TAIWANESE JOURNAL OF MATHEMATICS
Vol. 21, No. 3, pp. 549–567, June 2017
DOI: 10.11650/tjm/7753
This paper is available online at http://journal.tms.org.tw

On Surfaces of Maximal Sectional Regularity

Markus Brodmann, Wanseok Lee, Euisung Park* and Peter Schenzel

Abstract. We study projective surfaces $X \subset \mathbb{P}^r$ (with $r \geq 5$) of maximal sectional regularity and degree $d > r$, hence surfaces for which the Castelnuovo-Mumford regularity $\operatorname{reg}(\mathcal{C})$ of a general hyperplane section curve $\mathcal{C} = X \cap \mathbb{P}^{r-1}$ takes the maximally possible value $d - r + 3$. We use the classification of varieties of maximal sectional regularity of [5] to see that these surfaces are either particular divisors on a smooth rational 3-fold scroll $S(1,1,1) \subset \mathbb{P}^5$, or else admit a plane $\mathbb{F} = \mathbb{P}^2 \subset \mathbb{P}^r$ such that $X \cap \mathbb{F} \subset \mathbb{F}$ is a pure curve of degree $d - r + 3$. We show that our surfaces are either cones over curves of maximal regularity, or almost non-singular projections of smooth rational surface scrolls. We use this to show that the Castelnuovo-Mumford regularity of such a surface X satisfies the equality $\operatorname{reg}(X) = d-r+3$ and we compute or estimate various cohomological invariants as well as the Betti numbers of such surfaces.

1. Introduction

Throughout this paper, we work over an algebraically closed field $\Bbbk$ of arbitrary characteristic. We always write $S := \Bbbk[x_0, x_1, \ldots, x_r]$ and denote by $\mathbb{P}^r$ the projective r-space $\operatorname{Proj}(S)$ over $\Bbbk$.

There is a well-known conjecture concerning the upper bound of the Castelnuovo-Mumford regularity $\operatorname{reg}(X)$ of a nondegenerate irreducible projective variety $X \subset \mathbb{P}^r$ in terms of its degree d and codimension c:

Conjecture 1.1 (Eisenbud-Goto's conjecture)**.** [10] $\operatorname{reg}(X) \leq d - c + 1$.

This conjecture has been proved only for irreducible curves by Gruson-Lazarsfeld-Peskine [12] and for smooth complex surfaces by H. Pinkham [17] and R. Lazarsfeld [14]. But it is still open, even for singular surfaces. In their fundamental paper [12], the authors have also shown that if X is a curve then $\operatorname{reg}(X) = d-r+2$ if and only if either $d \leq r+1$

Received June 19, 2016; Accepted September 12, 2016.
Communicated by Wei-Ping Li.
2010 *Mathematics Subject Classification.* 14H45, 13D02.
Key words and phrases. Castelnuovo-Mumford regularity, Variety of maximal sectional regularity, Extremal locus, Extremal variety.
*Corresponding author.

or else $d \geq r+2$ and X is a smooth rational curve having a $(d-r+2)$-secant line. This result was extended in several directions. We first define the integer $\ell(X)$ by

$$\ell(X) := \max\left\{\text{length}(X \cap \mathbb{L}) \mid \mathbb{L} = \mathbb{P}^1 \nsubseteq X\right\}.$$

Obviously, it always holds that $\text{reg}(X) \geq \ell(X)$.

By a result in [12], if X is a curve of degree $d \geq r+2$ and $\text{reg}(X) = d-r+2$ then $\ell(X) = d-r+2$. Therefore it is natural to ask in general, whether the condition $\text{reg}(X) \geq d-c+1$ implies the existence of a $(d-c+1)$-secant line. In this direction, it seems worthwhile to mention that there are many examples of smooth varieties X with $\text{reg}(X) = d-c+1$ which admit no trisecant line.

When X is a smooth variety, A. Bertin in [3] proves that $\ell(X) \leq d-c+1$ and if equality is attained then X must be the image of an isomorphic linear projection of a smooth variety of minimal degree. She proves also that if $\ell(X)$ takes the possible maximal value $d-c+1$ then X satisfies the regularity conjecture.

Now, suppose that X is a singular variety. From the regularity conjecture, we expect that $\ell(X)$ satisfies the inequality $\ell(X) \leq d-c+1$. In [4], A. Bertin claims that equality holds only if X is a cone over a smooth variety having a $(d-c+1)$-secant line. Unfortunately, it turns out that there are many non-conic singular varieties which have a $(d-c+1)$-secant line. Up to now, to our best knowledge, the inequality $\ell(X) \leq d-c+1$ is verified when X is a locally Cohen-Macaulay variety. For details, we refer the reader to [15, Theorem 1.1].

The *extremal secant locus* $\Sigma(X)$ of X is defined as the closure of the set of all proper $(d-c+1)$-secant lines to X in the Grassmannian $\mathbb{G}(1, \mathbb{P}^r)$ of lines in $\mathbb{P}^r$. Obviously, if $\Sigma(X)$ is nonempty then $\text{reg}(X)$ is at least $d-c+1$. In [5], the authors prove that for $c \geq 3$, the dimension of $\Sigma(X)$ is at most equal to $2\dim(X)-2$. Moreover, equality is attained if and only if X is a *variety of maximal sectional regularity* (VMSR), that is, the regularity of a general linear curve section of X takes the possibly maximal value $d-c+1$. The authors also classify all VMSR's of codimension at least 3. But it is unknown yet if they satisfy Eisenbud-Goto's regularity conjecture.

Along this line, the main purpose of this paper is to study algebraic properties of *surfaces of maximal sectional regularity* (SMSR). Let $X \subset \mathbb{P}^r$, $r \geq 5$, be a SMSR of degree $d \geq r+1$. By [5, Theorem 6.2], X falls under one of the following cases:

(i) (Type I) $r = 5$ and X is contained in the rational normal threefold scroll $S(1,1,1)$ as a divisor linearly equivalent to $H + (d-3)F$ where H and F are respectively the hyperplane divisor and a ruling plane of $S(1,1,1)$;

(ii) (Type II) There exists a plane $\mathbb{F}(X)$ such that $X \cap \mathbb{F}(X)$ is of dimension one and of degree $\geq d-r+3$.

It is a strong condition that a variety is a divisor of a smooth rational normal scroll. Thus, the study of type I is much easier than of type II. For example we can compute the whole Betti diagram of X of type I (cf. Theorem 2.1 and Remark 2.2). The present paper concerns mainly the case where X is of type II. Note that a cone over a curve of maximal regularity is included in type II. From now on, we suppose that $X \subset \mathbb{P}^r$ is of type II and non-conic. Then X is determined by the triple $(\widetilde{X}, D, \Lambda)$ where

(a) $\widetilde{X} \subset \mathbb{P}^{d+1}$ is a smooth rational normal surface scroll,

(b) D is an effective divisor of $\widetilde{X}$ linearly equivalent to $H + (3-r)F$ for a hyperplane divisor H and a ruling line F of $\widetilde{X}$, and

(c) Λ is a $(d-r)$-dimensional subspace of the $(d-r+3)$-dimensional linear span $\langle D \rangle$ of D such that the linear projection $\pi_\Lambda \colon \mathbb{P}^{d+1} \setminus \Lambda \to \mathbb{P}^r$ is generically injective along D.

