ij})_{j \in J}$  of regular correspondences between  $X$  and  $X_i$ , such that both  $(A_{ij})_{j \in J}$  and  $(\overline{A_{ij}})_{j \in J}$  are upper approximating families for  $A_i$  and a family  $(F_{ij})_{j \in J}$  of regular correspondences between  $X$  and  $X_i$ , such that both  $(F_{ij})_{j \in J}$  and  $(\overline{F_{ij}})_{j \in J}$  are upper approximating families for  $F_i$ . The correspondences  $A_{ij}$  and  $F_{ij}$  are regular, and it is clear that  $A_{ij}$  and  $F_{ij}$  have an open graph and thus, they have open lower sections.

The hypotheses guarantee that  $A_i$  and  $F_i$  have closed graphs, and this allows us to deduce that the set  $W_i := \{(x, y) \in X \times X : x_i \in A_i(x) \text{ and } y_i \in F_i(x)\}$  is closed in  $X \times X$ . Therefore, the abstract economy  $\Gamma_j = (X_i, A_{ij}, P_i, F_{ij})_{i \in I}$  satisfies all hypotheses of Theorem 1. Moreover, Theorem 1 implies that  $\Gamma_j$  has an equilibrium  $(x^{*j}, y^{*j}) \in X \times X$  such that  $A_{ij}(x^{*j}) \cap P_i(x^{*j}, y^{*j}) = \emptyset$ ,  $x_i^{*j} \in A_{ij}(x^{*j})$  and  $y_i^{*j} \in F_{ij}(x^{*j})$  for all  $i \in I$ .

The inclusion  $A_i(x^{*j}) \subset A_{ij}(x^{*j})$  implies that  $A_i(x^{*j}) \cap P_i(x^{*j}, y^{*j}) = \emptyset$ . Therefore,  $\{(x^{*j}, y^{*j})\}_{j \in I} \subset U_i^C$  follows straightforward from the last assertion. We remark that  $U_i^C$  is closed in  $X \times X$  by condition iv).

Furthermore, we exploit the fact that  $(x^{*j}, y^{*j})_{j \in I}$  is a net in the compact set  $X \times X$ ; without loss of generality, we may assume that  $(x^{*j})_{j \in I}$  converges to  $x^* \in X$  and  $(y^{*j})_{j \in I}$  converges to  $y^* \in X$ . Then, for each  $i \in I$ ,  $x_i^* = \lim_{j \in I} x_i^{*j}$  and  $y_i^* = \lim_{j \in I} y_i^{*j}$ . Clearly,  $(x^*, y^*) \in U_i^C$  for all  $i \in I$ , and we conclude that  $A_i(x^*) \cap P_i(x^*, y^*) = \emptyset$ .

Now, the conclusion of the theorem follows from the key idea of the method: each abstract economy  $\Gamma_j$  has an equilibrium point  $(x^{*j}, y^{*j})$  satisfying the properties  $x_i^{*j} \in A_{ij}(x^{*j}) \subset \overline{A_{ij}}(x^{*j})$  and  $y_i^{*j} \in \overline{F_{ij}}(x^{*j}) \subset \overline{F_{ij}}(x^{*j})$ . Given the closedness of the graphs of the correspondences  $\overline{A_{ij}}$  and  $\overline{F_{ij}}$ , we obtain  $(x^*, x_i^*) \in \text{Gr}\overline{A_{ij}}$  and  $(x^*, y_i^*) \in \text{Gr}\overline{A_{ij}}$  for every  $i \in I$ . Recall that for each  $i \in I$ ,  $(A_{ij})_{j \in I}$  is an upper approximation family for  $A_i$ , which guarantees that  $\bigcap_{j \in I} \overline{A_{ij}}(x) \subset \overline{A_i}(x)$  for each

$\in X$ , and then,  $(x^*, x_i^*) \in \text{Gr}\overline{A_i}$ . Similarly,  $(x^*, y_i^*) \in \text{Gr}\overline{F_i}$ . Finally, we complete the proof by summarizing that for each  $i \in I$ ,  $A_i(x^*) \cap P_i(x^*, y^*) = \emptyset$ ,  $x_i^* \in \overline{A_i}(x^*)$  and  $y_i^* \in \overline{F_i}(x^*)$ .

**Remark 1.** The main difference between Theorem 1 and Theorem 2 is the nature of the constraint correspondences  $F_i$  and  $A_i$ , which have open lower sections in the first case and are upper semicontinuous in the second case. In practice, the assumptions over constraints can vary widely, and the upper semicontinuity is a very common hypothesis in the results concerning the equilibrium existence. This requirement, together with the compactness of the spaces  $X_i$ , implies that  $F_i$  and  $A_i$  have a closed graph, and therefore, the assumption iv) from Theorem 1 is fulfilled. This way, we conclude that the set  $W_i := \{(x, y) \in X \times X : x_i \in A_i(x) \text{ and } y_i \in F_i(x)\}$  is closed in  $X \times X$ . We notice that the assumption iv) in Theorem 2 is different, since it asks that the set  $U_i := \{(x, y) \in X \times X : P_i(x, y) \cap A_i(x) \neq \emptyset\}$  to be open. This condition is also common for the theorems which state the existence of equilibrium for cases when the correspondences are upper semicontinuous. Some proofs use Lemma 3, where the set  $D$  is precisely  $U_i$ , and  $T_1$  and  $T_2$  are defined using the correspondences of the model. The distinction between the two hypothesis iv) comes from the various approaches which are used for demonstrations. In [21], the authors use maximal element theorems for a family of correspondences in order to prove their result (which we present as Theorem 1). In the proof of Theorem 2, we exploit the closedness of  $U_i^C$  ( $i \in I$ ), since we have a sequence of points in  $U_i^C$ , converging to the equilibrium pair  $(x^*, y^*)$ , which must belong also to each  $U_i^C$ .

The next results in this section prove the existence of equilibrium for generalized abstract economies. Theorem 3 is a replicated form of Theorem 2, but all the correspondences are upper semicontinuous.

**Theorem 3.** Let  $\Gamma = \{X_i, A_i, F_i, P_i\}_{i \in I}$  be a generalized abstract economy, where  $I$  is any index set, such that, for each  $i \in I$ :

- (i)  $X_i$  is a nonempty convex subset of a Hausdorff locally convex space  $E_i$ ,  $D_i$  is a nonempty compact subset of  $X_i$  and denote  $X = \prod_{i \in I} X_i$ ,  $D = \prod_{i \in I} D_i$ ;
- (ii)  $A_i : X \rightarrow 2^{D_i}$  is upper semicontinuous such that for each  $x \in X$ ,  $A_i(x)$  is a nonempty closed convex subset of  $X_i$ ;
- (iii)  $P_i : X \times X \rightarrow 2^{X_i}$  is upper semicontinuous such that for each  $x \in X$ ,  $P_i(x)$  is a nonempty closed convex subset of  $X_i$ ;
- (iv) the set  $W_i = \{(x, y) \in X \times X : A_i(x) \cap P_i(x, y) \neq \emptyset\}$  is open;
- (v) for each  $x \in W_i$ ,  $x_i \notin P_i(x, y)$ .

Then, there exists  $(x^*, y^*) \in D \times D$  such that  $x_i^* \in A_i(x^*)$ ,  $y^* \in F_i(x^*)$  and  $A_i(x^*) \cap P_i(x^*, y^*) = \emptyset$  for each  $i \in I$ .

*Proof.* First note that assumption i) implies that  $D$  is a nonempty compact subset of  $X$ . Let us focus on any player  $i \in I$ . We first define the correspondence  $G_i : X \times X \rightarrow 2^{D_i}$  as follows:

$$G_i(x, y) = A_i(x) \cap P_i(x, y) \text{ for each } (x, y) \in X \times X.$$

Clearly,  $G_i$  has closed convex values and it is upper semicontinuous.

In order to prove our result, we construct the correspondence  $T_i : X \times X \rightarrow 2^{D_i}$  by

$$T_i(x, y) = \begin{cases} (A_i(x) \cap P_i(x, y)) \times F_i(x), & \text{if } x \in W_i; \\ A_i(x) \times F_i(x), & \text{if } x \notin W_i. \end{cases}$$

We can apply Lemma 3 to assert that  $T_i$  is upper semicontinuous and has nonempty closed convex values.

Let us define the correspondence  $T : X \times X \rightarrow 2^D$ , by

$$T(x, y) = \prod_{i \in I} T_i(x, y) \text{ for each } x \in X.$$

Then,  $T$  is also upper semicontinuous and nonempty closed convex valued. Himmelberg's fixed point Theorem [14] indicates the existence of a fixed point for  $T$ . Thus, there exists  $(x^*, y^*) \in T(x^*, y^*)$ , where  $(x^*, y^*) \in D \times D$ . Obviously,  $x_i^* \in T_i(x^*)$  for each  $i \in I$ . Now, assumption v) guarantees that  $x_i^* \notin P_i(x^*, y^*)$ , and this implies  $x_i^* \in A_i(x^*)$ ,  $y_i^* \in F_i(x^*)$  and  $(A_i \cap P_i)(x^*) = \emptyset$  for each  $i \in I$ .

Theorem 4 is distinguished by the fact that no continuity assumptions are made over the constraint correspondences. It improves the known results established in literature by the reduction of the hypotheses and the new technique of proof mainly based on Lemma 2. We emphasize that this approximation methodology differs essentially from Ionescu-Tulcea's one and it will be largely used in the following sections of this paper.

**Theorem 4.** Let  $\Gamma = \{X_i, A_i, F_i, P_i\}_{i \in I}$  be a generalized abstract economy, where  $I$  is any index set, such that, for each  $i \in I$ :

- (i)  $X_i$  is a nonempty convex subset of a Hausdorff locally convex space  $E_i$ ,  $D_i$  is a nonempty compact subset of  $X_i$  and denote  $X = \prod_{i \in I} X_i$ ,  $D = \prod_{i \in I} D_i$ ;
- (ii)  $\overline{A_i^{V_i}}, \overline{F_i^{V_i}} : X \rightarrow 2^{D_i}$  are correspondences with nonempty convex values for each open absolutely convex symmetric neighborhood  $V_i$  of 0 in  $E_i$ ;
- (iii)  $P_i : X \times X \rightarrow 2^{X_i}$  is upper semicontinuous with nonempty closed convex values;
- (iv) the set  $W_i = \{(x, y) \in X \times X : A_i(x) \cap P_i(x, y) \neq \emptyset\}$  is open;
- (v) for each  $(x, y) \in W_i$ ,  $x_i \notin P_i(x, y)$ .

