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On the Existence and Uniform Attractivity of the Solutions of a Class of Nonlinear Integral Equations on Unbounded Interval

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Abstract. In this paper, we prove the existence and uniform attractivity of the solutions of a class of functional integral equations which contain a number of classical nonlinear integral equations as special cases. Our investigations will be carried out in the space of continuous and bounded functions on an unbounded interval. The main tools here are the measure of noncompactness and the suitable fixed point theorem. We introduce also some examples and remarks showing the difference between our main result and some previous results.

1. Introduction

It is well known that integral equations have wide application in engineering, mechanics, physics, economics, optimization, queing theory and so on. The theory of integral equations is rapidly developing with the help of tools in functional analysis, topology and fixed-point theory.

Agarwal and O'Regan [1] gave the existence of the solutions for the nonlinear integral equation

$$x(t) = \int_0^\infty k(t,s) f(s,x(s))\, ds, \quad t \in \mathbb{R}^+, \tag{1.1}$$

in the space $\mathrm{C}_l[0,\infty)$, where $\mathrm{C}_l[0,\infty)$ denotes the space of bounded and continuous functions on $\mathbb{R}^+$ which have limit at infinity, in 2004.

Meehan and O'Regan [10, 11] discussed both the existence of the solutions for the nonlinear integral equation

$$x(t) = h(t) + \mu \int_0^\infty k(t,s) f(s,x(s))\, ds, \quad t \in \mathbb{R}^+, \tag{1.2}$$

in the space $\mathrm{C}_l[0,\infty)$ and the existence of the solutions for the nonlinear integral equation

$$x(t) = h(t) + \int_0^\infty k(t,s)\, [f(x(s)) + g(x(s))]\, ds, \quad t \in \mathbb{R}^+, \tag{1.3}$$

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in the space $\mathrm{BC}(\mathbb{R}^+,\mathbb{R})$, where $\mathrm{BC}(\mathbb{R}^+,\mathbb{R})$ denotes the space of bounded and continuous functions on $\mathbb{R}^+$, in 1999 and 2000, respectively. Later in [12] they established the existence of at least one positive solution of nonlinear integral equation

$$x(t) = h(t) + \int_0^\infty k(t,s) f(s,x(s))\, ds, \quad t \in \mathbb{R}^+, \tag{1.4}$$

in the space $L^p(\mathbb{R}^+)$ in 2001.

In 2004, Banaś and Południak [4] investigated the monotonic solutions for the nonlinear integral equation

$$x(t) = f(t) + \int_0^\infty u(t,s,x(s))\, ds, \quad t \in \mathbb{R}^+, \tag{1.5}$$

in the space of Lebesque integrable functions on unbounded interval by using the Darbo fixed point theorem and the measure of noncompactness defined in Definition 2.1.

Banaś and Martin [5] studied the existence and asymptotic stability of the solutions for the nonlinear integral equation

$$x(t) = g(t) + f(t,x(t)) \int_0^\infty K(t,s) h(s,x(s))\, ds, \quad t \in \mathbb{R}^+, \tag{1.6}$$

in the Banach space $\mathrm{BC}(\mathbb{R}^+,\mathbb{R})$, in 2006.

In 2004, Cabellaro and others [6], in 2008, Banaś and Olszowy [3] and more recently in 2013, Darwish and others [7] studied the existence of the solutions for the Urysohn integral equation defined on unbounded interval

$$x(t) = a(t) + f(t,x(t)) \int_0^\infty u(t,s,x(s))\, ds, \quad t \in \mathbb{R}^+, \tag{1.7}$$

with the help of measure of noncompactness and a fixed point theorem in the space $\mathrm{BC}(\mathbb{R}^+,\mathbb{R})$. Of course authors studied integral equation (1.7) under different assumptions and measure of noncompactness, also they have given rather different existence theorems.

Olszowy [13–15] studied (1.7) in the Fréchet space of real functions being defined and continuous on $\mathbb{R}^+$ and has given results about monotonicity of the solutions of the integral equation (1.7).

In 2010, Karoui and others [9] studied (1.7) in the space $L^p(\mathbb{R}^+)$ by means of Schauder's fixed point theorem.

Motivated by recent researches in this field, we study the more general nonlinear integral equation,

$$x(t) = (T_1x)(t) + (T_2x)(t) \int_0^\infty u(t,s,x(s))\, ds, \quad t \in \mathbb{R}^+, \tag{1.8}$$

where the functions $u(t,s,x)$ and the operators T_i, $(i = 1,2)$ appearing in (1.8) are given, while $x = x(t)$ is an unknown function. It is clear that (1.8) includes (1.1)–(1.7) as special cases. Using the technique of a suitable measure of noncompactness, we prove an existence theorem for (1.8). We give some examples satisfying the conditions given in this paper. The approach applied in this paper depends on extending and generalizing of the methods and tools used in the study of some nonlinear integral equations which are presented in the papers [4–7, 9]. It is worthwhile mentioning that the class of integral equations considered in this paper are more general then those investigated up to now.

2. Auxiliary facts and notations

In this section, we give a collection of auxiliary facts which will be needed in the sequel. Assume that $(E, \|\cdot\|)$ is a real Banach space with zero element θ. Let $B(x, r)$ denote the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(\theta, r)$. If X is a subset of E, then $\overline{X}$ and $\operatorname{Conv} X$ denote the closure and convex closure of X, respectively. With the symbols λX and $X+Y$, we denote the standard algebraic operations on sets. Moreover, we denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and $\mathfrak{N}_E$ its subfamily consisting of all relatively compact subsets. The definition of the concept of a measure of noncompactness presented below comes from [2].