Namely, $X = \pi_\Lambda(\widetilde{X})$ and $\mathbb{F}(X)$ is the plane $\pi_\Lambda(\langle D \rangle \setminus \Lambda)$. In particular, we can see that X contains the plane curve $\pi_\Lambda(D)$ of degree $d-r+3$. For details, we refer the reader to [5, Theorem 6.2]. This geometric description of X enables us to conclude that X has the following two interesting properties (cf. Definition and Remark 3.1(B) and Lemma 3.4):

(P1) X is an *almost nonsingular projection* of $\widetilde{X} \subset \mathbb{P}^{d+1}$. That is, the morphism $\pi_\Lambda \colon \widetilde{X} \to X$ has only finitely many singular points.

(P2) The intersection $X \cap \mathbb{F}(X)$ is a pure plane curve of degree $d-r+3$.

The following main result is essentially a consequence of (P1) and (P2).

Theorem 1.2. *Let* $5 \le r < d$ *and let* $X \subset \mathbb{P}^r$ *be a SMSR of degree* d *and of type II. Then*

(1) $\operatorname{reg}(X) = d-r+3$;

(2) X *is linearly normal;*

(3) $h^1(X, \mathcal{O}_X(d-r)) = 1$;

(4) $\mathrm{e}(X) \ge \binom{d-r+2}{2}$ *where*

$$\mathrm{e}(X) := \sum_{x \in X, \text{closed}} \operatorname{length}(H^1_{\mathfrak{m}_{X,x}}(\mathcal{O}_{X,x})).$$

The proofs of Theorem 1.2(1) and (2) are respectively provided after Corollary 3.3 and Lemma 3.4. Theorem 1.2(1) says that X satisfies Eisenbud-Goto's regularity conjecture. Theorem 1.2(2) is essentially induced by (P2).

For the proofs of Theorem 1.2(3) and (4), see Proposition 3.6. By Theorem 1.2(3), we can say that X is $(d-r+2)$-irregular since the cohomology group $H^1(X, \mathcal{O}_X(d-r))$ does not vanish. Comparing with the curve case, this leads us naturally to ask whether X fails to be $(d-r+1)$-normal or not. It seems an interesting phenomenon that for all our computational examples, X is $(d-r+1)$-normal. In Section 5, we find a few interesting conditions which are equivalent to the $(d-r+1)$-normality of X (cf. Theorem 5.1). So, one of the most important open question about X is whether it satisfies the $(d-r+1)$-normality or not.

In Theorem 1.2(4), note that the invariant $\mathrm{e}(X)$ counts the *number of non-Cohen-Macaulay points* of X in a weighted way. Concerning the inequality $\mathrm{e}(X) \geq \binom{d-r+2}{2}$, there are examples such that $\mathrm{e}(X)$ is strictly bigger than $\binom{d-r+2}{2}$. For details, see Remark 5.5(A). On the other hand, if the divisor $D \subset \widetilde{X}$ in the above condition (b) is a smooth irreducible curve and the projection center $\Lambda \subset \langle D \rangle$ is general enough, so that $C = \pi_\Lambda(D)$ is a plane curve with $\binom{d-r+2}{2}$ nodes, then we have $\mathrm{e}(X) = \binom{d-r+2}{2}$. We can regard such X as the general case among all SMSRs of type II. Along this line, Section 5 is devoted to show that the equality $\mathrm{e}(X) = \binom{d-r+2}{2}$ is closely related to the simplicity of the socle of the second cohomology module $H^2_*(\mathcal{I}_X)$ of X (cf. Proposition 5.4).

In Section 4, we study the arithmetic depth of X. Let Y be the union of X and $\mathbb{F}(X)$. We determine all pairs $\tau(X) := (\mathrm{depth}(X), \mathrm{depth}(Y))$. For $d \leq 2r-4$ it turns out that Y is always arithmetically Cohen-Macaulay and X is arithmetically normal (hence $H^1(\mathbb{P}^r, \mathcal{I}_X(j)) = 0$ for all $j > 0$). This allows to lift syzygetic information on the union of a general linear curve section $\mathcal{C}$ of X with its extremal secant line to Y and to get information on X (see [7]). For details see Theorem 4.1. Moreover, we construct explicit examples which illustrate that all the given pairs $\tau(X)$ occur.

2. Surfaces of type I

Throughout this section, let $X \subset \mathbb{P}^5$ be a surface of degree d and of maximal sectional regularity of type I. Thus X is contained in the threefold scroll $W := S(1,1,1)$ and its divisor class in W is $H + (d-3)F$. Let S, A_W and A_X denote the homogeneous coordinate rings of $\mathbb{P}^5$, W and X, respectively.

Theorem 2.1. *The minimal free resolution of A_X is of the form*

$$\begin{aligned} 0 &\to S(-d-2)^{\beta_5} \to S(-d-1)^{\beta_4} \to S(-d)^{\beta_3} \\ &\to S(-3)^2 \oplus S(-d+1)^{\beta_2} \to S(-2)^3 \oplus S(-d+2)^{\beta_1} \to S \to A_X \to 0 \end{aligned}$$

with

$$\beta_1 = \binom{d-1}{2}, \quad \beta_2 = 2(d-1)(d-3), \quad \beta_3 = 3(d^2-5d+5),$$
$$\beta_4 = 2(d-2)(d-4) \quad \textit{and} \quad \beta_5 = \binom{d-3}{2}.$$

In particular, it holds that $\operatorname{reg}(X) = d-2$.

Proof. First, note that as a consequence of [16, Remark 4.8(2)], the resolution of A_X must have the shape given in our statement. This implies the following form of the Hilbert series $H(A_X, t)$:

$$H(A_X,t) = \frac{1}{(1-t)^6}\left(1 - 3t^2 + 2t^3 - \beta_1 t^{d-2} + \beta_2 t^{d-1} - \beta_3 t^d + \beta_4 t^{d+1} - \beta_5 t^{d+2}\right).$$

Now, let $I(X/W)$ denote the kernel of the natural map from A_W to A_X. Then

$$I(X/W)_n \cong H^0(W, \mathcal{O}_W(-X+nH)) = H^0(W, \mathcal{O}_W((n-1)H-(d-3)F))$$

and hence it follows that

$$\dim_{\Bbbk} I(X/W)_n = \begin{cases} \binom{n+1}{2}(n-d+3) & \text{if } n \geq d-2, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the Hilbert series of $I(X/W)$ is given by

$$H(I(X/W),t) = \frac{t^{d-2}}{(1-t)^4}\left(\binom{d-1}{2} - (d-1)(d-4)t + \binom{d-3}{2}t^2\right).$$

Also, the Hilbert series of A_W is given by $H(A_W,t) = (1+2t)/(1-t)^4$. Now, consider the short exact sequence $0 \to I(X/W) \to A_W \to A_X \to 0$. By the additivity of the Hilbert series on short exact sequences, we have

$$H(A_X,t) = \frac{1}{(1-t)^4}\left(1 + 2t - \binom{d-1}{2}t^{d-2} + (d-1)(d-4)t^{d-1} - \binom{d-3}{2}t^d\right).$$

By comparing the above two expressions for the Hilbert series of A_X, we obtain the desired values for $\beta_1, \ldots, \beta_5$. $\square$

Remark 2.2. By using the short exact sequence $0 \to \mathcal{I}_W \to \mathcal{I}_X \to \mathcal{O}_W(-X) \to 0$, one can compute the values of $h^1(\mathbb{P}^5, \mathcal{I}_X(j))$ for all $j \in \mathbb{Z}$:

(a) $h^1(\mathbb{P}^5, \mathcal{I}_X(j)) = \binom{j+1}{2}(d-j-3)$ for $1 \leq j \leq d-4$ and zero else.