Then, there exists  $(x^*, y^*) \in D \times D$  such that  $x_i^* \in \overline{A_i}(x^*)$ ,  $y_i^* \in \overline{F_i}(x^*)$  and  $A_i(x^*) \cap P_i(x^*, y^*) = \emptyset$  for each  $i \in I$ .

*Proof.* We first note that assumption iv) asserts that  $W_i$  is open in  $X$  for each  $i \in I$ .

In order to use an approximation technique to prove our result, we will consider a basis  $\beta_i$  of open absolutely convex symmetric neighborhoods of 0 in  $E_i$  and we denote  $\beta = \prod_{i \in I} \beta_i$ .

The fixed point approach can be effectively exploited. To do this, we will construct several correspondences.

For each  $V = (V_i)_{i \in I} \in \prod_{i \in I} \beta_i$ , for each  $i \in I$ , let's define  $T_i^{V_i} : X \times X \rightarrow 2^{X_i}$  by

$$T_i^{V_i}(x, y) := \begin{cases} (\overline{A_i^{V_i}}(x) \cap P_i(x, y)) \times \overline{F_i^{V_i}}(x), & \text{if } (x, y) \in W_i, \\ \overline{A_i^{V_i}}(x) \times \overline{F_i^{V_i}}(x), & \text{if } (x, y) \notin W_i \end{cases} \quad \text{for each } (x, y) \in X \times X.$$

Assumption ii) ensures that each  $T_i^{V_i}$  is upper semicontinuous with nonempty closed convex values. Now, we define  $T^V : X \times X \rightarrow 2^D$  by

$$T^V(x, y) = \prod_{i \in I} T_i^{V_i}(x, y) \text{ for each } (x, y) \in X \times X.$$

The correspondence  $T^V$  is upper semicontinuous with nonempty closed convex values. The existence of a fixed point  $(x_V^*, y_V^*) = \prod_{i \in I} (x_V^*, y_V^*)_i \in D \times D$  for  $T^V$  is ensured by Himmelberg's fixed point Theorem [14]. Clearly,  $(x_V^*, y_V^*)_i \in \overline{T_i^{V_i}}(x_V^*, y_V^*)$  for each  $i \in I$ .

We will show the existence of the equilibrium points for the generalized abstract economy. Towards this end, we define for each  $V = (V_i)_{i \in I} \in \beta$ , the set

$$Q_V = \cap_{i \in I} \{(x, y) \in D \times D : (x_i, y_i) \in T_i^{V_i}(x, y)\}.$$

We note that  $Q_V$  is nonempty since  $(x_V^*, y_V^*) \in Q_V$ . The closedness of  $Q_V$  is also obvious.

We claim that the family  $\{Q_V : V \in \beta\}$  has the finite intersection property.

Indeed, let  $\{V^{(1)}, V^{(2)}, \dots, V^{(n)}\}$  be any finite set of  $\beta$  and let  $V^{(k)} = \prod_{i \in I} V_i^{(k)}$ ,  $k = 1, \dots, n$ . For each  $i \in I$ , let  $V_i = \bigcap_{k=1}^n V_i^{(k)}$ , then  $V_i \in \beta_i$ ; thus  $V = \prod_{i \in I} V_i \in \prod_{i \in I} \beta_i$ .

Clearly  $Q_V \subseteq \bigcap_{k=1}^n Q_{V^{(k)}}$  so that  $\bigcap_{k=1}^n Q_{V^{(k)}} \neq \emptyset$ .

Since  $D \times D$  is compact and the family  $\{Q_V : V \in \beta\}$  has the finite intersection property, the intersection over the entire collection of neighborhoods is nonempty:  $\cap\{Q_V : V \in \beta\} \neq \emptyset$ . Take any  $(x^*, y^*) \in \cap\{Q_V : V \in \beta\}$ , then for each  $V_i \in \beta_i$ ,  $(x_i^*, y_i^*) \in \overline{T_i^{V_i}}(x^*, y^*)$ . Lemma 2 implies that  $(x_i^*, y_i^*) \in \overline{T_i}(x^*, y^*)$  for each  $i \in I$ .

Finally, condition v) guarantees that  $x_i^* \in \overline{A_i}(x^*)$ ,  $y_i^* \in \overline{F_i}(x^*)$  and  $(A_i \cap P_i)(x^*, y^*) = \emptyset$  for each  $i \in I$ . This completes the proof.

#### 4. SYSTEMS OF VECTOR QUASI-EQUILIBRIUM PROBLEMS UNDER UPPER SEMICONTINUITY ASSUMPTIONS

This section is dedicated to establishing the main results of this paper. We state new theorems concerning the existence of the solutions for the systems of vector quasi-equilibrium problems which are presented below. Our research considers the continuity with respect to a cone of the correspondences  $f_i$ . Our study is split into two parts which are presented separately: Theorem 5 is proved under the assumption of lower  $(-C_i)$ -semicontinuity of  $f_i$  and in Theorem 6 we make the assumption of upper  $(C_i)$ -semicontinuity of  $f_i$ . Both theorems consider upper semicontinuous correspondences  $A_i$  and  $F_i : X \rightarrow 2^{X_i}$  and improve the existent results in literature. The last result of this section identifies a situation which has not been treated by prior studies: no continuity assumption is made over the correspondences  $A_i$  and  $F_i$ . An approximation technique, based on Lemma 2, is used in order to demonstrate our statement. The novelty of this paper can be measured in two standard procedures: the methodology used in our demonstrations and the assumptions used to formulate the theorems.

We start this section with the presentation of the problem we approach.

For each  $i \in I$ , let  $X_i$  be a nonempty subset of a topological vector space  $E_i$ ,  $Y_i$  a topological vector space and let  $X = \prod_{i \in I} X_i$  and  $C_i \subset X_i$  a closed cone with  $\text{int}C \neq \emptyset$ .

For each  $i \in I$ , let  $A_i, F_i : X \rightarrow 2^{X_i}$  and  $f_i : X \times X \times X_i \rightarrow 2^{X_i}$  be correspondences with nonempty values. We consider the following systems of generalized vector quasi-equilibrium problems (in short, SGVQEP (I)):

Find  $(x^*, y^*) \in X \times X$  such that for each  $i \in I$ ,  $x_i^* \in \overline{A_i}(x^*)$ ,  $y_i^* \in \overline{F_i}(x^*)$  and  $f_i(x^*, y^*, u_i) \subseteq C_i$  for each  $u_i \in A_i(x^*)$ .

The next theorems establish the existence of the solutions for systems of vector quasi-equilibrium problem.

The highlights of our first result of this section include the upper semicontinuity (resp. lower  $(-C_i)$ -semicontinuity) of  $F_i$  and  $A_i : X \rightarrow 2^{X_i}$  (resp.  $f_i : X \times X \times X_i \rightarrow 2^{X_i}$ ). The  $C_i$ -quasi-convexity of  $f_i(x, y, \cdot) : X_i \rightarrow 2^{X_i}$  is also assumed.

**Theorem 5.** *For each  $i \in I$  ( $I$  finite), let  $X_i$  be a nonempty compact convex subset of a locally convex Hausdorff topological vector space  $E_i$ . Let  $X = \prod_{i \in I} X_i$  be perfectly normal and  $C_i$  a closed cone with  $\text{int}C_i \neq \emptyset$ . Let  $f_i : X \times X \times X_i \rightarrow 2^{X_i}$  be a correspondence with nonempty values. For each  $i \in I$ , assume that:*

- (i)  $F_i, A_i : X \rightarrow 2^{X_i}$  are upper semicontinuous correspondences with nonempty closed convex values;
- (ii) for all  $x, y \in X$ ,  $f_i(x, y, x_i) \subseteq C_i$ ;
- (iii)  $f_i(\cdot, \cdot, \cdot)$  is lower  $(-C_i)$ -semicontinuous;
- (iv) for each  $x, y \in X$ , the correspondence  $f_i(x, y, \cdot)$  is  $C_i$ -quasi-convex;

(v)  $U_i = \{(x, y) \in X \times X : \text{there exists } u_i \in A_i(x) \text{ such that } f_i(x, y, u_i) \subseteq C_i\}$  is open.

Then, there exists a solution  $(x^*, y^*) \in X \times X$  of (SGVQEP)(I).

*Proof.* The key idea of the proof is to use Theorem 2. Towards this end, we need the construction of the following correspondences.

For each  $i \in I$ , let  $P_i : X \times X \rightarrow 2^{X_i}$  be defined by

$$P_i(x, y) = \{u_i \in X_i : f_i(x, y, u_i) \not\subseteq C_i\} \text{ for each } (x, y) \in X \times X.$$

We will show that  $P_i$  has an open graph and convex values.

Firstly we are proving the convexity of  $P_i(x_0, y_0)$ , where  $(x_0, y_0) \in X \times X$  is arbitrary fixed. Indeed, let us consider  $u_1, u_2 \in P_i(x_0, y_0)$  and  $\lambda \in [0, 1]$ . Since  $u_1, u_2 \in X_i$  and the set  $X_i$  is convex, the convex combination  $u = \lambda u_1 + (1 - \lambda)u_2$  is an element of  $X_i$ .

Further, by using the property of properly  $C$ -quasiconvexity of  $f_i(x_0, y_0, \cdot)$ , we can assume, without loss of generality, that  $f_i(x_0, y_0, u_1) \subset f_i(x_0, y_0, u) + C_i$ .