Definition 2.1. A function $\mu\colon \mathfrak{M}_E \to \mathbb{R}^+ = [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

(1) The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$;

(2) $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$;

(3) $\mu(\overline{X}) = \mu(\operatorname{Conv} X) = \mu(X)$;

(4) $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$;

(5) if $\{X_n\}$ is a sequence of nonempty, bounded, closed subsets of the set E such that $X_{n+1} \subset X_n$, $(n = 1, 2, \ldots)$ and $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

In the sequel, we will work in the Banach space $\mathrm{BC}(\mathbb{R}^+, \mathbb{R})$. The space $\mathrm{BC}(\mathbb{R}^+, \mathbb{R})$ is furnished with the standard norm $\|x\| = \sup\{|x(t)| : t \in \mathbb{R}^+\}$.

We will use a measure of noncompactness in the space $\mathrm{BC}(\mathbb{R}^+, \mathbb{R})$. In order to define this measure let us fix a nonempty and bounded subset X of $\mathrm{BC}(\mathbb{R}^+, \mathbb{R})$. For $x \in X$, $\varepsilon \ge 0$ and $L > 0$ denoted by $w^L(x, \varepsilon)$ the modulus of continuity of function x, i.e.,

$$w^L(x, \varepsilon) = \sup\{|x(s) - x(t)| : t, s \in [0, L] \text{ and } |t - s| \le \varepsilon\}.$$

Further let us put

$$w^L(X,\varepsilon) = \sup\left\{w^L(x,\varepsilon) : x \in X\right\},$$
$$w_0^L(X) = \lim_{\varepsilon\to 0} w^L(X,\varepsilon)$$

and

$$w_0(X) = \lim_{L\to\infty} w_0^L(X). \tag{2.1}$$

Moreover, if $t \in \mathbb{R}^+$ is a fixed number, let us denote

$$X(t) = \{x(t) : x \in X\}$$

and

$$\operatorname{diam} X(t) = \sup\left\{|x(t) - y(t)| : x, y \in X\right\}.$$

With help of the above mappings we define the following measure of noncompactness in $\mathrm{BC}(\mathbb{R}^+,\mathbb{R})$, [2]:

$$\mu(X) = w_0(X) + \limsup_{t\to\infty} \operatorname{diam} X(t). \tag{2.2}$$

The kernel of this measure consists of all nonempty and bounded subsets X of $\mathrm{BC}(\mathbb{R}^+,\mathbb{R})$ such that functions from X are locally equicontinuous on $\mathbb{R}^+$ and the thickness of the bundle formed by functions from X tends to zero at infinity.

Now we recall definitions of the concepts of local attractivity and asymptotic stability of the solutions of operator equations. Let us assume that Ω is a nonempty subset of the space $\mathrm{BC}(\mathbb{R}^+,\mathbb{R})$ and F is an operator defined on Ω with values in $\mathrm{BC}(\mathbb{R}^+,\mathbb{R})$. Let us consider the operator equation of the form

$$x(t) = (Fx)(t), \quad t \in \mathbb{R}^+. \tag{2.3}$$

Definition 2.2. We say that solutions of (2.3) are locally attractive if there exist an $x_0 \in \mathrm{BC}(\mathbb{R}^+,\mathbb{R})$ and an $r > 0$ such that for all solutions $x = x(t)$ and $y = y(t)$ of (2.3) belonging to $B(x_0,r) \cap \Omega$ we have that

$$\lim_{t\to\infty} (x(t) - y(t)) = 0.$$

In the case when limit is uniform with respect to the set $B(x_0,r) \cap \Omega$, that is, when for each $\varepsilon \geq 0$ there exists $L > 0$ such that

$$|x(t) - y(t)| \leq \varepsilon$$

for all $x, y \in B(x_0,r) \cap \Omega$ being solutions of (2.3) for any $t \geq L$, we will say that solutions of (2.3) are uniformly locally attractive (or equivalently asymptotically stable) on $\mathbb{R}^+$, [8].

Finally, we will make use of the following fixed-point theorem, [2].

Theorem 2.3. *Let Q be a nonempty, bounded, closed and convex subset of the Banach space E and let*

$$F\colon Q \to Q$$

be a continuous transformation such that $\mu(FX) \le k\mu(X)$ for any nonempty subset X of Q, where μ is a measure of noncompactness in E and $k \in [0,1)$ is a constant. Then F has a fixed point in the set Q.

Remark 2.4. Denote by $\operatorname{Fix} F$ the set of all fixed points of the operator F belonging to Q. It can be readily seen that the set $\operatorname{Fix} F$ belongs to the family $\ker \mu$, [2].

3. The main result

We will consider the existence of the solutions of (1.8) assuming that the following conditions are satisfied:

(i) The operators $T_i\colon \mathrm{BC}(\mathbb{R}^+,\mathbb{R}) \to \mathrm{BC}(\mathbb{R}^+,\mathbb{R})$ are continuous and there exist continuous nondecreasing functions $d_i\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|(T_i x)(t)| \le d_i(\|x\|)$$

for all $x \in \mathrm{BC}(\mathbb{R}^+,\mathbb{R})$ and $t \in \mathbb{R}^+$, $(i = 1,2)$.