In particular, this implies that the linearly normal embedding of X by the very ample line bundle $\mathcal{O}_X(1)$ is a surface of minimal degree in $\mathbb{P}^{d+1}$. This enables us to describe cohomological properties of X completely. Here we summarize them:

(b) $h^0(X, \mathcal{O}_X(j)) = (j+1)(dj+2)/2$ for all $j \geq 0$ and zero else.

(c) $h^1(X, \mathcal{O}_X(j)) = 0$ for all $j \in \mathbb{Z}$.

(d) $h^2(X, \mathcal{O}_X(-j)) = (j-1)(dj-2)/2$ for all $j \geq 2$ and zero else.

3. Surfaces of type II

This section is aimed to investigate algebraic properties of SMSRs of type II and to give proofs of the four statements of Theorem 1.2.

Let $5 \leq r < d$ and let $X \subset \mathbb{P}^r$ be a surface of degree d and of maximal sectional regularity of type II. That is, there is a plane $\mathbb{F}(X)$ such that the intersection $X \cap \mathbb{F}(X)$ is of dimension one and of degree $\geq d-r+3$. Throughout this section, we will use freely the triple $(\widetilde{X}, D, \Lambda)$ of section 1.

Definition and Remark 3.1. (A) We say that a morphism $f\colon Y \to Z$ is *almost non-singular* if its singular locus $\mathrm{Sing}(f)$ defined by

$$\mathrm{Sing}(f) := \left\{z \in Z \mid \mathrm{length}\left(f^{-1}(z)\right) \geq 2\right\}$$

is a finite set.

(B) Recall that X is obtained as the image of a birational linear projection morphism of a rational normal surface scroll $\widetilde{X} \subset \mathbb{P}^{d+1}$. To be precise, if $\widetilde{X}$ is singular then X is a cone over a curve of maximal regularity. On the other hand, if $\widetilde{X}$ is smooth then X is non-conic. Note that X has only finitely many singular points since its general hyperplane section is smooth. This implies that the linear projection $f\colon \widetilde{X} \to X$ is almost nonsingular.

Our first goal of this section is to prove Theorem 1.2(1), which addresses that X satisfies the conjectural Eisenbud-Goto's bound. Then we will prove the remaining three cohomological properties of X in Theorem 1.2 by using the equality $\mathrm{reg}(X) = d-r+3$.

Now, we shall prove the main result of this section, which will allow us to establish the announced regularity bound. Our proof is based on Definition and Remark 3.1(B) and the fact that the smooth rational normal surface scroll $X \subset \mathbb{P}^{d+1}$ has a very nice syzygetic property. Recall that a nondegenerate projective variety $\widetilde{V} \subset \mathbb{P}^N$ is said to satisfy *condition* $N_{2,p}$ for some $p \geq 1$, if its homogeneous ideal is generated by quadrics and the syzygies among them are generated by linear syzygies until the $(p-1)$th step. In particular, any 2-regular variety satisfies condition $N_{2,p}$ for all $p > 0$. We obtain the following general bounding result for the regularity of almost non-singular linear projections of a variety satisfying condition $N_{2,p}$ for some $p \geq 2$.

Theorem 3.2. *Let $\widetilde{V} \subset \mathbb{P}^N$ be a nondegenerate projective variety which satisfies condition $N_{2,p}$ for some $p \geq 2$. If $r > N-p$ and $\pi_\Lambda\colon \widetilde{V} \to \mathbb{P}^r$ is an almost nonsingular projection,* *then the subvariety* $\pi_\Lambda(\widetilde{V}) \subset \mathbb{P}^r$ *is* $\max\left\{\operatorname{reg}(\widetilde{V}), N-r+2\right\}$*-regular and its homogeneous ideal is generated by forms of degree* $\leq N-r+2$.

Proof. Let S be the homogeneous coordinate ring of $\mathbb{P}^r$. Also let $\widetilde{A}$ and A be respectively the homogeneous coordinate ring of $\widetilde{V}$ in $\mathbb{P}^N$ and $\pi_\Lambda(\widetilde{V})$ in $\mathbb{P}^r$. Then $\widetilde{A}$ is a finitely generated graded S-module and it follows by [1, Theorem 3.6] that the minimal free presentation of $\widetilde{A}$ has the shape

$$\cdots \to S^s(-2) \xrightarrow{v} S \oplus S^{N-r}(-1) \xrightarrow{q} \widetilde{A} \to 0$$

for some $s \in \mathbb{N}$. Moreover, A is the image $q(S)$. Therefore

$$\widetilde{A}/A \cong \operatorname{Coker}\left(u\colon S^s(-2) \to S^{N-r}(-1)\right),$$

where u is the map naturally induced by v. As π_Λ is an almost nonsingular projection, we have $\dim(\widetilde{A}/A) \leq 1$. So, it follows by [8, Corollary 2.4] that $\operatorname{reg}(\widetilde{A}/A) \leq N-r$. Now, the short exact sequence $0 \to A \to \widetilde{A} \to \widetilde{A}/A \to 0$ implies that

$$\operatorname{reg}\left(\pi_\Lambda(\widetilde{V})\right) = \operatorname{reg}(A)+1 \leq \max\left\{\operatorname{reg}(\widetilde{A})+1, N-r+2\right\} = \max\left\{\operatorname{reg}(\widetilde{V}), N-r+2\right\}.$$

To prove the second statement, consider the exact sequence

$$0 \to \operatorname{Ker}(u) \to S^s(-2) \xrightarrow{u} S^{N-r}(-1) \to \widetilde{A}/A \to 0.$$

Then we get $\operatorname{reg}(\operatorname{Ker}(u)) \leq N-r+2$. In particular, the graded S-module $\operatorname{Ker}(u)$ is generated by homogeneous elements of degree $\leq N-r+2$. Finally, by a diagram chasing, one can show that the homogeneous ideal of $\pi_\Lambda(\widetilde{V})$ is isomorphic to $\operatorname{Ker}(u)/\operatorname{Ker}(v)$, which completes the proof. □

Clearly the above result works most effectively if $\widetilde{V}$ satisfies condition $N_{2,p}$ for a large p.

Corollary 3.3. *Let* $V \subset \mathbb{P}^r$ *be an almost nonsingular projection of a nondegenerate* n*-dimensional projective variety* $\widetilde{V} \subset \mathbb{P}^N$ *of degree* d.

(1) *If* $\widetilde{V}$ *is a variety of minimal degree, then* $\operatorname{reg}(V) \leq d-(r-n)+1$.

(2) *If* $\widetilde{V}$ *is a del Pezzo variety and* $r < N$*, then* $\operatorname{reg}(V) \leq d-(r-n)$.

Proof. (1) This is clear as $\widetilde{V}$ is 2-regular.

(2) If $\widetilde{V}$ is del Pezzo it satisfies condition $N_{2,N-n-1}$ and $\operatorname{reg}(\widetilde{V}) = 3$. Then the assertion comes immediately from Theorem 3.2. □

Proof of Theorem 1.2(1). We have $\operatorname{reg}(X) \geq d-r+3$ since X admits $(d-r+3)$-secant lines. On the other hand, it holds by Corollary 3.3 that $\operatorname{reg}(X) \leq d-r+3$. □

Next, we aim to show that the intersection $X \cap \mathbb{F}(X)$ is a pure plane curve of degree $d-r+3$. This enables us to prove that X is linearly normal. Then, we will investigate some cohomological properties of X in terms of four natural short exact sequences induced from Definition and Remark 3.1(B) and Lemma 3.4 below. For details, see Notation and Remark 3.5.