We will prove that  $u \in P_i(x_0, y_0)$ . If, by contrary,  $u \notin P_i(x_0, y_0)$ , then,  $f_i(x_0, y_0, u) \subseteq C_i$  and, therefore,  $f_i(x_0, y_0, u_1) \subset f_i(x_0, y_0, u) + C_i \subseteq C_i + C_i \subseteq C_i$ , which contradicts  $u_1 \in P_i(x_0, y_0)$ . Hence,  $u \in P_i(x_0, y_0)$  and, consequently,  $P_i(x_0, y_0)$  is a convex set.

Assumption ii) asserts that  $x_i \notin P_i(x, y)$  for each  $(x, y) \in X \times X$ .

The closedness of the  $(\text{Gr}P_i)^C$  will be shown now.

We consider the net  $\{(x_\alpha, y_\alpha, u_\alpha) : \alpha \in \Lambda\} \subset (\text{Gr}P_i)^C$ , such that  $(x_\alpha, y_\alpha, u_\alpha) \rightarrow (x_0, y_0, u_0) \in X \times X \times X_i$ . Then,  $u_\alpha \notin P_i(x_\alpha, y_\alpha)$  for each  $\alpha \in \Lambda$ , i.e.  $f_i(x_\alpha, y_\alpha, u_\alpha) \subseteq C_i$ . We prove that  $(x_0, y_0, u_0) \in (\text{Gr}P_i)^C$ , that is  $u_0 \notin P_i(x_0, y_0)$ . We use the lower  $(-C_i)$ -continuity of  $F$  and we conclude that, for each neighborhood  $U$  of the origin in  $X_i$ , there exists a neighborhood  $V(x_0, y_0, u_0)$  of  $(x_0, y_0, u_0)$  such that,  $f_i(x_0, y_0, u_0) \subset f_i(x_\alpha, y_\alpha, u_\alpha) + U + C_i$  for each  $(x_\alpha, y_\alpha, u_\alpha) \in V(x_0, y_0, u_0)$ . Then, for each  $(x, y, u) \in V(x_0, y_0, u_0)$ ,  $f_i(x_0, y_0, u_0) \subset C_i + U + C_i \subset C_i + U$ .

We will prove that  $f_i(x_0, y_0, u_0) \subseteq C_i$ . If, by contrary, there exists  $a \in f_i(x_0, y_0, u_0)$  and  $a \notin C_i$ , then,  $0 \notin B := C_i - a$  and  $B$  is closed. Thus,  $B$  is open and  $0 \in X_i \setminus B$ . There exists an open symmetric neighborhood  $U_1$  of the origin in  $X_i$ , such that  $U_1 \subset X_i \setminus B$  and  $U_1 \cap B$  is closed. Therefore,  $0 \notin B + U_1$ , i.e.,  $a \notin C_i + U_1$ , which contradicts  $f_i(x_0, y_0, z_0) \subset U_1 + C_i$ . We conclude that  $f_i(x_0, y_0, z_0) \subseteq C_i$  and then,  $u_0 \in P_i(x_0, y_0)^C$  and  $(\text{Gr}P_i)^C$  is closed. Therefore,  $\text{Gr}P_i$  is open and  $P_i$  has open lower sections.

According to vi),  $U_i = \{(x, y) \in X \times X : A_i(x) \cap P_i(x, y) \neq \emptyset\}$  is open and according to ii),  $x_i \notin P_i(x, y)$  for each  $(x, y) \in X \times X$ .

All the assumptions of Theorem 2 are fulfilled. Then, there exists  $(x^*, y^*) \in X \times X$  such that for each  $i \in I$ ,  $A_i(x^*) \cap P_i(x^*, y^*) = \emptyset$ ,  $x_i^* \in \overline{A_i(x^*)}$  and  $y_i^* \in \overline{F_i(x^*)}$ . Consequently, there exist  $x^*, y^* \in X$  such that  $x_i^* \in \overline{A_i(x^*)}$ ,  $y_i^* \in \overline{F_i(x^*)}$  and  $f_i(x^*, y^*, u) \subseteq C_i$  for each  $u \in A_i(x^*)$ .

**Remark 2.** We note that Theorem 5 differs from Theorem 3.2.1 in [22] in the following way: the correspondences  $A_i$  and  $F_i$  are upper semicontinuous and  $f_i$  is lower  $(-C_i)$ -continuous for each  $i \in I$ .

Theorem 6 is stated in terms of upper  $(C_i)$ -semicontinuity for  $f_i : X \times X \times X_i \rightarrow 2^{X_i}$  and upper semicontinuity for  $F_i$  and  $A_i$ . Its proof is mainly based on Theorem 3.

**Theorem 6.** For each  $i \in I$ , let  $X_i$  be a nonempty convex subset of a Hausdorff locally convex space  $E_i$ ,  $D_i$  a nonempty compact subset of  $X_i$  and denote  $X = \prod_{i \in I} X_i$ ,  $D = \prod_{i \in I} D_i$ . Let  $f_i : X \times X \times X_i \rightarrow 2^{X_i}$  be correspondence with nonempty values. For each  $i \in I$ , assume that:

- (i)  $F_i, A_i : X \rightarrow 2^{X_i}$  are upper semicontinuous correspondences with nonempty closed convex values;
- (ii) for each  $x, y \in X$ ,  $f_i(x, y, x_i) \subseteq \text{int}C_i$ ;
- (iii) for each  $(x, y) \in X \times X$ ,  $f_i(x, y, \cdot)$  is upper semicontinuous;
- (iv)  $f_i(\cdot, \cdot, \cdot)$  is upper  $(C_i)$ -semicontinuous;
- (v) for each  $x, y \in X$ , the correspondence  $f_i(x, y, \cdot)$  is  $C_i$ -quasi-convex;
- (vi)  $U_i = \{(x, y) \in X \times X : \text{there exists } u_i \in A_i(x) \text{ such that } f(x, y, u_i) \subseteq \text{int}C_i\}$  is open.

Then, there exists  $(x^*, y^*) \in X \times X$  such that  $x_i^* \in \overline{A_i}(x^*)$ ,  $y_i^* \in \overline{F_i}(x^*)$  and  $f_i(x^*, y^*, u) \subseteq \text{int}C_i$  for each  $u \in A_i(x^*)$ .

*Proof.* The proof requires the application of Theorem 3 for a generalized abstract economy which we intend to construct. We need to define the preference correspondences. Actually, for each  $i \in I$ , let  $P_i : X \times X \rightarrow 2^{X_i}$  be defined by

$$P_i(x, y) = \{u_i \in X_i : f_i(x, y, u_i) \subseteq \text{int}C_i\} \text{ for each } (x, y) \in X \times X.$$

We will check that the assumptions of Theorem 3 are fulfilled.

Firstly, we will prove that  $P_i$  has a closed graph and nonempty closed convex values.

Let us fix  $(x_0, y_0) \in X \times X$ .

In order to show the convexity of  $P_i(x_0, y_0)$ , let us consider  $u_1, u_2 \in P_i(x_0, y_0)$  and  $\lambda \in [0, 1]$ . Let  $u$  be the convex combination  $u = \lambda u_1 + (1 - \lambda)u_2 \in X_i$ . Further, in virtue of the property of properly  $C$ -quasiconvexity of  $f_i(x_0, y_0, \cdot)$ , we can assume, without loss of generality, that  $f_i(x_0, y_0, u_1) \subset f_i(x_0, y_0, u) + C_i$ .

We will prove that  $u \in P_i(x_0, y_0)$ . If, by contrary,  $u \notin P_i(x_0, y_0)$ ,  $f_i(x_0, y_0, u) \subseteq \text{int}C_i$  and, consequently,  $f_i(x_0, y_0, u_1) \subset f_i(x_0, y_0, u) + C_i \subseteq \text{int}C_i + C_i \subseteq \text{int}C_i$ , which contradicts  $u_1 \in P_i(x_0, y_0)$ . We conclude that  $u \in P_i(x_0, y_0)$  which implies the convexity of  $P_i(x_0, y_0)$ .

Further, we will prove that  $P_i(x_0, y_0)$  is closed.

Towards this end, we consider the net  $\{u_\alpha : \alpha \in \Lambda\} \subseteq P_i(x_0, y_0)$  such that  $u_\alpha \rightarrow u_0$ . Then,  $u_\alpha \in X_i$  and  $f_i(x_0, y_0, u_\alpha) \not\subseteq \text{int}C_i$  for all  $\alpha \in \Lambda$ . It is clear from the closedness of  $X_i$  that  $u_0 \in X_i$ . We assume, by contrary, that  $f_i(x_0, y_0, u_0) \subseteq \text{int}C_i$ . Since  $f_i(x_0, y_0, \cdot)$  is upper semicontinuous, then,  $f_i(x_0, y_0, u_\alpha) \subset \text{int}C_i$  for  $\alpha \geq \alpha_0$ ,  $\alpha_0 \in \Lambda$ , which is a contradiction. Therefore, our assumption is false and  $f_i(x_0, y_0, u_0) \not\subseteq \text{int}C_i$ , i.e.  $u_0 \in P_i(x_0, y_0)$  and  $P_i(x_0, y_0)$  is a closed set.

Now, the closedness of  $P_i$  will be shown. We consider the net  $\{(x_\alpha, y_\alpha, u_\alpha) : \alpha \in \Lambda\} \subset \text{Gr}P_i$  such that  $(x_\alpha, y_\alpha, u_\alpha) \rightarrow (x_0, y_0, u_0) \in X \times X \times X_i$ . Then,  $u_\alpha \in P_i(x_\alpha, y_\alpha)$  for each  $\alpha \in \Lambda$  and we prove that  $(x_0, y_0, u_0) \in \text{Gr}P_i$ , that is  $u_0 \in P_i(x_0, y_0)$ . If, by contrary,  $u_0 \notin P_i(x_0, y_0)$ , then,  $f_i(x_0, y_0, u_0) \subseteq \text{int}C_i$ . This relation implies that there exists a neighborhood  $U_0$  of the origin in  $Z$  such that  $f_i(x_0, y_0, u_0) + U_0 \subset \text{int}C_i$ .

