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On Weak*-convergence in the Localized Hardy Spaces $H^1_\rho(\mathcal{X})$ and its Application

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Abstract. Let $(\mathcal{X}, d, \mu)$ be a complete RD-space. Let ρ be an admissible function on $\mathcal{X}$, which means that ρ is a positive function on $\mathcal{X}$ and there exist positive constants C_0 and k_0 such that, for any $x, y \in \mathcal{X}$,

$$\rho(y) \leq C_0[\rho(x)]^{1/(1+k_0)}[\rho(x) + d(x,y)]^{k_0/(1+k_0)}.$$

In this paper, we define a space $\mathrm{VMO}_\rho(\mathcal{X})$ and show that it is the predual of the localized Hardy space $H^1_\rho(\mathcal{X})$ introduced by Yang and Zhou [14]. Then we prove a version of the classical theorem of Jones and Journé [7] on weak*-convergence in $H^1_\rho(\mathcal{X})$. As an application, we give an atomic characterization of $H^1_\rho(\mathcal{X})$.

1. Introduction

It is a well-known and classical result (see [2]) that the space $\mathrm{BMO}(\mathbb{R}^n)$ is the dual of the Hardy space $H^1(\mathbb{R}^n)$ one of the few examples of separable, nonreflexive Banach space which is a dual space. In fact, let $C_c(\mathbb{R}^n)$ be the space of all continuous functions with compact support and denote by $\mathrm{VMO}(\mathbb{R}^n)$ the closure of $C_c(\mathbb{R}^n)$ in $\mathrm{BMO}(\mathbb{R}^n)$, Coifman and Weiss showed in [2] that $H^1(\mathbb{R}^n)$ is the dual space of $\mathrm{VMO}(\mathbb{R}^n)$, which gives to $H^1(\mathbb{R}^n)$ a richer structure than $L^1(\mathbb{R}^n)$. For example, the classical Riesz transforms $\nabla(-\Delta)^{-1/2}$ are not bounded on $L^1(\mathbb{R}^n)$, but are bounded on $H^1(\mathbb{R}^n)$. In addition, the weak*-convergence is true in $H^1(\mathbb{R}^n)$ (see [7]), which is useful in the application of Hardy spaces to compensated compactness (see [1]) and in the study of commutators of singular integral operators (see [8,10]). Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n$, $n \geq 3$, where V is a nonnegative function, $V \neq 0$, and belongs to the reverse Hölder class $\mathrm{RH}_{n/2}(\mathbb{R}^n)$. The Hardy space associated with the Schrödinger operator L, $H^1_L(\mathbb{R}^n)$, is then defined as the set of functions

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$f \in L^1(\mathbb{R}^n)$ such that $\|f\|_{H^1_L} := \|\mathcal{M}_L f\|_{L^1} < \infty$, where $\mathcal{M}_L f(x) := \sup_{t>0} \left|e^{-tL} f(x)\right|$. Recently, Ky [9] established that the weak*-convergence is true in $H^1_L(\mathbb{R}^n)$, which is useful in studying the endpoint estimates for commutators of singular integral operators related to L (see [10]).

Let $(\mathcal{X}, d, \mu)$ be an RD-space, which means that $(\mathcal{X}, d, \mu)$ is a space of homogeneous type in the sense of Coifman-Weiss with the additional property that a reverse doubling property holds in $\mathcal{X}$ (see Section 2). Typical examples for such RD-spaces include Euclidean spaces, Heisenberg groups, Lie groups of polynomial growth, or more generally, Carnot-Carathéodory spaces with doubling measures. We refer to the seminal paper of Han, Müller and Yang [4] for a systematic study of the theory of function spaces in harmonic analysis on RD-spaces. Recently, Yang and Zhou [14] introduced and studied the theory of localized Hardy spaces $H^1_\rho(\mathcal{X})$ related to the admissible functions ρ. There, they showed that this theory has a wide range of applications in studying the theory of Hardy spaces associated with Schrödinger operators or degenerate Schrödinger operators on $\mathbb{R}^n$, or associated with sub-Laplace Schrödinger operators on Heisenberg groups or connected and simply connected nilpotent Lie groups, see [14, Section 5] for details.