Lemma 3.4. *The intersection $\mathcal{C} := X \cap \mathbb{F}(X)$ is a plane curve of degree $d-r+3$.*

Proof. For a general hyperplane $\mathbb{H}$ of $\mathbb{P}^r$, the curve $\mathcal{C}_{\mathbb{H}} = X \cap \mathbb{H}$ is of maximal regularity and the line $\mathbb{L}_{\mathbb{H}} = \mathbb{F}(X) \cap \mathbb{H}$ is $(d-r+3)$-secant to $\mathcal{C}_{\mathbb{H}}$. This shows that $\mathcal{C}$ contains a plane curve of degree $\geq d-r+3$. On the other hand, if either the degree of the plane curve is strictly bigger than $d-r+3$ or if $\mathcal{C}$ has a closed associated point, one can easily see that X admits a proper multisecant line $\mathbb{L}$ such that $\operatorname{length}(X \cap \mathbb{L}) > d-r+3$. This is impossible since $\operatorname{reg}(X) = d-r+3$ by Theorem 1.2. $\square$

Proof of Theorem 1.2(2). Assume $h^1(\mathbb{P}^r, \mathcal{I}_X(1)) > 0$. Then X is a regular projection of a nondegenerate surface X' in $\mathbb{P}^{r+1}$. Note that X' is again an almost nonsingular projection of a smooth rational normal surface scroll. Therefore we have $\operatorname{reg}(X') \leq d-r+2$ by Corollary 3.3. On the other hand, the pre-image of the plane curve $X \cap \mathbb{F}(X)$ under this regular projection is again a plane curve of degree $(d-r+3)$. In particular, we have $\operatorname{reg}(X') \geq d-r+3$. This contradiction proves our claim. $\square$

We now introduce four basic exact sequences which are obtained from Definition and Remark 3.1(B) and Lemma 3.4. They will play a crucial role in all what follows.

Notation and Remark 3.5. (A) On use of [2, Proposition 5.2] we see that $\mathrm{e}(X) = h^1(X, \mathcal{O}_X(n))$ for all $n \leq 0$. Now, consider the almost non-singular projection $f\colon \widetilde{X} \to X$ from the rational normal surface scroll $\widetilde{X} \subset \mathbb{P}^{d+1}$. Then we have the exact sequence

$$0 \to \mathcal{O}_X \to f_*\mathcal{O}_{\widetilde{X}} \to \mathcal{F} := f_*\mathcal{O}_{\widetilde{X}}/\mathcal{O}_X \to 0 \tag{3.1}$$

with $h^0(X, \mathcal{F}) = \mathrm{e}(X)$, where the equality follows from the fact that $\operatorname{Supp}(\mathcal{F}) = \operatorname{Sing}(X)$ is a finite set and $\widetilde{X}$ is a Cohen-Macaulay variety. As an immediate consequence of (3.1), we get

$$h^3(\mathbb{P}^r, \mathcal{I}_X(j)) = h^2(X, \mathcal{O}_X(j)) = h^2(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(j)) = \begin{cases} \frac{(j+1)(dj+2)}{2} & \text{if } j \leq -2, \\ 0 & \text{if } j \geq -1. \end{cases} \tag{3.2}$$

(B) Let $\mathbb{H}$ be a general hyperplane of $\mathbb{P}^r$. Then the intersection $\mathcal{C}_{\mathbb{H}} = X \cap \mathbb{H}$ is an integral curve of maximal regularity and the line $\mathbb{L}_{\mathbb{H}} = \mathbb{F}(X) \cap \mathbb{H}$ is $(d-r+3)$-secant to $\mathcal{C}_{\mathbb{H}}$. We have the exact sequences

$$0 \to \mathcal{I}_X(-1) \to \mathcal{I}_X \to \mathcal{I}_{\mathcal{C}_{\mathbb{H}}} \to 0 \tag{3.3}$$

and

$$0 \to \mathcal{I}_Y(-1) \to \mathcal{I}_Y \to \mathcal{I}_{\mathcal{C}_{\mathbb{H}} \cup \mathbb{L}_{\mathbb{H}}} \to 0 \tag{3.4}$$

where $Y = X \cup \mathbb{F}(X)$. In [6, Proposition 2.7(c),(d)], it is shown that

$$H^1(\mathbb{P}^{r-1}, \mathcal{I}_{\mathcal{C}_{\mathbb{H}} \cup \mathbb{L}_{\mathbb{H}}}(j)) = 0 \text{ if } j \leq 1 \quad \text{and} \quad H^2(\mathbb{P}^{r-1}, \mathcal{I}_{\mathcal{C}_{\mathbb{H}} \cup \mathbb{L}_{\mathbb{H}}}(j)) = 0 \text{ if } j \geq 1. \tag{3.5}$$

So, by combining (3.4) with the second part of (3.5), we can easily show that

$$H^3(\mathbb{P}^r, \mathcal{I}_Y(j)) = 0 \quad \text{for all } j \geq 0. \tag{3.6}$$

(C) Note that the quotient sheaf $\mathcal{I}_X/\mathcal{I}_Y$ is isomorphic to the ideal sheaf of $\mathcal{C}$ in $\mathbb{F}(X)$. Therefore, by Lemma 3.4, we have the exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^2}(-d+r-3) \to 0. \tag{3.7}$$

Proposition 3.6. *For all $j \in \mathbb{Z}$, put $a_j := h^2(\mathbb{P}^r, \mathcal{I}_X(j))$ and $b_j := h^2(\mathbb{P}^r, \mathcal{I}_Y(j))$. Then it holds*

(1) *$a_j = \mathrm{e}(X)$ for all $j \leq 0$, $a_1 = \mathrm{e}(X) = r - d - 1$, $a_{d-r} = 1$, $a_j = 0$ for all $j > d - r$ and the sequence a_j is strictly decreasing in the range $1 \leq j \leq d - r$.*

(2) *$b_0 = b_1 = \mathrm{e}(X) - \binom{d-r+2}{2}$, hence $\mathrm{e}(X) \geq \binom{d-r+2}{2}$ and the sequence of the b_j is increasing in the range $j \leq 0$ and decreasing in the range $j \geq 1$.*

Proof. (1) As $h^1(X, f_*\mathcal{O}_{\widetilde{X}}(n)) = H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(n)) = 0$ for all $n \in \mathbb{Z}$ it follows from the sequence (3.1) that

$$a_j = h^2(\mathbb{P}^r, \mathcal{I}_X(j)) = h^1(X, \mathcal{O}_X(j)) = h^0(X, \mathcal{F}) = \mathrm{e}(X) \quad \text{for all } j \leq 0$$

and

$$a_1 = h^2(\mathbb{P}^r, \mathcal{I}_X(1)) = h^1(X, \mathcal{O}_X(1)) = (r+1) - (d+2) + \mathrm{e}(X) = \mathrm{e}(X) + r - d - 1.$$

We have $a_j = 0$ for all $j \geq d - r + 1$ since X is $(d - r + 3)$-regular. Now, consider the exact cohomology sequence induced by (3.3):

$$H^1(\mathbb{P}^{r-1}, \mathcal{I}_{\mathcal{C}_{\mathbb{H}}}(d-r+1)) \to H^2(\mathbb{P}^r, \mathcal{I}_X(d-r)) \to H^2(\mathbb{P}^r, \mathcal{I}_X(d-r+1)) = 0.$$