(ii) The function $u\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist continuous functions $a, b, \psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{t\to\infty} a(t) = 0$, $\|b\|_1 = \int_0^\infty |b(s)|\,ds < \infty$ and ψ is nondecreasing on $\mathbb{R}^+$ such that

$$|u(t,s,x)| \le a(t)b(s)\psi(|x|) \quad \text{for all } t, s \in \mathbb{R}^+ \text{ and } x \in \mathbb{R}.$$

(iii) There exists a positive real number r_0 satisfying the inequality

$$d_1(r_0) + d_2(r_0)\,\|a\|\,\|b\|_1\,\psi(r_0) \le r_0.$$

(iv) There exist the nonnegative constants k_{r_0} and q_{r_0} for r_0 such that the inequalities

$$\mu(T_1 X) \le k_{r_0}\mu(X)$$

and

$$\omega_0(T_2 X) \le q_{r_0}\omega_0(X)$$

hold for all nonempty and bounded subsets X of B_{r_0}, where ω_0 and μ are defined by (2.1) and (2.2).

(v) We assume that
$$k_{r_0} + q_{r_0} \|a\| \|b\|_1 \psi(r_0) < 1.$$

(vi) There exists a continuous nondecreasing function $\phi_{r_0} \colon \mathbb{R}^+ \to \mathbb{R}^+$ which holds $\phi_{r_0}(0) = 0$ and
$$|u(t_2, s, x) - u(t_1, s, x)| \leq \phi_{r_0}(|t_2 - t_1|)\tau(s)$$
for all $t_1, t_2, s \in \mathbb{R}^+$ and $x \in \mathbb{R}$ with $|x| \leq r_0$, where τ is an element of the space $\mathrm{BC}(\mathbb{R}^+, \mathbb{R}^+)$ such that $\int_0^\infty \tau(s)\, ds < \infty$.

(vii) There exists a continuous nondecreasing function $\eta_{r_0} \colon \mathbb{R}^+ \to \mathbb{R}^+$ which holds $\eta_{r_0}(0) = 0$ and
$$|u(t, s, x) - u(t, s, y)| \leq \eta_{r_0}(|x - y|)v(s)$$
for all $t, s \in \mathbb{R}^+$ and $x, y \in \mathbb{R}$ with $|x| \leq r_0$, $|y| \leq r_0$, where v is an element of the space $\mathrm{BC}(\mathbb{R}^+, \mathbb{R}^+)$ such that $\int_0^\infty v(s)\, ds < \infty$.

Now we can formulate an existence result concerning the functional integral equation (1.8).

Theorem 3.1. *Under assumptions* (i)–(vii), *there exists a positive real number r_0 such that the equation* (1.8) *has at least one solution $x = x(t)$ belonging to $B_{r_0} \subset \mathrm{BC}(\mathbb{R}^+, \mathbb{R})$. Moreover, the solutions of the equation* (1.8) *are uniformly locally attractive on $\mathbb{R}^+$.*

Proof. We define operator F on B_{r_0} in the following way:
$$(Fx)(t) = (T_1x)(t) + (T_2x)(t) \int_0^\infty u(t, s, x(s))\, ds.$$
Notice that in view of assumptions (i) and (ii), the function $t \to (Fx)(t)$ is well defined on the interval $\mathbb{R}^+$. At first we show that the function (Fx) is continuous on $\mathbb{R}^+$. To do this fix arbitrarily $L > 0$ and $\varepsilon \geq 0$. Take arbitrary numbers $t, t_0 \in [0, L]$ with $|t - t_0| \leq \varepsilon$. Then in view of assumptions we obtain that

$$\begin{aligned}
&|(Fx)(t) - (Fx)(t_0)| \\
&= \left|(T_1x)(t) + (T_2x)(t) \int_0^\infty u(t, s, x(s))\, ds - (T_1x)(t_0) - (T_2x)(t_0) \int_0^\infty u(t_0, s, x(s))\, ds\right| \\
&\leq |(T_1x)(t) - (T_1x)(t_0)| + |(T_2x)(t) - (T_2x)(t_0)| \left|\int_0^\infty u(t, s, x(s))\, ds\right| \\
&\quad + |(T_2x)(t_0)| \left|\int_0^\infty [u(t, s, x(s)) - u(t_0, s, x(s))]\, ds\right| \\
&\leq \omega^L(T_1x, \varepsilon) + \omega^L(T_2x, \varepsilon) \int_0^\infty |u(t, s, x(s))|\, ds \\
&\quad + d_2(\|x\|) \int_0^\infty |u(t, s, x(s)) - u(t_0, s, x(s))|\, ds \\
&\leq \omega^L(T_1x, \varepsilon) + \omega^L(T_2x, \varepsilon) a(t) \int_0^\infty b(s)\psi(|x(s)|)\, ds + d_2(\|x\|) \int_0^\infty \phi_{r_0}(|t - t_0|)\tau(s)\, ds \\
&\leq \omega^L(T_1x, \varepsilon) + \omega^L(T_2x, \varepsilon) \|a\| \psi(\|x\|) \|b\|_1 + d_2(\|x\|)\phi_{r_0}(\varepsilon) \|\tau\|_1,
\end{aligned} \tag{3.1}$$

where

$$\omega^L(T_ix,\varepsilon) = \sup\{|(T_ix)(s) - (T_ix)(t)| : t,s \in [0,L] \text{ and } |t-s| \le \varepsilon\}$$

for $i = 1,2$. By uniform continuity of the functions (T_ix) on the set $[0,L]$, we deduce that $\omega^L(T_ix,\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus we have that (Fx) is continuous on $[0,L]$. Since $L > 0$ is arbitrary, (Fx) is continuous on $\mathbb{R}^+$. Next we show that (Fx) is bounded on $\mathbb{R}^+$. By our assumptions, for arbitrarily fixed $t \in \mathbb{R}^+$, we derive that

$$\begin{aligned} |(Fx)(t)| &= \left|(T_1x)(t) + (T_2x)(t)\int_0^\infty u(t,s,x(s))\,ds\right| \\ &\le |(T_1x)(t)| + |(T_2x)(t)|\int_0^\infty |u(t,s,x(s))|\,ds \\ &\le d_1(\|x\|) + d_2(\|x\|)\int_0^\infty a(t)b(s)\psi(|x(s)|)\,ds. \end{aligned} \tag{3.2}$$