Given a complete RD-space $(\mathcal{X}, d, \mu)$ and an admissible function ρ, we denote by $\mathrm{BMO}_\rho(\mathcal{X})$ the dual space of $H^1_\rho(\mathcal{X})$ (see Section 2) and $\mathrm{VMO}_\rho(\mathcal{X})$ the closure in the BMO_ρ-norm of the space $C_c(\mathcal{X})$ of all continuous functions with compact support. The aim of the present paper is to show that $H^1_\rho(\mathcal{X})$ is a dual space and that the weak*-convergence is true in $H^1_\rho(\mathcal{X})$. Our main results can be read as follows:

Theorem 1.1. *The space $H^1_\rho(\mathcal{X})$ is the dual of the space $\mathrm{VMO}_\rho(\mathcal{X})$.*

Theorem 1.2. *Suppose that $\{f_j\}_{j\geq 1}$ is a bounded sequence in $H^1_\rho(\mathcal{X})$, and that $f_j(x) \to f(x)$ for almost every $x \in \mathcal{X}$. Then, $f \in H^1_\rho(\mathcal{X})$ and $\{f_j\}_{j\geq 1}$ weak*-converges to f, that is, for every $\varphi \in \mathrm{VMO}_\rho(\mathcal{X})$, we have*

$$\lim_{j\to\infty} \int_{\mathcal{X}} f_j(x)\varphi(x)\, d\mu(x) = \int_{\mathcal{X}} f(x)\varphi(x)\, d\mu(x).$$

It should be pointed out that when $\mathcal{X} \equiv \mathbb{R}^n$, $n \geq 3$, and

$$\rho(x) \equiv \sup\left\{r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)\, dy \leq 1\right\},$$

where V is in the reverse Hölder class $\mathrm{RH}_{n/2}(\mathbb{R}^n)$. Theorem 1.2 is just the main theorem in the paper of Ky [9, Theorem 1.1].

Throughout the whole paper, C denotes a positive geometric constant which is independent of the main parameters, but may change from line to line.

2. Preliminaries

Let d be a quasi-metric on a set $\mathcal{X}$, that is, d is a nonnegative function on $\mathcal{X}\times\mathcal{X}$ satisfying

(a) $d(x,y)=d(y,x)$,

(b) $d(x,y)>0$ if and only if $x\neq y$,

(c) there exists a constant $\kappa\geq 1$ such that for all $x,y,z\in\mathcal{X}$,

$$d(x,z)\leq \kappa(d(x,y)+d(y,z)). \tag{2.1}$$

A trip $(\mathcal{X},d,\mu)$ is called a *space of homogeneous type* in the sense of Coifman-Weiss if μ is a regular Borel measure satisfying *doubling property*, i.e., there exists a constant $C>1$ such that for all $x\in\mathcal{X}$ and $r>0$,

$$\mu(B(x,2r))\leq C\mu(B(x,r)).$$

Remark 2.1. By [2, Theorem (3.2)], we see that if $(\mathcal{X},d,\mu)$ is a complete space of homogeneous type, then the closure of B is a compact set for all ball $B\subset\mathcal{X}$.

Recall (see [4]) that a space of homogeneous type $(\mathcal{X},d,\mu)$ is called an *RD-space* if it satisfies *reverse doubling property*, i.e., there exists a constant $C>1$ such that

$$\mu(B(x,2r))\geq C\mu(B(x,r))$$

for all $x\in\mathcal{X}$ and $r\in(0,\operatorname{diam}(\mathcal{X})/2)$, where $\operatorname{diam}(\mathcal{X}):=\sup_{x,y\in\mathcal{X}}d(x,y)$.