Thus a_{d-r} is at most 1 since $h^1(\mathbb{P}^{r-1}, \mathcal{I}_{\mathcal{C}_{\mathbb{H}}}(d-r+1)) = 1$ (see [12, Remarks on page 504]). On the other hand, the equalities (3.6) and the sequence (3.7) enable us to show that a_{d-r} must be positive. This completes the proof that $a_{d-r} = 1$. Now, from the exact sequence $0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_{C_{\mathbb{H}}} \to 0$, we have

$$\bigoplus_{j \geq 0} H^0(C_{\mathbb{H}}, \mathcal{O}_{C_{\mathbb{H}}}(j)) \to \bigoplus_{j \geq 0} H^1(X, \mathcal{O}_X(j-1)) \to \bigoplus_{j \geq 0} H^1(X, \mathcal{O}_X(j)) \to 0.$$

Therefore $h^1(X,\mathcal{O}_X(j-1)) \geq h^1(X,\mathcal{O}_X(j))$ for all $j \geq 0$. Let E denote the image of $\bigoplus_{j\geq 0} H^0(C_{\mathbb{H}},\mathcal{O}_{C_{\mathbb{H}}}(j))$ in $\bigoplus_{j\geq 0} H^1(X,\mathcal{O}_X(j-1))$. Note that $\mathcal{O}_{C_{\mathbb{H}}}$ is 0-regular as a coherent sheaf on $\mathbb{P}^r$. Thus E is generated by E_1 as a graded S-module. This means that if $E_m = 0$ for some $m > 0$ then $E_j = 0$ for all $j \geq m$. So, if $h^1(X,\mathcal{O}_X(m-1)) = h^1(X,\mathcal{O}_X(m))$ for some $m > 0$ we have

$$h^1(X,\mathcal{O}_X(m-1)) = h_1(X,\mathcal{O}_X(m)) = h_1(X,\mathcal{O}_X(m+1)) = \cdots,$$

which happens only when $H^1(X,\mathcal{O}_X(m-1)) = H^1(X,\mathcal{O}_X(m)) = \cdots = 0$. This shows that the sequence $a_0, a_1, \ldots, a_{d-r+1}$ must strictly decrease.

(2) The first formula is obtained immediately from (3.6) and (3.7). The second statement about the sequence $\{b_j\}_{j\in\mathbb{Z}}$ can be shown by using (3.4) and the first part of (3.5). $\square$

4. The arithmetic depth of SMSR of type II

This section is devoted to the study of the arithmetic depth of X and $Y := X \cup \mathbb{F}(X)$, where X is a SMSR of type II. Our main result is

Theorem 4.1. *Let $\tau(X)$ denote the pair* $(\mathrm{depth}(X), \mathrm{depth}(Y))$. *Then*

(1) *If $r+1 \leq d \leq 2r-4$, then $\tau(X) = (2,3)$.*

(2) *If $2r-3 \leq d \leq 3r-7$, then $\tau(X)$ is equal to $(1,1)$ or $(2,2)$ or $(2,3)$.*

(3) *If $3r-6 \leq d$, then $\tau(X)$ is equal to $(1,1)$ or $(2,2)$.*

To prove this theorem we need some preparations. First we consider the exact sequence of graded S-modules which is induced by (3.7):

$$0 \to I_Y \to I_X \to E \to H^1_*(\mathcal{I}_Y) \to H^1_*(\mathcal{I}_X) \to 0. \tag{4.1}$$

So, the degree j piece E_j of E is equal to $H^0(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(j-d+r-3))$.

Lemma 4.2. (1) *The sequence $0 \to I_Y \to I_X \to E \to 0$ is exact.*

(2) *There exists a form $g \in S$ of degree $d-r+3$ such that $I_X = I_Y + \langle g\rangle$.*

(3) *$H^1_*(\mathcal{I}_Y) \cong H^1_*(\mathcal{I}_X)$. In particular,* $\mathrm{depth}(X) = 1$ *if and only if* $\mathrm{depth}(Y) = 1$.

(4) $\mathrm{reg}(Y) \leq d-r+3$.

Proof. Note that I_X has a form g of degree $d-r+3$ whose restriction to $\Bbbk[\mathbb{F}(X)]$, the homogeneous coordinate ring of $\mathbb{F}(X)$, is a defining equation of the plane curve $X \cap \mathbb{F}(X)$. This element g maps to a nonzero element, say $\widehat{g}$, of $E_{d-r+3} = H^0(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2})$. Obviously,

E is generated by $\widehat{g}$ and hence the map $I_X \to E$ is surjective. This proves (1) and (2). Then (3) comes from (1) and (4.1). Now, (4) can be shown by (3), Proposition 3.6 and Theorem 1.2(1). □

Proposition 4.3. *Set $m := \operatorname{reg}(Y)$. Then the following statements hold:*

(1) *For all $i \geq 1$ we have*

$$\beta_{i,j}(X) = \begin{cases} \beta_{i,j}(Y) & \text{for } 1 \leq j \leq m-1, \\ \beta_{i,j}(Y) = 0 & \text{for } m \leq j \leq d-r+1, \\ \beta_{i,d-r+2}(Y) + \binom{r-2}{i-1} & \text{for } j = d-r+2 \end{cases}$$

where $\beta_{i,j}(X)$ and $\beta_{i,j}(Y)$ are respectively the (i,j)th graded Betti numbers of the homogeneous coordinate rings of X and Y.

(2) *$m \leq d-r+2$ if and only if $\beta_{i,d-r+2}(X) = \binom{r-2}{i-1}$ for all $i \geq 1$.*

Proof. The structure of the minimal free resolution of $E = (S/I_{\mathbb{F}(X)})(-d+r-3)$ is evident. Now, the assertion comes immediately from the long exact sequence of Tor functors induced by the exact sequence $0 \to I_Y \to I_X \to E \to 0$. □

Proposition 4.4. *Let $\mathcal{C} \subset \mathbb{P}^r$, $r \geq 4$, be a curve of degree $d \geq 3r-3$ and of maximal regularity. Also let $\mathbb{L}$ be a $(d-r+2)$-secant line to $\mathcal{C}$. Then* $\operatorname{depth}(\mathcal{C} \cup \mathbb{L}) = 1$.

Proof. Note that $\mathbb{S} := \operatorname{Join}(\mathbb{L}, C) \subset \mathbb{P}^r$ is a rational normal 3-fold scroll of type $S(0,0,r-2)$ whose vertex $S(0,0)$ is exactly the line $\mathbb{L}$. Let $I_{\mathcal{C}}$ and $I_{\mathbb{S}}$ be respectively the homogeneous vanishing ideals of $\mathcal{C}$ and $\mathbb{S}$ in S. Then we have

$$\dim_{\Bbbk}(I_{\mathcal{C}})_2 \geq \dim_{\Bbbk}(I_{\mathbb{S}})_2 = \binom{r-2}{2}.$$

Assume now that $\operatorname{depth}(\mathcal{C} \cup \mathbb{L}) \neq 1$, so that $\mathcal{C} \cup \mathbb{L}$ is arithmetically Cohen-Macaulay. Then, by [6, Proposition 3.6] it follows that

$$\dim_{\Bbbk}(I_{\mathcal{C}})_2 = \binom{r+1}{2} - d - 1, \quad \text{whence} \quad \binom{r-2}{2} \leq \binom{r+1}{2} - d - 1,$$

and this yields the contradiction that $d \leq 3r-4$. □

Proof of Theorem 4.1. Note that $\operatorname{depth}(X)$ is at most 2 (cf. Proposition 3.6(1)). Furthermore, if $\operatorname{depth}(X) = 1$ then $\operatorname{depth}(Y) = 1$, and if $\operatorname{depth}(X) = 2$ then $\operatorname{depth}(Y)$ is 2 or 3 (cf. Lemma 4.2(3)). That is, $\tau(X) \in \{(1,1),(2,2),(2,3)\}$. This proves (2).