Hence, from (3.2), we obtain the following evaluation:

$$\|Fx\| \le d_1(\|x\|) + d_2(\|x\|)\,\|a\|\,\|b\|_1\,\psi(\|x\|) \tag{3.3}$$

which implies that the function (Fx) is bounded on $\mathbb{R}^+$. Combining this fact with the continuity of the function (Fx) on $\mathbb{R}^+$, we conclude that the operator F transforms the ball B_{r_0} into the space $\mathrm{BC}(\mathbb{R}^+,\mathbb{R})$. Moreover linking (3.3) and the assumption (iii) we deduce that the operator F maps the ball B_{r_0} into itself, where r_0 is a number indicated in assumption (iii). Now, we shall prove that operator F is continuous on B_{r_0}. To do this, consider $\varepsilon > 0$ and fixed $y_0 \in B_{r_0}$. Then,

(3.4)

$$\begin{aligned} &|(Fx)(t) - (Fy_0)(t)| \\ &= \left|(T_1x)(t) + (T_2x)(t)\int_0^\infty u(t,s,x(s))\,ds - (T_1y_0)(t) - (T_2y_0)(t)\int_0^\infty u(t,s,y_0(s))\,ds\right| \\ &\le |(T_1x)(t) - (T_1y_0)(t)| + |(T_2x)(t) - (T_2y_0)(t)|\int_0^\infty |u(t,s,x(s))|\,ds \\ &\quad + |(T_2y_0)(t)|\int_0^\infty |u(t,s,x(s)) - u(t,s,y_0(s))|\,ds. \end{aligned}$$

Hence from estimate (3.4) we get

$$\begin{aligned} \|Fx - Fy_0\| &\le \|T_1x - T_1y_0\| + \|T_2x - T_2y_0\|\,\|a\|\,\|b\|_1\,\psi(r_0) \\ &\quad + d_2(r_0)\int_0^\infty v(s)\eta_{r_0}(|x(s) - y_0(s)|)\,ds. \end{aligned} \tag{3.5}$$

Since the operators T_i are continuous for any $y_0 \in B_{r_0}$, there exist the numbers $\delta_i(\varepsilon) > 0$ with $\delta_i(\varepsilon) \le \varepsilon$ such that we have $\|T_ix - T_iy_0\| \le \varepsilon$ for all x satisfying $\|x - y_0\| < \delta_i$. Let us take $\delta(\varepsilon) = \min\{\delta_1(\varepsilon), \delta_2(\varepsilon)\}$. In this case if $\|x - y_0\| < \delta(\varepsilon)$, (3.5) becomes

$$\|Fx - Fy_0\| \le \varepsilon + \varepsilon\,\|a\|\,\|b\|_1\,\psi(r_0) + d_2(r_0)\,\|v\|_1\,\eta_{r_0}(\varepsilon). \tag{3.6}$$

Therefore from (3.6) and assumption (vii) we have that F is continuous on the ball B_{r_0}.

Further, we shall show that operator F satisfies the Darbo condition on the ball B_{r_0}. In order to do this, let us take a nonempty subset X of the ball B_{r_0}. Fix $\varepsilon \geq 0$, $L > 0$ and choose $x \in X$ and $t_1, t_2 \in [0, L]$ such that $|t_1 - t_2| \leq \varepsilon$. Then in view of (3.1) we have that

$$\omega^L(Fx, \varepsilon) \leq \omega^L(T_1 x, \varepsilon) + \omega^L(T_2 x, \varepsilon) \|a\| \|b\|_1 \psi(r_0) + d_2(r_0)\phi_{r_0}(\varepsilon) \|\tau\|_1 \tag{3.7}$$

and by (3.7), we get

$$\omega^L(FX, \varepsilon) \leq \omega^L(T_1 X, \varepsilon) + \omega^L(T_2 X, \varepsilon) \|a\| \|b\|_1 \psi(r_0) + d_2(r_0)\phi_{r_0}(\varepsilon) \|\tau\|_1$$

which yields

$$\omega_0^L(FX) \leq \omega_0^L(T_1 X) + \omega_0^L(T_2 X) \|a\| \|b\|_1 \psi(r_0). \tag{3.8}$$

If we take limit as $L \to \infty$, we have by (3.8) that

$$\omega_0(FX) \leq \omega_0(T_1 X) + \omega_0(T_2 X) \|a\| \|b\|_1 \psi(r_0). \tag{3.9}$$

Further let us take a nonempty subset X of the ball B_{r_0}. For $x, y \in X$ and $t \in \mathbb{R}^+$, from estimate (3.4) and the conditions (i) and (ii) we get that

$$\begin{aligned} \operatorname{diam}(FX)(t) \leq {} & \operatorname{diam}(T_1 X)(t) + \operatorname{diam}(T_2 X)(t) a(t) \|b\|_1 \psi(r_0) \\ & + 2 d_2(r_0) a(t) \|b\|_1 \psi(r_0). \end{aligned} \tag{3.10}$$