Here and what in follows, for $x,y\in\mathcal{X}$ and $r>0$, we denote $V_r(x):=\mu(B(x,r))$ and $V(x,y):=\mu(B(x,d(x,y)))$.

Definition 2.2. Let $x_0\in\mathcal{X}$, $r>0$, $0<\beta\leq 1$ and $\gamma>0$. A function f is said to belong to the space of test functions, $\mathcal{G}(x_0,r,\beta,\gamma)$, if there exists a positive constant C_f such that

(i) $|f(x)|\leq C_f\frac{1}{V_r(x_0)+V(x_0,x)}\left(\frac{r}{r+d(x_0,x)}\right)^\gamma$ for all $x\in\mathcal{X}$;

(ii) $|f(x)-f(y)|\leq C_f\left(\frac{d(x,y)}{r+d(x_0,x)}\right)^\beta\frac{1}{V_r(x_0)+V(x_0,x)}\left(\frac{r}{r+d(x_0,x)}\right)^\gamma$ for all $x,y\in\mathcal{X}$ satisfying that $d(x,y)\leq\frac{r+d(x_0,x)}{2\kappa}$.

For any $f\in\mathcal{G}(x_0,r,\beta,\gamma)$, we define

$$\|f\|_{\mathcal{G}(x_0,r,\beta,\gamma)}:=\inf\{C_f:\text{(i) and (ii) hold}\}.$$

Let ρ be a positive function on $\mathcal{X}$. Following Yang and Zhou [14], the function ρ is said to be *admissible* if there exist positive constants C_0 and k_0 such that, for any $x, y \in \mathcal{X}$,

$$\rho(y) \leq C_0[\rho(x)]^{1/(1+k_0)}[\rho(x) + d(x, y)]^{k_0/(1+k_0)}.$$

Throughout the whole paper, we always assume that $\mathcal{X}$ is a complete RD-space with $\mu(\mathcal{X}) = \infty$, and ρ is an admissible function on $\mathcal{X}$. Also we fix $x_0 \in \mathcal{X}$.

In Definition 2.2, it is easy to see that $\mathcal{G}(x_0, 1, \beta, \gamma)$ is a Banach space. For simplicity, we write $\mathcal{G}(\beta, \gamma)$ instead of $\mathcal{G}(x_0, 1, \beta, \gamma)$. Let $\epsilon \in (0, 1]$ and $\beta, \gamma \in (0, \epsilon]$, we define the space $\mathcal{G}_0^\epsilon(\beta, \gamma)$ to be the completion of $\mathcal{G}(\epsilon, \epsilon)$ in $\mathcal{G}(\beta, \gamma)$, and denote by $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$ the space of all continuous linear functionals on $\mathcal{G}_0^\epsilon(\beta, \gamma)$. We say that f is a *distribution* if f belongs to $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$.

Remark that, for any $x \in \mathcal{X}$ and $r > 0$, one has $\mathcal{G}(x, r, \beta, \gamma) = \mathcal{G}(x_0, 1, \beta, \gamma)$ with equivalent norms, but of course the constants are depending on x and r.

Let f be a distribution in $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$. We define *the grand maximal functions* $\mathcal{M}(f)$ and $\mathcal{M}_\rho(f)$ as follows

$$\mathcal{M}(f)(x) := \sup\left\{|\langle f, \varphi\rangle| : \varphi \in \mathcal{G}_0^\epsilon(\beta, \gamma), \|\varphi\|_{\mathcal{G}(x,r,\beta,\gamma)} \leq 1 \text{ for some } r > 0\right\},$$
$$\mathcal{M}_\rho(f)(x) := \sup\left\{|\langle f, \varphi\rangle| : \varphi \in \mathcal{G}_0^\epsilon(\beta, \gamma), \|\varphi\|_{\mathcal{G}(x,r,\beta,\gamma)} \leq 1 \text{ for some } r \in (0, \rho(x))\right\}.$$

Definition 2.3. Let $\epsilon \in (0, 1)$ and $\beta, \gamma \in (0, \epsilon)$.