(1) If $d \leq 2r-4 = 2(r-1)-2$, then the general hyperplane section $Y \cap \mathbb{H} = \mathcal{C}_{\mathbb{H}} \cup \mathbb{L}_{\mathbb{H}}$ of Y is arithmetically Cohen-Macaulay by [6, Proposition 3.5]. Thus we have $\operatorname{depth}(Y) = 3$ and hence $\operatorname{depth}(X) = 2$.

(2) If $d \geq 3r-6$, then $\operatorname{depth}(\mathcal{C}_{\mathbb{H}} \cup \mathbb{L}_{\mathbb{H}}) = 1$ by Proposition 4.4 and hence Y is not arithmetically Cohen-Macaulay. Therefore either $\tau(X) = (1,1)$ or $\tau(X) = (2,2)$. □

Next, we will construct a few examples, which show that $\tau(X)$ can take all possible pairs listed in Theorem 4.1.

Construction and Examples 4.5. We assume that the characteristic of the base field $\Bbbk$ is zero. Let $\widetilde{X} := S(a,b) \subset \mathbb{P}^{a+b+1}$ be a rational normal surface scroll for some integers $a, b \geq 3$. Now, let Λ be a $(b-3)$-dimensional subspace of $\langle S(b) \rangle = \mathbb{P}^b$ which avoids $S(b)$ and let $X \subset \mathbb{P}^{a+3}$ be the image of $\widetilde{X}$ under the linear projection $\pi_\Lambda \colon \mathbb{P}^{a+b+1} \setminus \Lambda \to \mathbb{P}^{a+1}$. Observe that this linear projection maps $\langle S(b) \rangle$ onto a plane $\mathbb{P}^2 = \mathbb{F}$. Suppose that this projection maps $S(b)$ birationally onto a plane curve $C_b \subset \mathbb{F}$ of degree b. Then, by [5, Theorem 6.3] we obtain:

($*$) X is a surface of maximal sectional regularity of type II and $\mathbb{F}$ is the extremal secant variety of X.

(A) If $b \leq a+2$, then $\deg(X) \leq 2(a+3)-4$ and hence we get $\tau(X) = (2,3)$ by Theorem 4.1(1).

(B) From now on, we assume that $b \geq a+3$, and we will vary the projection center Λ. To do so, we choose a homogeneous polynomial $f \in \Bbbk[s,t]$ of degree b which is not divisible by s and by t. Let $\Lambda_f = \mathbb{P}^{b-3}$ be a subspace of $\langle S(b) \rangle$ such that the plane curve C_b is parametrized by $[s^b : f : t^b]$. Then, the surface

$$X_f := \pi_{\Lambda_f}(\widetilde{X}) \subset \mathbb{P}^{a+3}$$

can be written as

$$X_f = \left\{ [us^a : us^{a-1}t : \ldots : ust^{a-1} : ut^a : vs^b : vf(s,t) : vt^b] \mid (s,t),(u,v) \in K^2 \setminus \{(0,0)\} \right\}.$$

After an appropriate choice of f, this latter presentation is accessible to syzygetic computations. The occurring Betti diagrams have been computed by means of the Computer Algebra System Singular [9].

Example 4.6. Let $(a,b) = (3,8)$ and consider $X_{f_i} \subset \mathbb{P}^6$ $(i = 1,2,3)$ for the following choices of $f = f_i$:

(1) $f_1 = s^7t + s^6t^2 + s^5t^3 + s^4t^4 + s^3t^5 + s^2t^6 + st^7$,

(2) $f_2 = s^7t + s^6t^2 + s^5t^3 + s^4t^4 + s^3t^5 + s^2t^6$, and

(3) $f_3 = s^7t + s^6t^2 + s^5t^3 + s^4t^4$.

Then $X_{f_i} \subset \mathbb{P}^6$ is of degree $d = 11$ $(= 2r-1 = 3r-7)$ for all $i = 1,2,3$. The graded Betti diagrams of X_{f_1}, X_{f_2} and X_{f_3} are given respectively in the three tables below.

	i	1	2	3	4	5	6
X_{f_1}	$\beta_{i,1}$	6	8	3	0	0	0
	$\beta_{i,2}$	0	0	0	0	0	0
	$\beta_{i,3}$	4	12	12	4	0	0
	$\beta_{i,4}$	0	0	0	0	0	0
	$\beta_{i,5}$	1	4	6	4	1	0
	$\beta_{i,6}$	0	0	0	0	0	0
	$\beta_{i,7}$	1	4	6	4	1	0
X_{f_2}	$\beta_{i,1}$	5	5	0	0	0	0
	$\beta_{i,2}$	1	0	1	0	0	0
	$\beta_{i,3}$	1	9	11	4	0	0
	$\beta_{i,4}$	4	18	32	28	12	2
	$\beta_{i,5}$	0	0	0	0	0	0
	$\beta_{i,6}$	0	0	0	0	0	0
	$\beta_{i,7}$	1	4	6	4	1	0
X_{f_3}	$\beta_{i,1}$	3	2	0	0	0	0
	$\beta_{i,2}$	10	27	24	7	0	0
	$\beta_{i,3}$	0	0	0	0	0	0
	$\beta_{i,4}$	0	0	0	0	0	0
	$\beta_{i,5}$	0	0	0	0	0	0
	$\beta_{i,6}$	0	0	0	0	0	0
	$\beta_{i,7}$	1	4	6	4	1	0

By Proposition 4.3(a) we can see from these tables that

$$\tau(X_{f_1}) = (2,2), \quad \tau(X_{f_2}) = (1,1) \quad \text{and} \quad \tau(X_{f_3}) = (2,3).$$

Example 4.7. Let $(a,b) = (3,9)$ and consider $X_{f_i} \subset \mathbb{P}^6$, $(i = 1,2)$ for the two choices

(1) $f_1 = s^8t + s^7t^2 + s^6t^3 + s^5t^4 + s^4t^5 + s^3t^6 + s^2t^7 + st^8$ and

(2) $f_2 = s^8t + s^7t^2 + s^6t^3 + s^5t^4 + s^4t^5 + s^3t^6 + s^2t^7$.

Then $X_{f_i} \subset \mathbb{P}^6$ is of degree $d = 12$ $(= 3r - 6)$ for $i = 1, 2$. The graded Betti diagrams of X_{f_1} and X_{f_2} are given respectively in the tables below.

	i	1	2	3	4	5	6
	$\beta_{i,1}$	6	8	3	0	0	0
	$\beta_{i,2}$	0	0	0	0	0	0
	$\beta_{i,3}$	2	4	0	0	0	0
X_{f_1}	$\beta_{i,4}$	1	4	10	6	1	0
	$\beta_{i,5}$	0	0	0	0	0	0
	$\beta_{i,6}$	1	4	6	4	1	0
	$\beta_{i,7}$	0	0	0	0	0	0
	$\beta_{i,8}$	1	4	6	4	1	0
	$\beta_{i,1}$	5	5	0	0	0	0
	$\beta_{i,2}$	0	0	1	0	0	0
	$\beta_{i,3}$	5	15	15	5	0	0
X_{f_2}	$\beta_{i,4}$	0	0	0	0	0	0
	$\beta_{i,5}$	5	23	42	38	17	3
	$\beta_{i,6}$	0	0	0	0	0	0
	$\beta_{i,7}$	0	0	0	0	0	0
	$\beta_{i,8}$	1	4	6	4	1	0

By Proposition 4.3(a) we can verify that

$$\tau(X_{f_1}) = (2, 2) \quad \text{and} \quad \tau(X_{f_2}) = (1, 1).$$

5. Two more issues on SMSR's of type II

This section is devoted to the discussion of two issues which come naturally from results in our main Theorem 1.2.