If we take limitsupremum as $t \to \infty$ in (3.10) we have the inequality

$$\limsup_{t\to\infty} \operatorname{diam}(FX)(t) \leq \limsup_{t\to\infty} \operatorname{diam}(T_1 X)(t). \tag{3.11}$$

By linking (3.9) and (3.11) we derive that

$$\begin{aligned} \mu(FX) &\leq \mu(T_1 X) + q_{r_0}\omega_0(X) \|a\| \|b\|_1 \psi(r_0) \\ &\leq k_{r_0}\mu(X) + q_{r_0} \|a\| \|b\|_1 \psi(r_0)\mu(X) \\ &\leq l_{r_0}\mu(X) \end{aligned} \tag{3.12}$$

from (iv) and the inequality $\omega_0(X) \leq \mu(X)$, where $l_{r_0} = k_{r_0} + q_{r_0} \|a\| \|b\|_1 \psi(r_0)$.

Now let us observe that by assumption (v) and (3.12) we have that F is a contraction with respect to the measure of noncompactness μ. Hence by Theorem 2.3 the operator F has at least one fixed point x in the ball B_{r_0}. Obviously, every function $x = x(t)$ being a fixed point of the operator F is a solution of (1.8). Further, keeping in mind Remark 2.4, we conclude that the set $\operatorname{Fix} F$ of all fixed points of the operator F belonging to the ball B_{r_0} is a member of the $\ker \mu$. Hence, in view of the description of the $\ker \mu$ we infer that all of solutions of (1.8) belonging to the ball B_{r_0} are uniformly attractive on $\mathbb{R}^+$. This step completes the proof of our theorem. □

4. Examples

Example 4.1. Consider the following integral equation:

$$x(t) = \alpha \sin(x(t)+1) + \beta x^2(t) \int_0^\infty \frac{\arctan x(s)}{\exp(t)(s^2+1)}\, ds, \tag{4.1}$$

where $t \in \mathbb{R}^+$ and α, β are any constants such that $0 < |\alpha| < 1$. Notice that (4.1) is a special case of (1.8) if we put

$$(T_1 x)(t) = \alpha \sin(x(t)+1), \quad (T_2 x)(t) = \beta x^2(t)$$

and

$$u(t,s,x) = \frac{\arctan x}{\exp(t)(s^2+1)}.$$

It is easily verified that the assumptions of Theorem 3.1 are satisfied. T_1 and T_2 are continuous operators on the space $\mathrm{BC}(\mathbb{R}^+, \mathbb{R})$. Further for all $t \in \mathbb{R}^+$ and $x \in \mathrm{BC}(\mathbb{R}^+, \mathbb{R})$,

$$|(T_1 x)(t)| \le |\alpha|\, |\sin(x(t)+1)| \le |\alpha| \tag{4.2}$$

and

$$|(T_2 x)(t)| \le |\beta| \left|x^2(t)\right| \le |\beta|\, \|x\|^2. \tag{4.3}$$

Hence assumption (i) is satisfied with $d_1(x) = |\alpha|$ and $d_2(x) = |\beta|\, x^2$. Now notice that the function u is continuous on the set $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$. Moreover, we get

$$|u(t,s,x)| = \left|\frac{\arctan x}{\exp(t)(s^2+1)}\right| \le \frac{\pi}{2\exp(t)(s^2+1)} \tag{4.4}$$

for all $t, s \in \mathbb{R}^+$ and $x \in \mathbb{R}$. Thus, according to assumption (ii) we may put $a(t) = \pi/(2\exp(t))$, $b(s) = 1/(s^2+1)$ and $\psi(x) = 1$. Further we get

$$\|a\| = \sup\left\{\left|\frac{\pi}{2\exp(t)}\right| : t \ge 0\right\} = \frac{\pi}{2}, \quad \|b\|_1 = \int_0^\infty \frac{ds}{s^2+1} = \frac{\pi}{2}$$

and obviously, we have that $a(t) \to 0$ as $t \to \infty$. Now if we consider (4.2)–(4.4), the assumption (iii) takes the following form:

$$|\alpha| + |\beta|\, r_0^2 \frac{\pi^2}{4} \le r_0. \tag{4.5}$$

Apart from this, fixing a nonempty and bounded subset X of the ball B_{r_0}, let $x \in X$, $\varepsilon \ge 0$, $L > 0$ and $t, s \in [0, L]$ such that $|t - s| \le \varepsilon$. Then

$$\begin{aligned} |(T_1 x)(t) - (T_1 x)(s)| &= |\alpha|\, |\sin(x(t)+1) - \sin(x(s)+1)| \\ &\le |\alpha|\, |x(t) - x(s)| \end{aligned} \tag{4.6}$$

and

$$
\begin{aligned}
|(T_2x)(t)-(T_2x)(s)| &= \left|\beta x^2(t)-\beta x^2(s)\right| \\
&= |\beta|\,|x(t)+x(s)|\,|x(t)-x(s)| \\
&\leq 2\,|\beta|\,r_0\,|x(t)-x(s)|\,.
\end{aligned} \tag{4.7}
$$