(i) The Hardy space $H^1(\mathcal{X})$ is defined by

$$H^1(\mathcal{X}) = \left\{f \in (\mathcal{G}_0^\epsilon(\beta, \gamma))' : \|f\|_{H^1} := \|\mathcal{M}(f)\|_{L^1} < \infty\right\}.$$

(ii) The Hardy space $H_\rho^1(\mathcal{X})$ is defined by

$$H_\rho^1(\mathcal{X}) = \left\{f \in (\mathcal{G}_0^\epsilon(\beta, \gamma))' : \|f\|_{H_\rho^1} := \|\mathcal{M}_\rho(f)\|_{L^1} < \infty\right\}.$$

Remark 2.4. It was established in [3] that the space $H^1(\mathcal{X})$ coincides with the atomic Hardy $H^1_{\rm at}(\mathcal{X})$ of Coifman and Weiss [2]. Moreover, for all $f \in H^1(X)$,

$$\|f\|_{L^1} \leq C\|f\|_{H_\rho^1} \leq C\|f\|_{H^1}.$$

Recall (see [2]) that a function $f \in L^1_{\rm loc}(\mathcal{X})$ is said to be in BMO$(\mathcal{X})$ if

$$\|f\|_{\rm BMO} := \sup_B \frac{1}{\mu(B)} \int_B \left| f(x) - \frac{1}{\mu(B)} \int_B f(y)\, d\mu(y)\right| d\mu(x) < \infty,$$

where the supremum is taken all over balls $B \subset \mathcal{X}$. Denote by VMO$(\mathcal{X})$ the closure in BMO norm of $C_c(\mathcal{X})$. The following is well-known (see [2]).

Theorem 2.5. (i) *The space* $\mathrm{BMO}(\mathcal{X})$ *is the dual space of* $H^1(\mathcal{X})$*.*

(ii) *The space* $H^1(\mathcal{X})$ *is the dual space of* $\mathrm{VMO}(\mathcal{X})$*.*

Definition 2.6. Let ρ be an admissible function and $\mathcal{D} := \{B(x,r) \subset \mathcal{X} : r \geq \rho(x)\}$. A function $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ is said to be in $\mathrm{BMO}_\rho(\mathcal{X})$ if

$$\|f\|_{\mathrm{BMO}_\rho} := \|f\|_{\mathrm{BMO}} + \sup_{B \in \mathcal{D}} \frac{1}{\mu(B)} \int_B |f(x)|\, d\mu(x) < \infty.$$

It was established in [13] that

Theorem 2.7. *The space* $\mathrm{BMO}_\rho(\mathcal{X})$ *is the dual space of* $H^1_\rho(\mathcal{X})$*.*

3. Proofs of Theorems 1.1 and 1.2

We begin by recalling the following (see [14, Proposition 3.1]).

Lemma 3.1. *Let ρ be an admissible function. Then, there exists a function* $K_\rho\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ *and a positive constant C such that*

(i) $K_\rho(x,y) \geq 0$ *for all* $x, y \in \mathcal{X}$*, and* $K_\rho(x,y) = 0$ *if* $d(x,y) > C \min\{\rho(x), \rho(y)\}$*;*

(ii) $K_\rho(x,y) \leq C \frac{1}{\mu(B(x,\rho(x))) + \mu(B(y,\rho(y)))}$ *for all* $x, y \in \mathcal{X}$*;*

(iii) $K_\rho(x,y) = K_\rho(y,x)$ *for all* $x, y \in \mathcal{X}$*;*

(iv) $|K_\rho(x,y) - K_\rho(x,y')| \leq C \frac{d(y,y')}{\rho(x)} \frac{1}{\mu(B(x,\rho(x))) + \mu(B(y,\rho(y)))}$ *for all* $x, y, z \in \mathcal{X}$ *with* $d(y,y') \leq [\rho(x) + d(x,y)]/2$*;*