Further, the upper  $C_i$ -continuity of  $f_i$  yields the existence of a neighborhood  $V(x_0, y_0, u_0)$  of  $(x_0, y_0, u_0)$  such that,  $f_i(x, y, u) \subset f_i(x_0, y_0, u_0) + U_0 + C$  for each  $(x, y, u) \in V(x_0, y_0, u_0)$ . The last assertion implies that, for each  $(x, y, u) \in V(x_0, y_0, u_0)$ ,  $f_i(x, y, u) \subset \text{int}C_i + C_i \subset \text{int}C_i$ , which guarantees the existence of  $\alpha_0 \in \Lambda$  such that for each  $\alpha \geq \alpha_0$ ,  $f_i(x_\alpha, y_\alpha, u_\alpha) \subset \text{int}C_i$ .

The last relation contradicts  $u_\alpha \in P_i(x_\alpha, y_\alpha)$ . Consequently, the assumption that  $u_0 \notin P_i(x_0, y_0)$  is false. Since  $u_0 \in P_i(x_0, y_0)$ ,  $\text{Gr}P_i$  is closed, and, since  $X_i$  is compact, it is clear that  $P_i$  is upper semicontinuous.

According to vi),  $U_i = \{(x, y) \in X \times X : A_i(x) \cap P_i(x, y) \neq \emptyset\}$  is open and according to ii),  $x_i \notin P_i(x, y)$  for each  $(x, y) \in X \times X$ .

All the assumptions of Theorem 3 are fulfilled and we can apply it to assert the existence of a pair  $(x^*, y^*)$  such that for each  $i \in I$ ,  $A_i(x^*) \cap P_i(x^*, y^*) = \emptyset$ ,  $x_i^* \in \overline{A_i}(x^*)$  and  $y_i^* \in \overline{F_i}(x^*)$ . This means that there exist  $x^*, y^* \in X$  such that  $x_i^* \in \overline{A_i}(x^*)$ ,  $y_i^* \in \overline{F_i}(x^*)$  and  $f(x^*, y^*, u) \subseteq \text{int}C_i$  for each  $u \in A_i(x^*)$ . The proof is complete.

In order to demonstrate the last theorem of this section, an approximating technique will be used. The proof is based on Theorem 4. It is shown that the continuity assumptions over the correspondences  $F_i$  and  $A_i$  can be avoided. New types of hypotheses lead to a result which is weaker than the previous ones.

**Theorem 7.** For each  $i \in I$ , let  $X_i$  be a nonempty convex subset of a Hausdorff locally convex space  $E_i$ ,  $D_i$  a nonempty compact subset of  $X_i$  and denote  $X = \prod_{i \in I} X_i$ ,  $D = \prod_{i \in I} D_i$ . Let  $f_i : X \times X \times X_i \rightarrow 2^{X_i}$  be a lower semicontinuous correspondence with nonempty values. For each  $i \in I$ , assume that:

- (i)  $\overline{F_i^{V_i}}, \overline{A_i^{V_i}} : X \rightarrow 2^{X_i}$  are correspondences with nonempty convex values for each open absolutely convex symmetric neighborhood  $V_i$  of 0 in  $E_i$ ;
- (ii) for all  $x, y \in X$ ,  $f_i(x, y, x_i) \subseteq \text{int}C_i$ ;
- (iii) for each  $(x, y) \in X \times X$ ,  $f_i(x, y, \cdot)$  is upper semicontinuous;
- (iv)  $f_i(\cdot, \cdot, \cdot)$  is upper  $(C_i)$ -semicontinuous;

- (v) for each  $x, y \in X$ , the correspondence  $f_i(x, y, \cdot)$  is  $C_i$ -quasi-convex;  
 (vi)  $U_i = \{(x, y) \in X \times X : \text{there exists } u_i \in A_i(x) \text{ such that } f(x, y, u_i) \subseteq \text{int}C_i\}$   
 is open.

Then, there exists  $(x^*, y^*) \in X \times X$  such that  $x_i^* \in \overline{A_i}(x^*)$ ,  $y_i^* \in \overline{F_i}(x^*)$  and  $f_i(x^*, y^*, u) \subseteq \text{int}C_i$  for each  $u \in A_i(x^*)$ .

*Proof.* We intend to derive the conclusion by using Theorem 4. In order to be in the setting of this last theorem, we have to construct the preference correspondences  $P_i$ .

Therefore, for each  $i \in I$ , let  $P_i : X \times X \rightarrow 2^{X_i}$  be defined by

$$P_i(x, y) = \{u_i \in X_i : f_i(x, y, u_i) \not\subseteq \text{int}C_i\} \text{ for each } (x, y) \in X \times X.$$

By the same reasoning as in the proof of Theorem 5, we can easily show that  $P_i$  is upper semicontinuous with nonempty closed convex values.

Firstly we note that assumption vi) asserts that  $U_i = \{(x, y) \in X \times X : A_i(x) \cap P_i(x, y) \neq \emptyset\}$  is open and assumption ii) assures that  $x_i \notin P_i(x, y)$  for each  $(x, y) \in X \times X$ .

All the assumptions of Theorem 4 are fulfilled. We shall complete the proof by applying Theorem 4 to show the existence of a pair  $(x^*, y^*) \in X \times X$  such that for each  $i \in I$ ,  $A_i(x^*) \cap P_i(x^*, y^*) = \emptyset$ ,  $x_i^* \in \overline{A_i}(x^*)$  and  $y_i^* \in \overline{F_i}(x^*)$ . Together with the fact that  $f_i(x^*, y^*, u) \subseteq \text{int}C_i$  for each  $u \in A_i(x^*)$ , the last assertion implies the existence of the solutions for the considered vector equilibrium problem.

## 5. STRONG VECTOR QUASI-EQUILIBRIUM PROBLEMS

This section is dedicated to the study of the strong vector equilibrium problem. Its originality consists in a new manner of treating this topic, by considering correspondences which fulfill no continuity assumptions. Instead, new types of convexities assumptions are made. In order to establish the announced results, some new types of generalized convexities introduced by the author are used.

In this particular section, we examine the following problem.

Let us consider  $E_1, E_2$  and  $Z$  be topological vector spaces, let  $K \subset E_1$ ,  $D \subset E_2$  be subsets and  $C_i \subset Z$  a nonempty closed convex cone. Let us also consider the correspondences  $A : K \rightarrow 2^K$ ,  $F : K \rightarrow 2^D$  and  $f : K \times D \times K \rightarrow 2^Z$ .

We will study the existence of the solutions for the following extension of the generalized strong vector quasi-equilibrium problem (shortly, GSVQEP): finding  $x^* \in K$  and  $y^* \in \overline{F}(x^*)$  such that  $x^* \in \overline{A}(x^*)$  and  $f_i(x^*, y^*, z) \subset C_i, \forall z \in \overline{A}(x^*)$ , where the correspondence  $\overline{A}$  is defined by  $\overline{A}(x) = \{y \in Y : (x, y) \in \text{cl}_{X \times Y} \text{Gr}A\}$ . Note that  $\text{cl}A(x) \subset \overline{A}(x)$  for each  $x \in X$ .

The element  $x^*$  will be called a strong solution for the GSVQEP and the set of all strong solutions for the GSVQEP will be denoted by  $V_A(f)$ .

**5.1. Strong vector quasi-equilibrium problems without continuity assumptions**

In this subsection, we demonstrate that the set  $V_A(f)$  of all strong solutions for the GSVQEP is nonempty also when the correspondences do not satisfy continuity assumptions defined explicitly. Instead, we will use some conditions concerning generalized convexity, mainly, the weakly naturally quasi-concavity property of correspondences. Other assumptions refer to correspondences with weakly convex graphs. We present these notions below.

Let us denote  $\Delta_{n-1} = \left\{ (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n : \sum_{i=1}^n \lambda_i = 1 \text{ and } \lambda_i \geq 0, i = 1, 2, \dots, n \right\}$  the standard (n-1)-dimensional simplex in  $\mathbb{R}^n$ .

The correspondence  $F : X \rightarrow 2^Y$  is said to have *weakly convex graph* [8] if for each  $n \in N$  and for each finite set  $\{x_1, x_2, \dots, x_n\} \subset X$ , there exists  $y_i \in F(x_i)$ , ( $i = 1, 2, \dots, n$ ) such that

$$(1.1) \quad \text{co}(\{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\}) \subset \text{Gr}(F)$$

The relation (1.1) is equivalent to

$$(1.2) \quad \sum_{i=1}^n \lambda_i y_i \in F\left(\sum_{i=1}^n \lambda_i x_i\right) \quad (\forall (\lambda_1, \lambda_2, \dots, \lambda_n) \in \Delta_{n-1}).$$

We introduced in [29] the weakly naturally quasi-concave correspondences.

**Definition 3.** (see [29]). Let  $X$  be a nonempty convex subset of a topological vector space  $E$  and  $Y$  a nonempty subset of a topological vector space  $Z$ . The correspondence  $F : X \rightarrow 2^Y$  is said to be *weakly naturally quasi-concave (WNQ)* iff for each  $n$  and for each finite set  $\{x_1, x_2, \dots, x_n\} \subset X$ , there exists  $y_i \in F(x_i)$ ,  $i \in \{1, \dots, n\}$  and  $g = (g_1, g_2, \dots, g_n) : \Delta_{n-1} \rightarrow \Delta_{n-1}$  a mapping with  $g_i$  continuous,  $g_i(1) = 1$  and  $g_i(0) = 0$  for each  $i \in \{1, 2, \dots, n\}$ , such that  $\sum_{i=1}^n g_i(\lambda_i) y_i \in F(\sum_{i=1}^n \lambda_i x_i)$  for every  $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \Delta_{n-1}$ .

**Example 1.** (see [20]) Let  $F : [0, 4] \rightarrow 2^{[-2, 2]}$  be defined by

$$F(x) = \begin{cases} [0, 2] & \text{if } x \in [0, 2); \\ [-2, 0] & \text{if } x = 2; \\ (0, 2] & \text{if } x \in (2, 4]. \end{cases}$$

$F$  is neither upper semicontinuous, nor lower semicontinuous in 2.  $F$  is weakly naturally quasi-concave.