From estimates (4.6), (4.7) and in view of (2.1), we get

$$\omega_0(T_1X)\leq|\alpha|\,\omega_0(X) \tag{4.8}$$

and

$$\omega_0(T_2X)\leq 2r_0\,|\beta|\,\omega_0(X). \tag{4.9}$$

(4.9) implies that the second inequality of assumption (iv) is satisfied with the constant $q_{r_0}=2r_0\,|\beta|$. For $x,y\in X$, we get

$$
\begin{aligned}
|(T_1x)(t)-(T_1y)(t)| &= |\alpha\,[\sin(x(t)+1)-\sin(y(t)+1)]| \\
&\leq |\alpha|\,|x(t)-y(t)|\,.
\end{aligned} \tag{4.10}
$$

Using (4.10), we have

$$\limsup_{t\to\infty}\operatorname{diam}(T_1X)(t)\leq|\alpha|\limsup_{t\to\infty}\operatorname{diam}X(t). \tag{4.11}$$

From (4.8) and (4.11), we get

$$\mu(T_1X)\leq|\alpha|\,\mu(X). \tag{4.12}$$

So, we have by (4.12) that the first inequality of assumption (iv) is satisfied with $k_{r_0}=|\alpha|$. Now if we consider (4.9) and (4.12), the inequality of assumption (v) takes the following form:

$$|\alpha|+\frac{r_0\,|\beta|\,\pi^2}{2}<1. \tag{4.13}$$

We consider two cases.

Case 1: $\beta\neq0$. Since $0<|\alpha|<1$, it can be easily verified that if

$$1-|\alpha|\,|\beta|\,\pi^2>0 \quad\text{and}\quad |\alpha|^2+|\alpha|\,|\beta|\,\pi^2-1<0,$$

then

$$r_0\in\left(0,\frac{2(1-|\alpha|)}{|\beta|\,\pi^2}\right)\bigcap\left[\frac{2-2\sqrt{1-|\alpha|\,|\beta|\,\pi^2}}{|\beta|\,\pi^2},\frac{2+2\sqrt{1-|\alpha|\,|\beta|\,\pi^2}}{|\beta|\,\pi^2}\right]$$

which is equivalent to $r_0\in\left[\frac{2-2\sqrt{1-|\alpha||\beta|\pi^2}}{|\beta|\pi^2},\frac{2(1-|\alpha|)}{|\beta|\pi^2}\right)$ as the positive solution of the system of inequalities (4.5) and (4.13).

Case 2: $\beta = 0$. Then, since $0 < |\alpha| < 1$, it is clear that $r_0 \in [|\alpha|, \infty)$ is the solution of the inequalities (4.5) and (4.13).

Additionally, without loss of generality assume that $t_1 < t_2$, for all t_1, t_2 and $s \in \mathbb{R}^+$ and $x \in \mathbb{R}$ with $|x| \le r_0$ we have

$$\begin{aligned} |u(t_1, s, x) - u(t_2, s, x)| &= \left| \frac{\arctan x}{\exp(t_1)(s^2+1)} - \frac{\arctan x}{\exp(t_2)(s^2+1)} \right| \\ &= \frac{|\arctan x|}{s^2+1} \frac{|\exp(t_2) - \exp(t_1)|}{\exp(t_1+t_2)} \\ &\le \frac{|\arctan x|}{s^2+1} \frac{\exp(\xi)\,|t_2 - t_1|}{\exp(t_1+t_2)} \\ &\le \frac{\pi}{2(s^2+1)} |t_2 - t_1|, \end{aligned}$$

where $\xi \in (t_1, t_2)$. If we put $\phi_{r_0}(t) = t$ and $\tau(s) = \pi/[2(s^2+1)]$, the assumption (vi) is satisfied. Finally, without loss of generality assume that $x < y$, for all $t \in \mathbb{R}^+$ and $x, y \in \mathbb{R}$ with $|x| \le r_0$, $|y| \le r_0$ we get

$$|u(t, s, x) - u(t, s, y)| = \left| \frac{\arctan x - \arctan y}{\exp(t)(s^2+1)} \right| \le \frac{|x-y|}{\exp(t)(1+\xi^2)(1+s^2)} \le \frac{|x-y|}{1+s^2},$$

where $\xi \in (x, y)$. If we take $\eta_{r_0}(t) = t$ and $v(s) = 1/(s^2+1)$, the assumption (vii) is satisfied.

Since all of the assumptions of Theorem 3.1 are fullfilled, we deduce that the integral equation (4.1) has at least one solution belonging to the ball B_{r_0} of the space $\mathrm{BC}(\mathbb{R}^+, \mathbb{R})$. Taking into account Remark 2.4 and the measure of noncompactness μ given by (2.2), we infer easily that any solutions of (4.1) which belong to the ball B_{r_0} are asymptotically stable on $\mathbb{R}^+$ as defined in Definition 2.2.

Remark 4.2. Notice that none of the existence theorems given in [1,3–7,9–15] are applicable to (4.1), since the integral equation (4.1) cannot be derived from any of the integral equations handled in mentioned papers.

Example 4.3. Let us consider the following integral equation:

$$x(t) = h(t) + \frac{x^2(t)}{1+t} \int_0^\infty \frac{\exp(-t)(e-1)x(s)}{(t+1)(s+e)(s+1)}\, ds, \tag{4.14}$$

where h is an element of the space $\mathrm{BC}(\mathbb{R}^+, \mathbb{R})$ such that $\|h\| \le 1/\sqrt{7}$ and $t \in \mathbb{R}^+$. Observe that $(T_1 x)(t) = h(t)$, $(T_2 x)(t) = \frac{x^2(t)}{1+t}$ and $u(t, s, x) = \frac{\exp(-t)(e-1)x}{(t+1)(s+e)(s+1)}$.