(v) *for any* $x, x', y, y' \in \mathcal{X}$ *satisfying* $d(x,x') \leq [\rho(y) + d(x,y)]/3$ *and* $d(y,y') \leq [\rho(x) + d(x,y)]/3$*, we have*

$$\begin{aligned} &\big|[K_\rho(x,y) - K_\rho(x,y')] - [K_\rho(x',y) - K_\rho(x',y')]\big| \\ &\leq C \frac{d(x,x')}{\rho(y)} \frac{d(y,y')}{\rho(x)} \frac{1}{\mu(B(x,\rho(x))) + \mu(B(y,\rho(y)))}; \end{aligned}$$

(vi) $\int_{\mathcal{X}} K_\rho(x,y)\, d\mu(x) = 1$ *for all* $y \in \mathcal{X}$*.*

Given a function f in $L^1(\mathcal{X})$, following [14], we define

$$K_\rho(f)(x) = \int_{\mathcal{X}} K_\rho(x,y) f(y)\, d\mu(y)$$

for all $x \in \mathcal{X}$. It follows directly from Lemma 3.1 that

$$\int_{\mathcal{X}} K_\rho(f)(x) g(x)\, d\mu(x) = \int_{\mathcal{X}} K_\rho(g)(x) f(x)\, d\mu(x) \tag{3.1}$$

for all $f \in L^1(\mathcal{X})$ and $g \in L^\infty(\mathcal{X})$. Moreover, by Remark 2.1,

$$K_\rho(\phi) \in C_c(\mathcal{X}) \quad \text{for all } \phi \in C_c(\mathcal{X}), \tag{3.2}$$

and, for any $x \in \mathcal{X}$, the function $\mathbb{K}_\rho(x,\cdot)\colon \mathcal{X} \to \mathbb{R}$, defined by

$$\mathbb{K}_\rho(x,z) := \int_{\mathcal{X}} K_\rho(x,y) K_\rho(y,z)\, d\mu(y), \tag{3.3}$$

is in $C_c(\mathcal{X})$. Remark that $K_\rho(K_\rho(f))(x) = \int_{\mathcal{X}} \mathbb{K}_\rho(x,z) f(z)\, d\mu(z)$.

The following lemma is due to Yang and Zhou [14].

Lemma 3.2. *There exists a positive constant C such that*

(i) *for any $f \in L^1(\mathcal{X})$,*

$$\|K_\rho(f)\|_{H^1_\rho} \le C \|f\|_{L^1};$$

(ii) *for any $g \in H^1_\rho(\mathcal{X})$,*

$$\|g - K_\rho(g)\|_{H^1} \le C \|g\|_{H^1_\rho}.$$

As a consequence of Lemma 3.2 and (3.1), for any $\phi \in C_c(\mathcal{X})$,

$$\|\phi - K_\rho(K_\rho(\phi))\|_{\mathrm{BMO}_\rho} \le C \|\phi\|_{\mathrm{BMO}}. \tag{3.4}$$

Now we are ready to give the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Since $\mathrm{VMO}_\rho(\mathcal{X})$ is a subspace of $\mathrm{BMO}_\rho(\mathcal{X})$, which is the dual space of $H^1_\rho(\mathcal{X})$, every function f in $H^1_\rho(\mathcal{X})$ determines a bounded linear functional on $\mathrm{VMO}_\rho(\mathcal{X})$ of norm bounded by $\|f\|_{H^1_\rho}$.