The next theorem is our first result concerning the existence of the strong solutions for the GSVQEP. The proof consists in the construction of a continuous selection for a correspondence and then, Brouwer’s fixed point Theorem is applied. The theorems consider the weakly naturally quasi-concave correspondences.

**Theorem 8.** Let  $E_1, E_2, Z$  be topological vector spaces,  $K \subset E_1$  and  $D \subset E_2$  be subsets. Let  $L$  be a simplex in  $K \times D$  and denote  $L_K = \text{pr}_K L$ . Let  $(A, F) : L_K \rightarrow 2^L$  be weakly naturally quasi-concave. Let us suppose that, for each  $n \in \mathbb{N}$ ,  $\lambda \in \Delta_{n-1}$  and  $x_1, x_2, \dots, x_n \in L_K$ ,  $A(\sum_{i=1}^n \lambda_i x_i) \subset \bigcap_{i=1}^n A(x_i)$ . Let  $f : L \times K \rightarrow 2^K$  a correspondence such that the following assumptions are fulfilled:

- (i)  $\forall (x, y) \in L, f(x, y, A(x)) \subset C$ ;
- (ii) for each  $z \in K$ , for each  $n \in \mathbb{N}$ ,  $\lambda, \lambda' \in \Delta_{n-1}$  and  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n) \in L$ ,  $f(\sum_{i=1}^n \lambda_i x_i, \sum_{i=1}^n \lambda'_i y_i, z) \subset \bigcap_{i=1}^n f(x_i, y_i, z)$ .

Then,  $V_f \neq \emptyset$ .

*Proof.* The fixed point approach is effectively exploited to solve the announced problem. The proof is based on the construction of a continuous selection of a weakly naturally quasi-concave correspondence which we will firstly define and also on the application of the Brouwer fixed point Theorem.

Let us start by defining the correspondences  $P : L \rightarrow 2^K$  and  $M : L \rightarrow 2^L$  as follows:

$$P(x, y) = \{u \in A(x) : f(u, y, z) \subset C \ \forall z \in A(x)\} \text{ and}$$

$$M(x, y) = (P(x, y), F(x)).$$

We claim that  $M$  is weakly naturally quasi-concave.

Indeed, let us consider  $n \in \mathbb{N}$ ,  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n) \in L$ . For each  $i = 1, 2, \dots, n$ , there exists  $(u_i, v_i) \in M(x_i, y_i)$ , that is  $u_i \in A(x_i)$  and  $f(u_i, y_i, z) \subset C \ \forall z \in A(x_i)$  and  $v_i \in F(x_i)$ . Let  $\lambda \in \Delta_{n-1}$  such that  $\sum_{i=1}^n \lambda_i (x_i, y_i) \in L$  and let us denote  $(u, v) = \sum_{i=1}^n \lambda_i (u_i, v_i)$ .

The hypotheses assures that  $(A, F)$  is weakly naturally quasiconcave and in the virtue of this property, there exists  $g = (g_1, g_2, \dots, g_n) : \Delta_{n-1} \rightarrow \Delta_{n-1}$  a function depending on  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$  with  $g_i$  continuous,  $g_i(1) = 1, g_i(0) = 0$  for each  $i = 1, 2, \dots, n$ , such that for every  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \Delta_{n-1}$ ,  $\sum_{i=1}^n g_i(\lambda_i) (u_i, v_i) \in$

$$(A, F)(\sum_{i=1}^n \lambda_i x_i).$$

Given assumption ii) in the statement of the theorem, we can write:

$$f(\sum_{i=1}^n g_i(\lambda_i) u_i, \sum_{i=1}^n \lambda_i y_i, z) \subset f(u_i, y_i, z) \subset C.$$

Naturally,  $f(\sum_{i=1}^n g_i(\lambda_i) u_i, \sum_{i=1}^n \lambda_i y_i, z) \subset C \ \forall z \in A(\sum_{i=1}^n \lambda_i x_i)$ . Now, having shown

that  $A(\sum_{i=1}^n \lambda_i x_i) \subset A(x_i)$  for each  $i \in \{1, 2, \dots, n\}$ , we obtain  $f(\sum_{i=1}^n g_i(\lambda_i)u_i, \sum_{i=1}^n \lambda_i y_i, z) \subset C \forall z \in A(x_i)$ .

Therefore,  $P$  is weakly naturally quasi-concave.

We will use the last assertion in order to prove the weakly naturally quasi-concavity of  $M$ .

Towards this end, we first notice that

$$\sum_{i=1}^n g_i(\lambda_i)u_i \in A(\sum_{i=1}^n \lambda_i x_i), f(\sum_{i=1}^n g_i(\lambda_i)u_i, \sum_{i=1}^n \lambda_i y_i, z) \subset C \forall z \in A(\sum_{i=1}^n \lambda_i x_i) \text{ and}$$

$$\sum_{i=1}^n g_i(\lambda_i)v_i \in F(\sum_{i=1}^n \lambda_i x_i), \text{ that is}$$

$$\sum_{i=1}^n g_i(\lambda_i)u_i \in P(\sum_{i=1}^n \lambda_i x_i, \sum_{i=1}^n \lambda_i y_i) \text{ and } \sum_{i=1}^n g_i(\lambda_i)v_i \in F(\sum_{i=1}^n \lambda_i x_i).$$

$$\text{Consequently, } \sum_{i=1}^n g_i(\lambda_i)(u_i, v_i) \in M(\sum_{i=1}^n \lambda_i(x_i, y_i)).$$

Hence, we proved that  $M$  is also weakly naturally quasi-concave.

Further, we will show that  $M$  has a continuous selection on  $L$ . We exploit the fact that  $L$  is a simplex. Let us suppose that it is the convex hull of the affinely independent set  $\{(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)\}$ .

On appealing to the last assumption, we state that there exist unique continuous functions  $\lambda_i : L \rightarrow \mathbb{R}, i = 1, 2, \dots, n$  such that, for each  $(x, y) \in L$ , we get  $(\lambda_1(x, y), \lambda_2(x, y), \dots, \lambda_n(x, y)) \in \Delta_{n-1}$  and  $(x, y) = \sum_{i=1}^n \lambda_i(x, y)(a_i, b_i)$ .

Let us define  $h : L \rightarrow L$  by

$$h(a_i, b_i) = (c_i, d_i) \quad (i = 1, \dots, n) \text{ and}$$

$$h(\sum_{i=1}^n \lambda_i(a_i, b_i)) = \sum_{i=1}^n \lambda_i(c_i, d_i) \in M(x, y).$$

We show that  $h$  is continuous. For this purpose, let us consider  $(x_m, y_m)_{m \in \mathbb{N}}$  a sequence which converges to  $(x_0, y_0) \in L$ , where  $(x_m, y_m) = \sum_{i=1}^n \lambda_i(x_m, y_m)(a_i, b_i)$

and  $(x_0, y_0) = \sum_{i=1}^n \lambda_i(x_0)(a_i, b_i)$ . The continuity of  $\lambda_i$ , implies that, for each  $i = 1, 2, \dots, n, \lambda_i(x_m, y_m) \rightarrow \lambda_i(x_0, y_0)$  as  $m \rightarrow \infty$ . Hence,  $h(x_m, y_m) \rightarrow h(x_0, y_0)$  as  $m \rightarrow \infty$ , i.e.  $h$  is continuous.

We proved that  $M$  has a continuous selection on  $B$ .

Finally, we apply the Brouwer fixed point theorem to guarantee the existence of a fixed point for the function  $h$ . Therefore, there exists  $(x^*, y^*) \in L$  such that  $h(x^*, y^*) = (x^*, y^*)$  and the last statement assures that  $(x^*, y^*) \in M(x^*, y^*)$ . Clearly,  $x^* \in P(x^*, y^*)$  and  $y^* \in F(x^*)$ , which imply that there exist  $x^* \in K$  and  $y^* \in F(x^*)$  such that  $x^* \in A(x^*)$  and  $f(x^*, y^*, x) \subset C, \forall x \in A(x^*)$ . We are thus lead to the

conclusion that  $x^* \in V_A(f)$ .

We have shown that  $V_A(f)$  is nonempty and the proof is completed.

We will prove a similar result in case of biconvexity. The biconvex sets were introduced by Aumann [3]. For the reader's convenience, we present below the most important notions concerning biconvexity.

Let  $X \subset E_1$  and  $Y \subset E_2$  be two nonempty convex sets,  $E_1, E_2$  be topological vector spaces and let  $B \subset X \times Y$ .

The set  $B \subset X \times Y$  is called a *biconvex set* on  $X \times Y$  if the section  $B_x = \{y \in Y : (x, y) \in B\}$  is convex for every  $x \in X$  and the section  $B_y = \{x \in X : (x, y) \in B\}$  is convex for every  $y \in Y$ . Let  $(x_i, y_i) \in X \times Y$  for  $i = 1, 2, \dots, n$ . A convex combination  $(x, y) = \sum_{i=1}^n \lambda_i(x_i, y_i)$ , (with  $\sum_{i=1}^n \lambda_i = 1$ ,  $\lambda_i \geq 0$   $i = 1, 2, \dots, n$ ) is called a *biconvex combination* if  $x_1 = x_2 = \dots = x_n = x$  or  $y_1 = y_2 = \dots = y_n = y$ . Let  $D \subseteq X \times Y$  be a given set. The set  $H := \bigcap \{D_I : D \subseteq D_I, D_I \text{ is biconvex}\}$  is called *biconvex hull of  $D$*  and is denoted  $\text{biconv}(D)$ .

**Theorem 9.** (Aumann and Hart [3]). *A set  $B \subseteq X \times Y$  is biconvex if and only if  $B$  contains all biconvex combinations of its elements.*

**Theorem 10.** (Aumann and Hart [3]). *The biconvex hull of a set  $P$  is biconvex. Furthermore, it is the smallest biconvex set (in the sense of set inclusion), which contains  $P$ .*

**Lemma 4.** (Gorski, Pfeuffer and Klamroth [13]). *Let  $D \subseteq X \times Y$  be a given set. Then  $\text{biconv}(D) \subseteq \text{conv}(D)$ .*

Now we introduce the following definition.