For a projective variety $V \subset \mathbb{P}^N$, we define the *index of normality* $N(V)$ of V as the largest integer j such that V fails to be j-normal. When V is arithmetically normal, we define $N(V) = -\infty$ as convention. Now, let $X \subset \mathbb{P}^r$ be a SMSR of type II and of degree d. Then from Theorem 1.2(1) and (3), it is possible that the value of $N(X)$ is strictly smaller than $d - r + 1$. Indeed, this happens for all our computational examples (cf. Problem and Remark 5.2). Along this line, our first main result in this section is about the relations among the index of normality, the Betti numbers of X and the Castelnuovo-Mumford regularity of the union $Y = X \cup \mathbb{F}(X)$.

Theorem 5.1. *Let $X \subset \mathbb{P}^r$, $r \geq 5$, be a SMSR of type II and of degree d. Then the following statements are equivalent:*

(a) $N(X) \leq d - r$.

(b) $\operatorname{reg}(Y) \leq d - r + 2$.

(c) $\beta_{i,d-r+2}(X) = \binom{r-2}{i-1}$ *for all* $i \geq 1$.

(d) $\beta_{r,d-r+2}(X) = 0$.

Proof. (a) $\Rightarrow$ (b): Since X is $(d-r+1)$-normal, we have $H^1(\mathbb{P}^r, \mathcal{I}_Y(d-r+1)) = 0$ by Lemma 4.2(3). Then Proposition 3.6(2) and (3.6) complete the proof that $\operatorname{reg}(Y) \leq d - r + 2$.

(b) $\Rightarrow$ (c): This follows immediately by Proposition 4.3(2).

(c) $\Rightarrow$ (d): This is obviously true.

(d) $\Rightarrow$ (a): If $\operatorname{depth}(X) > 1$, then our claim is obvious. So, we assume that $\operatorname{depth}(X) = 1$ and consider the exact sequence

$$0 \to A_X \to B \to F \to 0$$

where $A_X = S/I_X$ is the homogeneous coordinate ring of X, $B = \bigoplus_{n \in \mathbb{Z}} H^0(\mathbb{P}^r, \mathcal{O}_X(n))$ denotes the total ring of sections of X and $F = H^1_*(\mathcal{I}_X)$. In the long exact sequence

$$\operatorname{Tor}^S_{r+1}(B, \Bbbk)_{d+2} \to \operatorname{Tor}^S_{r+1}(F, \Bbbk)_{d+2} \to \operatorname{Tor}^S_r(A_X, \Bbbk)_{d+2},$$

the first and the third module vanish since $\operatorname{depth}(B)$ is positive and $\beta_{r,d-r+2}(X) = \dim_{\Bbbk} \operatorname{Tor}^S_r(A_X, \Bbbk)_{d+2}$. Therefore we have $\operatorname{Tor}^S_{r+1}(F, \Bbbk)_{d+2} = 0$ and hence $\beta_{r+1,d-r+1}(F) = 0$. This implies that $F_j = H^1(\mathbb{P}^r, \mathcal{I}_X(j))$ vanishes for all $j \geq d - r + 1$ since F is a graded Artinian module over S. □

Problem and Remark 5.2. (A) In Example 4.6, $X_{f_i} \subset \mathbb{P}^6$, $i = 1, 2, 3$, is of degree 11. Also we can see $\beta_{6,7}(X_{f_i}) = 0$ from their graded Betti diagrams. So, X_{f_i} is 6-normal by Theorem 5.1.

(B) In Example 4.7, $X_{f_i} \subset \mathbb{P}^6$, $i = 1, 2$, is of degree 12. Also we can see $\beta_{6,8}(X_{f_i}) = 0$ from their graded Betti diagrams. So, X_{f_i} is 7-normal by Theorem 5.1.

(C) For all our computational examples, it holds that $\beta_{r,d-r+2}(X) = 0$. So we aim to pose the following

Conjecture. *For $X \subset \mathbb{P}^r$ in Theorem* 5.1, *it holds always that* $N(X) \leq d - r$.

Our next issue in this section is about the value of $\mathrm{e}(X)$ and the socle of the second cohomology module $H^2_*(\mathcal{I}_X))$ of X. To be precise, Theorem 1.2(4) says that $\mathrm{e}(X)$ is always greater than or equal to $\binom{d-r+2}{2}$. Concerning the triple $(\widetilde{X}, D, \Lambda)$ of section 1, a general situation will be that $D \subset \widetilde{X}$ is a smooth irreducible curve and the projection center $\Lambda \subset \langle D \rangle$ is general enough so that $C = \pi_\Lambda(D)$ is a plane curve with exactly $\binom{d-r+2}{2}$ nodes. In this case, we get the equality $\mathrm{e}(X) = \binom{d-r+2}{2}$. Along this line, we study the "general case" in which $\mathrm{e}(X)$ takes its minimally possible value $\binom{d-r+2}{2}$. We will show that the condition $\mathrm{e}(X) = \binom{d-r+2}{2}$ is equivalent to the simplicity of the socle of the second total cohomology module of $\mathcal{I}_X$. To formulate our result, we recall the following notation.

Notation and Reminder 5.3. Let $T = \bigoplus_{n\in\mathbb{Z}} T_n$ be a graded S-module. Then, we denote the *socle* of T by $\mathrm{Soc}(T)$, thus:

$$\mathrm{Soc}(T) := (0 :_T S_+) \cong \mathrm{Hom}_S(\Bbbk, T) = \mathrm{Hom}_S(S/S_+, T).$$

Keep in mind that the socle of a graded Artinian S-module T is a $\Bbbk$-vector space of finite dimension, which vanishes if and only if T does.

Proposition 5.4. *The following statements are equivalent:*

(a) $\mathrm{e}(X)$ *takes its minimally possible value* $\binom{d-r+2}{2}$*;*

(b) $h^2(\mathbb{P}^r, \mathcal{I}_Y(j)) = 0$ *for all* $j \in \mathbb{Z}$*;*

(c) $h^2(\mathbb{P}^r, \mathcal{I}_X(j)) = \binom{-j+d-r+2}{2}$ *for all* $j \geq 0$*;*

(d) $\dim_\Bbbk \big(\mathrm{Soc}(H^2_*(\mathcal{I}_X))\big) = 1$.