Conversely, given a bounded linear functional L on $\mathrm{VMO}_\rho(\mathcal{X})$. Then,

$$|L(\phi)| \le \|L\| \, \|\phi\|_{\mathrm{VMO}_\rho} \le \|L\| \, \|\phi\|_{L^\infty}$$

for all $\phi \in C_c(\mathcal{X})$. This implies (see [12]) that there exists a finite signed Radon measure ν on $\mathcal{X}$ such that, for any $\phi \in C_c(\mathcal{X})$,

$$L(\phi) = \int_{\mathcal{X}} \phi(x)\, d\nu(x),$$

moreover, the total variation of ν, $|\nu|(\mathcal{X})$, is bounded by $\|L\|$. Therefore,

$$\|K_\rho(K_\rho(\nu))\|_{H^1_\rho} \le C \|K_\rho(\nu)\|_{L^1} \le C |\nu|(\mathcal{X}) \le C \|L\| \tag{3.5}$$

by Lemma 3.2, where $K_\rho(\nu)(x) := \int_{\mathcal{X}} K_\rho(x,y)\, d\nu(y)$ for all $x \in \mathcal{X}$.

On the other hand, by (3.4) and (3.2), we have

$$\begin{aligned}|(L - K_\rho(K_\rho(L)))(\phi)| &= |L(\phi - K_\rho(K_\rho(\phi)))| \\ &\le \|L\| \, \|\phi - K_\rho(K_\rho(\phi))\|_{\mathrm{VMO}_\rho} \\ &\le C \, \|L\| \, \|\phi\|_{\mathrm{BMO}}\end{aligned}$$

for all $\phi \in C_c(\mathcal{X})$, where $K_\rho(K_\rho(L))(\phi) := \int_{\mathcal{X}} K_\rho(K_\rho(\nu))(x)\phi(x)\, d\mu(x)$. Consequently, by Theorem 2.5(ii), there exists a function h belonging to $H^1(\mathcal{X})$ such that $\|h\|_{H^1} \le C \, \|L\|$ and

$$(L - K_\rho(K_\rho(L)))(\phi) = \int_{\mathcal{X}} h(x)\phi(x)\, d\mu(x)$$

for all $\phi \in C_c(\mathcal{X})$. This, together with (3.5), allows us to conclude that

$$L(\phi) = \int_{\mathbb{R}^d} f(x)\phi(x)\, d\mu(x)$$

for all $\phi \in C_c(\mathcal{X})$, where $f := h + K_\rho(K_\rho(\nu))$ is in $H^1_\rho(\mathcal{X})$ and satisfies that $\|f\|_{H^1_\rho} \le \|h\|_{H^1_\rho} + \|K_\rho(K_\rho(\nu))\|_{H^1_\rho} \le C \, \|L\|$. The proof of Theorem 1.1 is thus completed. □

Proof of Theorem 1.2. Let $\{f_{n_k}\}_{k=1}^\infty$ be an arbitrary subsequence of $\{f_n\}_{n=1}^\infty$. As $\{f_{n_k}\}_{k=1}^\infty$ is a bounded sequence in $H^1_\rho(\mathcal{X})$, by Theorem 1.1 and the Banach-Alaoglu theorem, there exists a subsequence $\left\{f_{n_{k_j}}\right\}_{j=1}^\infty$ of $\{f_{n_k}\}_{k=1}^\infty$ such that $\left\{f_{n_{k_j}}\right\}_{j=1}^\infty$ weak*-converges to g for some $g \in H^1_\rho(\mathcal{X})$. Therefore, by (3.3), for any $x \in \mathcal{X}$,

$$\begin{aligned}\lim_{j\to\infty} K_\rho(K_\rho(f_{n_{k_j}}))(x) &= \lim_{j\to\infty} \int_{\mathcal{X}} \mathbb{K}_\rho(x,z) f_{n_{k_j}}(z)\, d\mu(z) \\ &= \int_{\mathcal{X}} \mathbb{K}_\rho(x,z) g(z)\, d\mu(z) \\ &= K_\rho(K_\rho(g))(x).\end{aligned}$$