**Definition 4.** Let  $B \subset X \times Y$  be a biconvex set,  $Z$  a nonempty convex subset of a topological vector space  $F$  and  $T : B \rightarrow 2^Z$  a correspondence.  $T$  is called *weakly biconvex* if for each finite set  $\{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\} \subset B$ , there exists  $z_i \in T(x_i, y_i)$ , ( $i = 1, 2, \dots, n$ ) such that for every biconvex combination  $(x, y) = \sum_{i=1}^n \lambda_i(x_i, y_i) \in B$  (with  $\sum_{i=1}^n \lambda_i = 1$ ,  $\lambda_i \geq 0$   $i = 1, 2, \dots, n$ ),  $\sum_{i=1}^n \lambda_i z_i \in T(\sum_{i=1}^n \lambda_i(x_i, y_i))$ .

The second important statement of this section concerns the existence of the solutions for the strong vector quasi-equilibrium problems in case of correspondences having an weakly convex graph. The proof relies on the construction of a continuous selection of a biconvex correspondence. The Brouwer fixed point theorem is also used.

**Theorem 11.** *Let  $E_1, E_2, Z$  be topological vector spaces and  $K \subset E_1$ ,  $D \subset E_2$  be subsets. Let  $B$  be the biconvex hull of  $\{(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)\} \subset K \times D$  (a biconvex subset of  $K \times D$ ) and denote  $B_K = \text{pr}_K B$ .*

Let  $(A, F) : B_K \rightarrow 2^B$ , such that and  $A : B_K \rightarrow 2^K$  and  $F : B_K \rightarrow 2^D$  have weakly convex graphs and for each  $n \in N$ ,  $\lambda \in \Delta_{n-1}$  and  $x_1, x_2, \dots, x_n \in B_k$ ,  $A(\sum_{i=1}^n \lambda_i x_i) \subset \bigcap_{i=1}^n A(x_i)$ .

Let  $f : B \times K \rightarrow 2^K$  such that:

- (a)  $\forall (x, y) \in B, f(x, y, A(x)) \subset C$
- (b) for each  $z \in K$ , for each  $n \in N$ ,  $\lambda \in \Delta_{n-1}$  and  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n) \in B$ ,  $f(\sum_{i=1}^n \lambda_i x_i, \sum_{i=1}^n \lambda_i y_i, z) \subset \bigcap_{i=1}^n f(x_i, y_i, z)$ .

Then,  $V_f \neq \emptyset$ .

*Proof.* The approach is similar to one of the proof of the previous theorem. It relies on Brouwer’s fixed point theorem, which will be applied to a continuous selection of a weakly biconvex correspondence.

The key idea of the method is to define the correspondence  $P : B \rightarrow 2^K$  as follows:

$$P(x, y) = \{u \in A(x) : f(u, y, z) \subset C \forall z \in A(x)\}.$$

Let us focus on the correspondence  $P$ . We will prove that it is weakly biconvex. Towards this end, let  $n \in N$  and  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n) \in B$ .

For each  $i = 1, 2, \dots, n$ , there exists  $u_i \in P_i(x_i, y_i)$ , that is  $u_i \in A(x_i)$  and  $f(u_i, y_i, z) \subset C \forall z \in A(x_i)$ . Let  $\lambda \in \Delta_{n-1}$  be such that  $\sum_{i=1}^n \lambda_i(x_i, y_i) \in B$  and

let us denote  $u = \sum_{i=1}^n \lambda_i u_i$ . Knowing that  $A$  has a weakly convex graph, we obtain

$$u \in A(\sum_{i=1}^n \lambda_i x_i).$$

It is clear that  $f(\sum_{i=1}^n \lambda_i u_i, \sum_{i=1}^n \lambda_i y_i, z) \subset f(u_i, y_i, z) \subset C \forall z \in A(\sum_{i=1}^n \lambda_i x_i)$ .

The inclusion  $A(\sum_{i=1}^n \lambda_i x_i) \subset A(x_i)$  for each  $i \in \{1, 2, \dots, n\}$ , implies that

$$f(\sum_{i=1}^n \lambda_i u_i, \sum_{i=1}^n \lambda_i y_i, z) \subset C \forall z \in A(x_i) \forall z \in A(x_i), i = 1, 2, \dots, n.$$

Therefore,  $P$  is weakly biconvex.

Next we define the correspondence  $M : B \rightarrow 2^B$  by

$$M(x, y) = (P(x, y), F(x)) \forall (x, y) \in B.$$

We claim that  $M$  is also weakly biconvex.

Indeed, let  $n \in N$  and  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n) \in B$ . The weakly biconvexity of  $P$  assures the existence, for each  $i = 1, 2, \dots, n$ , of an element  $u_i \in P(x_i, y_i)$  such that for each  $\lambda \in \Delta_{n-1}$ ,  $\sum_{i=1}^n \lambda_i u_i \in P(\sum_{i=1}^n \lambda_i(x_i, y_i))$ .

The property fulfilled by  $F$ , of having a weakly convex graph, in turn guarantees that, for each  $i = 1, 2, \dots, n$ , there exists  $v_i \in F(x_i, y_i)$  such that for each  $\lambda \in \Delta_{n-1}$ ,  $\sum_{i=1}^n \lambda_i v_i \in F(\sum_{i=1}^n \lambda_i x_i)$ . We conclude that for each  $i = 1, 2, \dots, n$ , there exists  $(u_i, v_i) \in M(x_i, y_i)$  such that for each  $\lambda \in \Delta_{n-1}$ ,  $\sum_{i=1}^n \lambda_i (u_i, v_i) \in M(\sum_{i=1}^n \lambda_i (x_i, y_i))$ .

Further, it will be shown that  $M$  has a continuous selection on  $B$ .

Since  $B$  is the biconvex hull of  $(a_1, b_1), \dots, (a_n, b_n)$ , there exist unique continuous functions  $\lambda_i : K \rightarrow \mathbb{R}$ ,  $i = 1, 2, \dots, n$  such that for each  $(x, y) \in B$ , we have  $(\lambda_1(x, y), \lambda_2(x, y), \dots, \lambda_n(x, y)) \in \Delta_{n-1}$  and  $(x, y) = \sum_{i=1}^n \lambda_i(x, y)(a_i, b_i)$ .

Define  $h : B \rightarrow B$  by

$$h(a_i, b_i) = (c_i, d_i) \quad (i = 1, \dots, n) \text{ and}$$

$$h(\sum_{i=1}^n \lambda_i(a_i, b_i)) = \sum_{i=1}^n \lambda_i(c_i, d_i) \in M(x, y).$$

To obtain the continuity of  $f$ , let  $(x_m, y_m)_{m \in \mathbb{N}}$  be a sequence which converges to  $(x_0, y_0) \in B$ , where  $(x_m, y_m) = \sum_{i=1}^n \lambda_i(x_m, y_m)(a_i, b_i)$  implies  $a_1 = a_2 = \dots = a_n = a$  or  $b_1 = b_2 = \dots = b_n = b$  and  $(x_0, y_0) = \sum_{i=1}^n \lambda_i(x_0)(a_i, b_i)$  with  $a_1 = a_2 = \dots = a_n = a$  or  $b_1 = b_2 = \dots = b_n = b$ . In the virtue of the continuity of  $\lambda_i$ , for each  $i = 1, 2, \dots, n$ , the convergence is achieved:  $\lambda_i(x_m, y_m) \rightarrow \lambda_i(x_0, y_0)$  as  $m \rightarrow \infty$ . Hence  $h(x_m, y_m) \rightarrow h(x_0, y_0)$  as  $m \rightarrow \infty$ , which means that  $h$  is continuous.

We proved that  $M$  has a continuous selection on  $B$ . According to Brouwer's fixed point theorem,  $h$  has a fixed point  $(x^*, y^*) \in B$ , i.e.  $h(x^*, y^*) = (x^*, y^*)$ . We obtain that  $(x^*, y^*) \in M(x^*, y^*)$ . Therefore,  $x^* \in P(x^*, y^*)$  and  $y^* \in F(x^*)$ , which implies that there exists  $x^* \in K$  and  $y^* \in F(x^*)$  such that  $x^* \in A(x^*)$  and  $f(x^*, y^*, x) \subset C$ ,  $\forall x \in A(x^*)$ . The proof provides an element  $x^* \in V_A(f)$  and then,  $V_A(f)$  is nonempty.

**5.2. Strong vector quasi-equilibrium problems under weak continuity assumptions**

The aim of this subsection is to generalize the results concerning the problems with lower semicontinuous correspondences. The technique of proof we use is an approximation one.

Let us recall the following notation. If  $A : X \rightarrow 2^Y$  is a correspondence and  $D, V \subset Y$ , then  $A^V : X \rightarrow 2^Y$  is defined by  $A^V(x) = (A(x) + V) \cap D$ ,  $\forall x \in X$ . The hypotheses of Theorem 12 regards the lower semicontinuity of  $A^V$ , where  $V$  is any absolutely convex symmetric neighborhood of 0 in  $X$ .

The next example shows that the assumption we made over  $A$  can be fulfilled by correspondences which are not lower semicontinuous. Thus, our result is indeed an extension of an already studied case (please see [24]).

**Example 2.** Let  $A : (0, 2) \rightarrow 2^{[1,4]}$  be the correspondence defined by

$$A(x) = \begin{cases} [2 - x, 2], & \text{if } x \in (0, 1); \\ \{4\} & \text{if } x = 1; \\ [1, 2] & \text{if } x \in (1, 2). \end{cases} \quad A \text{ is not lower semicontinuous on } (0, 2).$$

Let  $D = [1, 2]$ . For each  $V = (-\varepsilon, \varepsilon)$  with  $\varepsilon > 0$ , the correspondence  $\overline{A^V}$  is lower semicontinuous and  $\overline{A^V}$  has nonempty convex values.