Proof. Proposition 3.6(2) shows the equivalence of (a), (b) and (c). So, it remains to show the equivalence (b) $\Leftrightarrow$ (d). Consider the exact sequence of graded S-modules

$$0 \to H^2_*(\mathcal{I}_Y) \to H^2_*(\mathcal{I}_X) \to H^2_*(\mathcal{O}_{\mathbb{P}^2}(-d+r-3)).$$

As

$$\mathrm{Soc}\big(H^3_*(\mathcal{I}_{\mathbb{F}}(-d+r-3))\big) = \Bbbk(d-r),$$
$$h^2(\mathbb{P}^r, \mathcal{I}_Y(j)) = 0 \text{ for all } j \geq d-r \quad \text{and} \quad h^2(\mathbb{P}^r, \mathcal{I}_X(j)) = 0 \text{ for all } j \geq d-r+1,$$

we get an isomorphism of graded S-modules

$$\mathrm{Soc}\big(H^2_*(\mathcal{I}_Y)\big) \cong \mathrm{Soc}\big(H^2_*(\mathcal{I}_X)\big)_{\leq d-r-1}.$$

From this isomorphism, we see that

$$H^2_*(\mathcal{I}_Y) = 0 \quad \text{if and only if} \quad \mathrm{Soc}\big(H^2_*(\mathcal{I}_X)\big)_{\leq d-r-1} = 0.$$

By Proposition 3.6(2) the module $H^2_*(\mathcal{I}_Y)$ vanishes if and only if the number $h^2(\mathbb{P}^r, \mathcal{I}_Y)$ does. So, condition (b) holds if and only if $\mathrm{Soc}(H^2_*(\mathcal{I}_X))$ is concentrated in degrees $\geq d-r$. By Proposition 3.6(1) this is the case if and only if condition (d) holds. □

Remark 5.5. Let X be an arbitrary SMSR of type II.

(A) If Y is arithmetically Cohen-Macaulay and hence $\tau(X) = (2,3)$, then the above equivalent conditions (a) and (b) hold. On the other hand, if $\tau(X) = (2,2)$ then $\mathrm{e}(X)$ is strictly bigger than $\binom{d-r+2}{2}$. There exist such surfaces. See Examples 4.6 and 4.7.

(B) From the exact sequence (3.1), we have

$$h^0(X, \mathcal{O}_X(j)) = d\binom{j+1}{2} + j + 1 + h^2(\mathbb{P}^r, \mathcal{I}_X(j)) - \mathrm{e}(X) \quad \text{for all } j \in \mathbb{N}_0.$$

Now, if X is general in the sense specified above, we have

$$h^0(X, \mathcal{O}_X(j)) = \begin{cases} d\binom{j+1}{2} + j + 1 + \binom{d-r+2-j}{2} - \binom{d-r+2}{2} & \text{for } 0 \leq j \leq d-r, \\ d\binom{j+1}{2} + j + 1 - \binom{d-r+2}{2} & \text{for } d-r < j. \end{cases}$$

Acknowledgments

The first named author thanks to the Korea University Seoul, to the Pukyong National University Busan, to the Mathematisches Forschungsinstitut Oberwolfach, to the Martin-Luther Universität Halle and to the Deutsche Forschungsgemeinschaft for their hospitality and the financial support provided during the preparation of this work. The second named author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2014R1A1A4A01008404). The third named author was supported by the Korea Research Foundation Grant funded by the Korean Government (NRF-2016R1D1A1A09918462). The fourth named author thanks to the Korea University Seoul, to the Pukyong National University Busan, to the Mathematisches Forschungsinstitut Oberwolfach and to the Deutsche Forschungsgemeinschaft for their hospitality respectively financial support offered during the preparation of this work.

References

[1] J. Ahn and S. Kwak, *Graded mapping cone theorem, multisecants and syzygies*, J. Algebra **331** (2011), no. 1, 243–262.
https://doi.org/10.1016/j.jalgebra.2010.07.030

[2] C. Albertini and M. Brodmann, *A bound on certain local cohomology modules and application to ample divisors*, Nagoya Math. J. **163** (2001), 87–106.
https://doi.org/10.1017/s0027763000007923

[3] M.-A. Bertin, *On the regularity of varieties having an extremal secant line*, J. Reine Angew. Math. **545** (2002), 167–181. https://doi.org/10.1515/crll.2002.032

[4] ———, *On singular varieties having an extremal secant line*, Comm. Algebra **34** (2006), no. 3, 893–909. https://doi.org/10.1080/00927870500441817

[5] M. Brodmann, W. Lee, E. Park and P. Schenzel, *On projective varieties of maximal sectional regularity*, J. Pure Appl. Algebra **221** (2017), no. 1, 98–118. https://doi.org/10.1016/j.jpaa.2016.05.028

[6] M. Brodmann and P. Schenzel, *On projective curves of maximal regularity*, Math. Z. **244** (2003), no. 2, 271–289. https://doi.org/10.1007/s00209-003-0496-0

[7] ———, *Projective curves with maximal regularity and applications to syzygies and surfaces*, Manuscripta Math. **135** (2011), no. 3-4, 469–495. https://doi.org/10.1007/s00229-011-0426-0

[8] M. Chardin, A. L. Fall and U. Nagel, *Bounds for the Castelnuovo-Mumford regularity of modules*, Math. Z. **258** (2008), no. 1, 69–80. https://doi.org/10.1007/s00209-007-0157-9

[9] M. Decker, G. M. Greuel and H. Schönemann, *Singular* $3-1-2$ – *A computer algebra system for polynomial computations*, (2011). http://www.singular.uni-kl.de

[10] D. Eisenbud and S. Goto, *Linear free resolutions and minimal multiplicity*, J. Algebra **88** (1984), no. 1, 89–133. https://doi.org/10.1016/0021-8693(84)90092-9

[11] T. Fujita, *Classification Theories of Polarized Varieties*, London Mathematical Society Lecture Note Series **155**, Cambridge University Press, Cambridge, 1990. https://doi.org/10.1017/cbo9780511662638

[12] L. Gruson, R. Lazarsfeld and C. Peskine, *On a theorem of Castelnuovo, and the equations defining space curves*, Invent. Math. **72** (1983), no. 3, 491–506. https://doi.org/10.1007/bf01398398

[13] S. Kwak and E. Park, *Some effects of property N_p on the higher normality and defining equations of nonlinearly normal varieties*, J. Reine Angew. Math. **582** (2005), 87–105. https://doi.org/10.1515/crll.2005.2005.582.87

[14] R. Lazarsfeld, *A sharp Castelnuovo bound for smooth surfaces*, Duke Math. J. **55** (1987), no. 2, 423–429. https://doi.org/10.1215/s0012-7094-87-05523-2

[15] A. Noma, *Multisecant lines to projective varieties*, in *Projective Varieties with Unexpected Properties*, A volume in memory of Giuseppe Veronese, Proceedings of the international conference 'Varieties with Unexpected Properties', Siena, Italy, June 8–13, 2004. Edited by Ciliberto, Ciro et al., Walter de Gruyter, Berlin, (2005), 349–359. https://doi.org/10.1515/9783110199703.349

[16] E. Park, *On syzygies of divisors on rational normal scrolls*, Math. Nachr. **287** (2014), no. 11-12, 1383–1393. https://doi.org/10.1002/mana.201300085

[17] H. C. Pinkham, *A Castelnuovo bound for smooth surfaces*, Invent. Math. **83** (1986), no. 2, 321–332. https://doi.org/10.1007/bf01388966

Markus Brodmann
Universität Zürich, Institut für Mathematik, Winterthurerstrasse 190, CH – Zürich, Switzerland
E-mail address: brodmann@math.uzh.ch

Wanseok Lee
Pukyong National University, Department of applied Mathematics, Daeyeon Campus 45, Yongso-ro, Nam-Gu, Busan, Republic of Korea
E-mail address: wslee@pknu.ac.kr

Euisung Park
Korea University, Department of Mathematics, Anam-dong, Seongbuk-gu, Seoul 136-701, Republic of Korea
E-mail address: euisungpark@korea.ac.kr

Peter Schenzel
Martin-Luther-Universität Halle-Wittenberg, Institut für Informatik,
Von-Secken-dorff-Platz 1, D – 06120 Halle (Saale), Germany
E-mail address: schenzel@informatik.uni-halle.de