This implies that $\lim_{j\to\infty}[f_{n_{k_j}}(x) - K_\rho(K_\rho(f_{n_{k_j}}))(x)] = f(x) - K_\rho(K_\rho(g))(x)$ for almost every $x \in \mathcal{X}$. Hence, by Lemma 3.2 and [6, Theorem 1.1],

$$\|f - K_\rho(K_\rho(g))\|_{H^1} \le \sup_{j\ge1} \left\| f_{n_{k_j}} - K_\rho(K_\rho(f_{n_{k_j}}))\right\|_{H^1} \le C \sup_{j\ge1} \left\| f_{n_{k_j}} \right\|_{H^1_\rho} < \infty,$$

moreover,

$$\lim_{j\to\infty} \int_{\mathcal{X}} [f_{n_{k_j}}(x) - K_\rho(K_\rho(f_{n_{k_j}}))(x)]\phi(x)\, d\mu(x) = \int_{\mathcal{X}} [f(x) - K_\rho(K_\rho(g))(x)]\phi(x)\, d\mu(x)$$

for all $\phi \in C_c(\mathcal{X})$. As a consequence, we obtain that

$$\begin{aligned}\|f\|_{H^1_\rho} &\le \|f - K_\rho(K_\rho(g))\|_{H^1_\rho} + \|K_\rho(K_\rho(g))\|_{H^1_\rho} \\ &\le C \, \|f - K_\rho(K_\rho(g))\|_{H^1} + C \, \|g\|_{H^1_\rho} \\ &\le C \sup_{j\ge1} \left\| f_{n_{k_j}} \right\|_{H^1_\rho} \\ &< \infty,\end{aligned}$$

moreover, by $\left\{f_{n_{k_j}}\right\}_{j=1}^{\infty}$ weak*-converges to g in $H^1_\rho(\mathcal{X})$, (3.1) and (3.2),

$$\begin{aligned}
&\lim_{j\to\infty}\int_{\mathcal{X}} f_{n_{k_j}}(x)\phi(x)\,d\mu(x)\\
&=\lim_{j\to\infty}\int_{\mathcal{X}}[f_{n_{k_j}}(x)-K_\rho(K_\rho(f_{n_{k_j}}))(x)]\phi(x)\,d\mu(x)+\lim_{j\to\infty}\int_{\mathcal{X}} f_{n_{k_j}}(x)K_\rho(K_\rho(\phi))(x)\,d\mu(x)\\
&=\int_{\mathcal{X}}[f(x)-K_\rho(K_\rho(g))(x)]\phi(x)\,d\mu(x)+\int_{\mathcal{X}} g(x)K_\rho(K_\rho(\phi))(x)\,d\mu(x)\\
&=\int_{\mathcal{X}} f(x)\phi(x)\,d\mu(x).
\end{aligned}$$

This, since $\{f_{n_k}\}_{k=1}^{\infty}$ is an arbitrary subsequence of $\{f_n\}_{n=1}^{\infty}$, allows us to complete the proof of Theorem 1.2. □

4. An application

The purpose of this section is to give an atomic characterization of $H^1_\rho(\mathcal{X})$ by using Theorems 1.1 and 1.2. First, we define the concept of atoms of log-type.

Definition 4.1. Given $1 < q \le \infty$. A measurable function a is called an (H^1_ρ, q)-atom of log-type related to the ball $B(x_0, r)$ if

(i) $\operatorname{supp} a \subset B(x_0, r)$,

(ii) $\|a\|_{L^q(\mathcal{X})} \le [\mu(B(x_0, r))]^{1/q-1}$,

(iii) $\left|\int_{\mathcal{X}} a(x)\,d\mu(x)\right| \le 1/\log\left(e + \frac{\rho(x_0)}{r}\right)$.

The main result in this section can be read as follows:

Theorem 4.2. *Let $1 < q \le \infty$. A function f is in $H^1_\rho(\mathcal{X})$ if and only if it can be written as $f = \sum_j \lambda_j a_j$, where a_j are (H^1_ρ, q)-atoms of log-type and $\sum_j |\lambda_j| < \infty$. Moreover, there exists a constant $C > 0$ such that, for any $f \in H^1_\rho(\mathcal{X})$,*

$$\|f\|_{H^1_\rho} \le C \inf\left\{\sum_j |\lambda_j| : f = \sum_j \lambda_j a_j\right\} \le C\,\|f\|_{H^1_\rho}.$$

Before giving the proof of Theorem 4.2, let us recall the definition of H^1_ρ atoms introduced by Yang and Zhou [14].