**Theorem 12.** Let  $E_1, E_2, Z$  be Hausdorff locally convex topological vector spaces,  $K \subset E_1$  and  $D \subset E_2$  be nonempty convex compact subsets and  $C$  be a nonempty closed convex cone. Let  $A : K \rightarrow 2^K$  be a correspondence such that  $\overline{A}$  and  $\overline{A^{V_1}}$  are lower semicontinuous with nonempty convex values for each absolutely convex symmetric neighborhood  $V_1$  of 0 in  $X$ . Let  $F : K \rightarrow 2^D$  be such that  $\overline{F}$  and  $\overline{F^{V_2}}$  has nonempty convex values for each absolutely convex symmetric neighborhood  $V_2$  of 0 in  $Y$ . Let  $f : K \times D \times K \rightarrow 2^Z$  such that the following assumptions are satisfied:

- (i) for all  $(x, y) \in K \times D$ ,  $f(x, y, \overline{A}(x)) \subset C$  and  $f(x, y, \overline{A^{V_1}}(x)) \subset C$  for each absolutely convex symmetric neighborhood  $V_1$  of 0 in  $X$ ;
- (ii) for all  $(y, z) \in D \times K$ ,  $f(\cdot, y, z)$  is properly  $C$ -quasiconvex;
- (iii)  $f(\cdot, \cdot, \cdot)$  is upper  $C$ -continuous;
- (iv) for all  $y \in D$ ,  $f(\cdot, y, \cdot)$  is lower  $(-C)$ -continuous.

Then,  $V_A(f) \neq \emptyset$ .

*Proof.* An approximation method will be used in order to demonstrate our last result.

In order to accomplish our proof, we need to define the correspondences  $P : K \times D \rightarrow 2^K$  and  $M : K \times D \rightarrow 2^{K \times D}$  by

$$P(x, y) = \{u \in A(x) : f(u, y, z) \subset C, \forall z \in A(x)\} \quad \forall (x, y) \in K \times D \text{ and}$$

$$M(x, y) = (P(x, y), F(x)) \quad \forall (x, y) \in K \times D.$$

Let  $V_1$  be an open absolutely convex symmetric neighborhood of 0 in  $X$ .

Firstly, we will show that  $\overline{P^{V_1}}$  is an upper semicontinuous correspondence with nonempty closed convex values. The elements of  $V_A(f)$  will be obtained as a consequence of the existence of the fixed points for  $\overline{M}$ .

Let  $P^{V_1}$  be defined by

$$P^{V_1}(x, y) = \{u \in A^{V_1}(x) : f(u, y, z) \subset C, \forall z \in A^{V_1}(x)\} \quad \forall (x, y) \in K \times D.$$

We will show that the values of  $\overline{P^{V_1}}$  can be described by

$$\overline{P^{V_1}}(x, y) = \{u \in \overline{A^{V_1}}(x) : f(u, y, z) \subset C, \forall z \in \overline{A^{V_1}}(x)\} \quad \forall (x, y) \in K \times D.$$

In order to do this, the closedness of  $\overline{P^{V_1}}$  will be shown firstly. We consider the net  $\{(x_\alpha, y_\alpha, u_\alpha) : \alpha \in \Lambda\} \subset \text{Gr} \overline{P^{V_1}}$  such that  $(x_\alpha, y_\alpha, u_\alpha) \rightarrow (x_0, y_0, u_0) \in K \times D \times K$ . Then,  $u_\alpha \in \overline{P^{V_1}}(x_\alpha, y_\alpha)$  for each  $\alpha \in \Lambda$  and we prove that  $(x_0, y_0, u_0) \in \text{Gr} \overline{P^{V_1}}$ ,

that is  $u_0 \in \overline{P^{V_1}}(x_0, y_0)$ . Since  $\overline{A^{V_1}}$  is upper semicontinuous and  $u_\alpha \in \overline{A^{V_1}}(x_\alpha)$ , then  $u_0 \in \overline{A^{V_1}}(x_0)$ . If, by contrary,  $u_0 \notin \overline{P^{V_1}}(x_0, y_0)$ , there exists  $z_0 \in \overline{A^{V_1}}(x_0)$  such that  $f(u_0, y_0, z_0) \not\subseteq C$ . This relation is exploited to assert the existence of a neighborhood  $U_0$  of the origin in  $Z$  such that  $f(u_0, y_0, z_0) + U_0 \not\subseteq C$ .

Further, the upper  $C$ -continuity of  $f$  guarantees that there exists a neighborhood  $V(u_0, y_0, z_0)$  of  $(u_0, y_0, z_0)$  such that,  $f(u, y, z) \subset f(u_0, y_0, z_0) + U_0 + C$  for each  $(u, y, z) \in V(u_0, y_0, z_0)$ . Then, for each  $(u, y, z) \in V(u_0, y_0, z_0)$ ,  $f(u, y, z) \not\subseteq C + C \subset C$ , which assures the existence of  $\alpha_0 \in \Lambda$  such that for each  $\alpha \geq \alpha_0$ ,  $f(u_\alpha, y_\alpha, z_\alpha) \not\subseteq C$ .

The last relation contradicts  $u_\alpha \in \overline{P^{V_1}}(x_\alpha, y_\alpha)$ . Consequently, the assumption that  $u_0 \notin \overline{P^{V_1}}(x_0, y_0)$  is false. Since  $u_0 \in \overline{P^{V_1}}(x_0, y_0)$ ,  $\text{Gr}P^{V_1}$  is closed, and, since  $K$  is compact, the upper semicontinuity of  $\overline{P^{V_1}}$  is guaranteed.

Now, we note that  $\overline{A^{V_1}}(x)$  is nonempty for each  $x \in K$  and this last remark, together with the assumption i), implies the non-emptiness of  $\overline{P^{V_1}}(x, y)$ .

Let us fix  $(x_0, y_0) \in K \times D$ . We will prove secondly that  $\overline{P^{V_1}}(x_0, y_0)$  is closed. Towards this end, we consider the net  $\{u_\alpha : \alpha \in \Lambda\} \subseteq \overline{P^{V_1}}(x_0, y_0)$  such that  $u_\alpha \rightarrow u_0$ . Then,  $u_\alpha \in \overline{A^{V_1}}(x_0)$  and  $f(u_\alpha, y_0, z) \subseteq C$  for all  $z \in \overline{A^{V_1}}(u_\alpha)$ . The closedness of  $\overline{A^{V_1}}(x_0)$  allows us to deduce that  $u_0 \in \overline{A^{V_1}}(x_0)$ .

On appealing to the lower semicontinuity of  $\overline{A^{V_1}}$ , we observe that, for any  $z_0 \in \overline{A^{V_1}}(u_0)$  and  $\{u_\alpha\} \rightarrow u_0$ , there exists a net  $\{z_\alpha\}$  such that  $z_\alpha \in \overline{A^{V_1}}(u_\alpha)$  and  $z_\alpha \rightarrow z_0$ . The last assertion guarantees, for each  $\alpha$ , the existence of  $z_\alpha \in \overline{A^{V_1}}(u_\alpha)$  with the property that  $f(u_\alpha, y_0, z_\alpha) \subset C$ . Since  $f(\cdot, y, \cdot)$  is lower  $(-C)$ -continuous, for each neighborhood  $U$  of the origin in  $Z$ , there exists a subnet  $\{u_\beta, z_\beta\}$  of  $\{u_\alpha, z_\alpha\}$  such that  $f(u_0, y_0, z_0) \subset f(u_\beta, y_0, z_\beta) + U + C$ . Consequently,  $f(u_0, y_0, z_0) \subset U + C$ .

Further, we prove that  $f(u_0, y_0, z_0) \subset C$ . If, by contrary, there exists  $a \in f(u_0, y_0, z_0)$  and  $a \notin C$ , then,  $0 \notin B := C - a$  and  $B$  is closed. Thus,  $Z \setminus B$  is open and  $0 \in Z \setminus B$ . There exists an open symmetric neighborhood  $U_1$  of the origin in  $Z$ , such that  $U_1 \subset Z \setminus B$  and  $U_1 \cap B = \emptyset$ . Therefore,  $0 \notin B + U_1$ , i.e.,  $a \notin C + U_1$ , which contradicts  $f(u_0, y_0, z_0) \subset U_1 + C$ . We conclude that  $f(u_0, y_0, z_0) \subset C$  and then,  $u_0 \in \overline{P^{V_1}}(x_0, y_0)$  and  $\overline{P^{V_1}}(x_0, y_0)$  is closed.

Therefore,  $\overline{P^{V_1}}(x, y) = \{u \in \overline{A^{V_1}}(x) : f(u, y, z) \subset C, \forall z \in \overline{A^{V_1}}(x)\} \forall (x, y) \in K \times D$ .

Now, we claim that  $\overline{P^{V_1}}(x_0, y_0)$  is convex, where  $(x_0, y_0) \in X \times X$  is arbitrary fixed. Indeed, let us consider  $u_1, u_2 \in \overline{P^{V_1}}(x_0, y_0)$  and  $\lambda \in [0, 1]$ . Since  $u_1, u_2 \in \overline{A^{V_1}}(x_0)$  and the set  $\overline{A^{V_1}}(x_0)$  is convex, the convex combination  $u = \lambda u_1 + (1 - \lambda)u_2$  is an element of  $\overline{A^{V_1}}(x_0)$ . Further, by using the property of properly  $C$ -quasiconvexity of  $f(\cdot, y, z)$ , we can assume, without loss of generality, that  $f(u_1, y_0, z) \subset f(u, y_0, z) + C$ . We will prove that  $u \in \overline{P^{V_1}}(x_0, y_0)$ . If, by contrary,  $u \notin \overline{P^{V_1}}(x_0, y_0)$ , there exists  $z_0 \in \overline{A^{V_1}}(x_0)$  such that  $f(u, y_0, z_0) \not\subseteq C$  and, consequently,  $f(u_1, y_0, z) \subset f(u, y_0, z) + C \not\subseteq C + C \subset C$ , which contradicts  $u_1 \in \overline{P^{V_1}}(x_0, y_0)$ . Thus,  $u \in \overline{P^{V_1}}(x_0, y_0)$ , which implies

that  $\overline{P^{V_1}}(x_0, y_0)$  is a convex set. The claim is shown.