It is clear that T_1 and T_2 are continuous operators on the space $\mathrm{BC}(\mathbb{R}^+, \mathbb{R})$. Additionally for all $t \in \mathbb{R}^+$ and for all $x \in \mathrm{BC}(\mathbb{R}^+, \mathbb{R})$,

$$|(T_1 x)(t)| = |h(t)| \le \|h\|$$

and

$$|(T_2x)(t)| \leq \left| \frac{x^2(t)}{1+t} \right| \leq \|x\|^2 .$$

Hence assumption (i) is satisfied with $d_1(x) = \|h\|$ and $d_2(x) = x^2$. The function $u(t,s,x)$ is continuous on the set $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$. Further we get

$$|u(t,s,x)| = \left| \frac{\exp(-t)(e-1)x}{(t+1)(s+e)(s+1)} \right| = \frac{\exp(-t)(e-1)\,|x|}{(t+1)(s+e)(s+1)}$$

for all $t, s \in \mathbb{R}^+$ and $x \in \mathbb{R}$. If we choose $a(t) = \frac{\exp(-t)}{t+1}$, $b(s) = \frac{e-1}{(s+e)(s+1)}$ and $\psi(x) = x$, assumption (ii) is satisfied. In fact, we have that $\|a\| = a(0) = 1$ and $\|b\|_1 = \int_0^\infty \frac{(e-1)\,ds}{(s+e)(s+1)} = 1$ and obviously $a(t) \to 0$ as $t \to \infty$. Now, if we consider the previous functions, then the inequality in the assumption (iii) is

$$\|h\| + r_0^3 \leq r_0. \tag{4.15}$$

Apart from this for $\varepsilon \geq 0$, $L > 0$, $\|x\| \leq r_0$ and $t, s \in [0, L]$ such that $|t - s| \leq \varepsilon$, we have that

$$|(T_1x)(t) - (T_1x)(s)| = |h(t) - h(s)| \tag{4.16}$$

and

$$\begin{aligned} |(T_2x)(t) - (T_2x)(s)| &= \left| \frac{x^2(t)}{1+t} - \frac{x^2(s)}{1+s} \right| \\ &\leq \frac{\left|(1+s)\left[x^2(t) - x^2(s)\right] + \left[(1+s) - (1+t)\right] x^2(s)\right|}{(1+t)(1+s)} \\ &\leq \frac{2r_0}{1+t}\,|x(t) - x(s)| + \frac{r_0^2\,|t-s|}{(1+t)(1+s)} \\ &\leq 2r_0\,|x(t) - x(s)| + r_0^2\varepsilon. \end{aligned} \tag{4.17}$$

Taking into account that the function h is uniformly continuous on the set $[0, L]$, we obtain by (4.16) that

$$\omega_0(T_1X) = 0.$$

In view of (2.1) we have by (4.17) that

$$\omega_0(T_2X) \leq 2r_0\omega_0(X).$$

Moreover, fixing a nonempty and bounded subset X of the ball B_{r_0}, for $x, y \in X$, we get

$$|(T_1x)(t) - (T_1y)(t)| = 0. \tag{4.18}$$

Using (4.18), we have

$$\limsup_{t\to\infty} \operatorname{diam}(T_1X)(t) = 0.$$

Thus we deduce that the inequalities of the assumption (iv) are satisfied with constants $k_{r_0} = 0$ and $q_{r_0} = 2r_0$. Next we have that the inequality of the assumption (v) is the equivalent to

$$2r_0^2 < 1. \tag{4.19}$$

It can be calculated that the number $r_0 \in (0.5128, 0.6395)$ is the solution of the inequalities (4.15) and (4.19). Further, without loss of generality assume that $t_1 < t_2$, for all t_1, t_2 and $s \in \mathbb{R}^+$ and $x \in \mathbb{R}$ with $|x| \le r_0$ we have

$$\begin{aligned}
|u(t_1,s,x) - u(t_2,s,x)| &= \left| \frac{\exp(-t_1)(e-1)x}{(t_1+1)(s+e)(s+1)} - \frac{\exp(-t_2)(e-1)x}{(t_2+1)(s+e)(s+1)} \right| \\
&\le \frac{(e-1)\,|x|}{(s+e)(s+1)} \left| \frac{\exp(-t_1)}{t_1+1} - \frac{\exp(-t_2)}{t_2+1} \right| \\
&\le \frac{(e-1)r_0}{(s+e)(s+1)} \left| \frac{(t_2+1)\exp(t_2) - (t_1+1)\exp(t_1)}{(t_1+1)(t_2+1)\exp(t_1+t_2)} \right| \\
&\le \frac{(e-1)r_0}{(s+e)(s+1)} \left| \frac{(t_2+1)\left(\exp(t_2) - \exp(t_1)\right) + \exp(t_1)(t_1-t_2)}{(t_1+1)(t_2+1)\exp(t_1+t_2)} \right| \\
&\le \frac{(e-1)r_0}{(s+e)(s+1)} \frac{\left[(t_2+1)\exp(\xi) + \exp(t_1)\right]|t_1-t_2|}{(t_1+1)(t_2+1)\exp(t_1+t_2)} \\
&\le \frac{(e-1)r_0\,|t_1-t_2|}{(s+e)(s+1)},
\end{aligned}$$

where $\xi \in (t_1, t_2)$. If we put $\phi_{r_0}(t) = r_0 t$ and $\tau(s) = \frac{e-1}{(s+e)(s+1)}$ assumption (vi) is satisfied. Next, let us observe that for all $t \in \mathbb{R}^+$ and $x, y \in \mathbb{R}$ with $|x| \le r_0$, $|y| \le r_0$ we have that

$$|u(t,s,x) - u(t,s,y)| = \frac{\exp(-t)(e-1)\,|x-y|}{(t+1)(s+e)(s+1)} \le \frac{(e-1)\,|x-y|}{(s+e)(s+1)}.$$

If we take $\eta_{r_0}(t) = t$ and $v(s) = \frac{e-1}{(s+e)(s+1)}$, the assumption (vii) is satisfied.