Definition 4.3. Given $1 < q \le \infty$. A measurable function a is called an (H^1_ρ, q)-atom related to the ball $B(x_0, r)$ if $r < \rho(x_0)$ and

(i) $\operatorname{supp} a \subset B(x_0, r)$,

(ii) $\|a\|_{L^q(\mathcal{X})} \le [\mu(B(x_0, r))]^{1/q-1}$,

(iii) if $r < \rho(x_0)/4$, then $\int_{\mathcal{X}} a(x)\, d\mu(x) = 0$.

Remark 4.4. If a is an (H^1_ρ, q)-atom, then $\frac{1}{\log(e+4)} a$ is an (H^1_ρ, q)-atom of log-type.

Proof of Theorem 4.2. By Remark 4.4 and [14, Theorems 3.2], it suffices to prove that there exists a constant $C > 0$ such that if f can be written as $f = \sum_j \lambda_j a_j$, where a_j are (H^1_ρ, q)-atoms of log-type related to the balls $B(x_j, r_j)$ and $\sum_j |\lambda_j| < \infty$, then $\|f\|_{H^1_\rho} \le C \sum_j |\lambda_j|$. Since Theorem 1.2, we only need to prove that

$$\|a_j\|_{H^1_\rho} \le C$$

for all j. This is reduced to showing that, for any $\phi \in C_c(\mathcal{X})$,

$$\left| \int_{\mathcal{X}} a_j(x)\phi(x)\, d\mu(x) \right| \le C \|\phi\|_{\mathrm{BMO}_\rho} \tag{4.1}$$

by Theorem 1.1. To prove (4.1), let us consider the following two cases:

Case 1: $r_j \ge \rho(x_j)$. Then, by the Hölder inequality and [14, Lemma 2.2],

$$\begin{aligned}\left| \int_{\mathcal{X}} a_j(x)\phi(x)\, d\mu(x) \right| &\le \|a_j\|_{L^q(B(x_j,r_j))} \|\phi\|_{L^{q'}(B(x_j,r_j))} \\ &\le [\mu(B(x_j, r_j))]^{1/q-1} C [\mu(B(x_j, r_j))]^{1/q'} \|\phi\|_{\mathrm{BMO}_\rho} \\ &\le C \|\phi\|_{\mathrm{BMO}_\rho},\end{aligned}$$

where and hereafter $1/q' + 1/q = 1$.

Case 2: $r_j < \rho(x_j)$. Then, by the Hölder inequality, [14, Lemma 2.2] and [11, Lemma 2.1],

$$\begin{aligned}\left| \int_{\mathcal{X}} a_j(x)\phi(x)\, d\mu(x) \right| &\le \left| \int_{\mathcal{X}} a_j(\phi - \phi_{B(x_j,r_j)})\, d\mu \right| + \left|\phi_{B(x_j,r_j)}\right| \left| \int_{\mathcal{X}} a_j\, d\mu \right| \\ &\le \|a_j\|_{L^q(B(x_j,r_j))} \left\| \phi - \phi_{B(x_j,r_j)} \right\|_{L^{q'}(B(x_j,r_j))} + C \|\phi\|_{\mathrm{BMO}_\rho} \\ &\le [\mu(B(x_j, r_j))]^{1/q-1} C [\mu(B(x_j, r_j))]^{1/q'} \|\phi\|_{\mathrm{BMO}_\rho} + C \|\phi\|_{\mathrm{BMO}_\rho} \\ &\le C \|\phi\|_{\mathrm{BMO}_\rho},\end{aligned}$$

where $\phi_{B(x_j,r_j)} := \frac{1}{\mu(B(x_j,r_j))} \int_{B(x_j,r_j)} \phi\, d\mu$. This ends the proof of Theorem 4.2. □

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