In order to prove the existence of the solutions for SVQEP, let us consider  $\beta_i$  a basis of open absolutely convex symmetric neighborhoods of zero in  $E_i$  for each  $i \in \{1, 2\}$  and let  $\beta = \beta_1 \times \beta_2$ . For each system of neighborhoods  $V = V_1 \times V_2 \in \beta_1 \times \beta_2$ , let's define the correspondence  $M^V : K \times D \rightarrow 2^{K \times D}$ , by

$$M^V(x, y) = (P^{V_1}(x, y), F^{V_2}(x)) = ((P(x, y) + V_1) \cap K, (F(x) + V_2) \cap D), (x, y) \in K \times D.$$

It is not difficult to see that  $\overline{M^V}$  is upper semicontinuous with nonempty closed convex values. Furthermore, by appealing to Ky Fan fixed point theorem [10], we are able to find a pair  $(x_{V_1}^*, y_{V_2}^*) \in K \times D$  such that  $(x_{V_1}^*, y_{V_2}^*) \in \overline{M^V}(x_{V_1}^*, y_{V_2}^*)$ .

For each  $V = V_1 \times V_2 \in \beta$ , let's define

$$Q_V = \{(x, y) \in K \times D : x \in \overline{P^{V_1}}(x, y)\} \cap \{(x, y) \in K \times D : y \in \overline{F^{V_2}}(x, y)\}.$$

$Q_V$  is nonempty since  $(x_{V_1}^*, y_{V_2}^*) \in Q_V$ , then  $Q_V$  is nonempty and closed. We prove that the family  $\{Q_V : V \in \beta\}$  has the finite intersection property. Let  $\{V^{(1)}, V^{(2)}, \dots, V^{(n)}\}$  be any finite set of  $\beta$  and let  $V^{(k)} = V_1^{(k)} \times V_2^{(k)}$ ,  $k = 1, 2, \dots, n$ . Let  $V_1 = \bigcap_{k=1}^n V_1^{(k)}$  and  $V_2 = \bigcap_{k=1}^n V_2^{(k)}$ . Then,  $V_1 \in \beta_1$  and  $V_2 \in \beta_2$ . Thus,  $V = V_1 \times V_2 \in \beta_1 \times \beta_2$ .

Clearly,  $Q_V \subseteq \bigcap_{k=1}^n Q_{V^{(k)}}$ , so that  $\bigcap_{k=1}^n Q_{V^{(k)}} \neq \emptyset$ .

Since  $K \times D$  is compact and the family  $\{Q_V : V \in \beta\}$  has the finite intersection property, we conclude that  $\bigcap \{Q_V : V \in \beta\} \neq \emptyset$ . Take any  $(x^*, y^*) \in \bigcap \{Q_V : V \in \beta\}$ , then for each  $V \in \beta$ ,  $(x^*, y^*) \in \overline{M^V}(x^*, y^*)$ . We can now appeal to Lemma 2 to assert that  $(x^*, y^*) \in \overline{M}(x^*, y^*)$ , which implies  $x^* \in \overline{P}(x^*, y^*)$  and  $y^* \in \overline{F}(x^*)$ .

By using the above technique, we can show that the values of  $\overline{P}$  can be described by

$$\overline{P}(x^*, y^*) = \{u \in \overline{A}(x) : f(u, y, z) \subset C, \forall z \in \overline{A}(x)\} \forall (x, y) \in K \times D.$$

Consequently, we are guaranteed that there exist  $x^* \in K$  and  $y^* \in \overline{F}(x^*)$  such that  $x^* \in \overline{A}(x^*)$  and  $f(x^*, y^*, x) \subset C, \forall x \in \overline{A}(x^*)$ . It is clear that  $x^* \in V_A(f)$  and then, the nonemptiness of  $V_A(f)$  is proven.

**Remark 3.** Theorem 2 generalizes Theorem 3.1 in [24], since the correspondences  $A$  and  $F$  verify assumptions which are weaker than the ones in [24].

#### REFERENCES

1. Q. H. Ansari and J. C. Yao, An existence result for the generalized vector equilibrium, *Appl. Math. Lett.*, **12** (1999) 53-56.
2. Q. H. Ansari, A. P. Farajzadeh and S. Schaible, Existence of solutions of strong vector equilibrium problems, *Taiwanese Journal of Mathematics*, **16(1)** (2012), 165-178.
3. R. Aumann and S. Hart, Bi-convexity and bi-martingales, *Isr. J. Math.*, **54(2)** (1986), 159-180.

4. G. Bigi, A. Capătă and G. Kassay, Existence results for strong vector equilibrium problems and their applications, *Optimization*, **61(5)** (2012), 567-583.
5. A. Borglin and H. Keiding, Existence of equilibrium actions and of equilibrium: a note on the 'new' existence theorem, *J. Math. Econom.*, **3** (1976), 313-316.
6. G. Debreu, A social equilibrium existence theorem, *Proc. Nat. Acad. Sci.*, **38** (1952), 886-893.
7. X. P. Ding and G. X. Z. Yuan, The study of existence of equilibria for generalized games without lower semicontinuity in locally topological vector spaces, *J. Math. Anal. Appl.*, **227** (1998), 420-438.
8. X. Ding and He Yiran, Best approximation theorem for set-valued mappings without convex values and continuity, *Appl Math. and Mech. English Edition*, **19(9)** (1998), 831-836.
9. X. P. Ding and J. L. Ho, New iterative algorithm for solving a system of generalized mixed implicit equilibrium problems in Banach spaces, *Taiwanese Journal of Mathematics*, **15(2)**, (2011), 673-695.
10. K. Fan, Fixed-point and minimax theorems in locally convex topological linear spaces, *Proc. Nat. Acad. Sci. U.S.A.*, **38** (1952), 121-126.
11. A. P. Farajzadeh and J. Zafarani, Equilibrium problems and variational inequalities in topological vector spaces, *Optimization*, **59(4)**, (2010), 485-499.
12. X. H. Gong, Strong vector equilibrium problem, *J. Global Optim.*, **36** (2006), 339-349.
13. J. Gorski, F. Pfeuffer and K. Klamroth, Biconvex sets and optimization with biconvex functions: a survey and extension, *Math. Meth. Oper. Res.*, **66** (2007), 373-407.
14. C. J. Himmelberg, Fixed points of compact multifunctions, *J. Math. Anal. Appl.*, **38** (1972), 205-207.
15. S. H. Hou, X. H. Gong and X. M. Yang, Existence and stability of solutions for generalized strong vector equilibrium problems with trifunctions, *J. Optim. Theory Appl.*, **146(2)**, (2010), 387-398.
16. R. Hu, Y. P. Fang and N. J. Huang, Well-posedness of systems of equilibrium problems, *Taiwanese Journal of Mathematics*, **14(6)** (2010), 2435-2446.
17. G. Kassay, On vector equilibrium problems and applications, *Taiwanese Journal of Mathematics*, **15(1)** (2011), 365-380.
18. W. K. Kim and K. K. Tan, New existence theorems of equilibrium and applications, *Nonlinear Anal.*, **47** (2001), 531-542.
19. S. Kum and M.-M. Wong, An extension of a generalized equilibrium problem, *Taiwanese Journal of Mathematics*, **15(4)** (2011), 1667-1675.
20. Y. C. Lin, On generalized vector equilibrium problems, *Nonlinear Analysis*, **70** (2009), 1040-1048.

21. L. Lin, Z. Yu, Q. Ansari and L. Lai, Fixed point and maximal element theorems with applications to abstract economies and minimax inequalities, *J. of Math. Anal. Appl.*, **284**(2) (2003), 656-671.
22. L. J. Lin, L. F. Chen and Q. H. Ansari, Generalized abstract economy and systems of generalized vector quasi-equilibrium problems, *J. Comput. Appl. Math.*, **208** (2007), 341-352.
23. Q.-M. Liu, L. Fan and G. Wang, Generalized vector quasi-equilibrium problems with set-valued mappings, *Appl. Math. Lett.*, **21**(9) (2008), 946-950.
24. X. J. Long, N. J. Huang and K. L. Teo, Existence and stability of solutions for generalized strong vector quasi-equilibrium problem, *Math. Comp. Model.*, **47** (2008), 445-451.
25. X.-J. Long, J.-W. Peng and S.-Y. Wu, Generalized vector variational-like inequalities and nonsmooth vector optimization problems, *Optimization*, **61**(9) (2012), 1075-1086.
26. D. T. Luc, Theory of vector optimization, in: *Lecture Notes in Economics and Mathematics Systems*, Vol. 319, Springer-Verlag, New York, 1989.
27. B. C. Ma and X. H. Gong, Optimality conditions for vector equilibrium problems in normed spaces, *Optimization*, **60**(12) (2011), 1441-1455.
28. F. Morgana, O. Jacinto and S. Scheimberg, Duality for generalized equilibrium problem, *Optimization*, **57**(6) (2008), 795-805.
29. M. Patriche, Fixed point theorems and applications in theory of games, *Fixed Point Theory*, **15**(1) (2014), 199-212.
30. W. Shafer and H. Sonnenschein, Equilibrium in abstract economies without ordered preferences. *J. Math. Econom.*, **2** (1975), 345-348.
31. S.-Q. Shan and N.-J. Huang, An iterative method for generalized mixed vector equilibrium problems and fixed point of nonexpansive mappings and variational inequalities, *Taiwanese Journal of Mathematics*, **16**(5) (2012), 1681-1705.
32. C. I. Tulcea, On the approximation of upper semi-continuous correspondences and the equilibriums of generalized games, *J. Math. An. Appl.*, **136** (1988), 267-289.
33. X. Z. Yuan, *The Study of Minimax Inequalities and Applications to Economies and Variational Inequalities*, Memoirs of the American Society, 1988, p. 625.

Monica Patriche  
University of Bucharest  
Faculty of Mathematics and Computer Science  
14 Academiei Street  
010014 Bucharest  
Romania  
E-mail: monica.patriche@yahoo.com