Thus, on the basis of Theorem 3.1 we conclude that (4.14) has at least one solution in the space $\mathrm{BC}(\mathbb{R}^+, \mathbb{R})$ belonging to the ball B_{r_0}. Obviously, all of the solutions of (4.14) from the ball B_{r_0} are asymptotic stable on $\mathbb{R}^+$.

Remark 4.4. Let us observe that (4.14) is a special case of the functional integral equation which is handled in [3, 6, 7, 15], where $f(t,x) = x^2/(1+t)$.

It is easily seen that assumptions (i)–(v) of Theorem 2 in [6] are fulfilled with $k_r = 2r$, $\|a\| = 1$, $\|b\|_1 = 1$, $M = 0$ and $\psi(r) = r$. Hence the inequality of assumption (vi), given in [6], takes the following form:

$$\|h\| + 2r^3 \le r. \tag{4.20}$$

It can be checked that if $\|h\| \in [1/\sqrt{13}, 1/\sqrt{7}]$, then (4.20) does not have a positive solution.

Further, if we choose $h(t) = (t+1)/(4t+3)$, then we have $\|h\| = 1/3$ and assumptions of Theorem 3.1 are satisfied as considered above, but since $\lim_{t\to\infty} h(t) \neq 0$, assumption (i) of Theorem 3.2 in [3] is not satisfied.

Besides, being h a decreasing function on $\mathbb{R}^+$, assumption (i) of Theorem 3.1 in [15] is not satisfied.

On the other hand, we have

$$|f(t,x) - f(t,y)| = \left| \frac{x^2}{1+t} - \frac{y^2}{1+t} \right| \leq \frac{|x+y|}{1+t} |x-y| .$$

It can be seen that there is not any constant number k such that the function $f(t,x)$ satisfies the Lipschitz condition with respect to the second variable. Hence, the assumption (iii) of Theorem 8 in [7] is not satisfied.

Therefore existence theorems in [3, 6, 7, 15] are inapplicable to (4.14)**.**

Example 4.5. Let us consider the following integral equation:

$$x(t) = t\exp(-2t) + \sqrt{x^2(t)+5} \int_0^\infty \frac{\sqrt{1+|x(s)|}}{\exp(t+s+1)}\, ds, \tag{4.21}$$

where $t \in \mathbb{R}^+$. Observe that $(T_1x)(t) = t\exp(-2t)$, $(T_2x)(t) = \sqrt{x^2(t)+5}$ and $u(t,s,x) = \sqrt{1+|x|}/\exp(t+s+1)$.

It is clear that T_1 and T_2 are continuous operators on the space $\mathrm{BC}(\mathbb{R}^+, \mathbb{R})$. Moreover for all $t \in \mathbb{R}^+$ and for all $x \in \mathrm{BC}(\mathbb{R}^+, \mathbb{R})$, we have

$$|(T_1x)(t)| = |t\exp(-2t)| \leq \frac{1}{2e}$$

and

$$|(T_2x)(t)| = \left|\sqrt{x^2(t)+5}\right| \leq \sqrt{\|x\|^2 + 5}.$$

Hence assumption (i) is satisfied with $d_1(x) = 1/(2e)$ and $d_2(x) = \sqrt{x^2+5}$, respectively.

The function $u(t,s,x)$ is continuous on the set $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$. Further, we get

$$|u(t,s,x)| = \left| \frac{\sqrt{1+|x|}}{\exp(t+s+1)} \right| = \frac{\sqrt{1+|x|}}{\exp(t+s+1)}$$

for all $t, s \in \mathbb{R}^+$ and $x \in \mathbb{R}$. Thus the functions appearing in assumption (ii) have the form $a(t) = \exp(-t-1)$, $b(s) = \exp(-s)$ and $\psi(x) = \sqrt{x+1}$. Clearly we have that $a(t) \to 0$ as $t \to \infty$, $\|a\| = 1/e$ and $\|b\|_1 = \int_0^\infty \exp(-s)\, ds = 1$. In order to verify assumption (iii) which has the form:

$$\frac{1}{2e} + \frac{1}{e}\sqrt{r_0^2+5}\sqrt{r_0+1} \leq r_0. \tag{4.22}$$

Moreover the operators T_1 and T_2 satisfy the assumption (iv). Indeed for $\varepsilon \geq 0$, $L > 0$, $\|x\| \leq r_0$ and $t, s \in [0, L]$ such that $|t - s| \leq \varepsilon$, we obtain

$$|(T_1x)(t) - (T_1x)(s)| = |t \exp(-2t) - s \exp(-2s)| \tag{4.23}$$

and without loss of generality, assuming that $x(t) < x(s)$, we get

$$\begin{aligned} |(T_2x)(t) - (T_2x)(s)| &= \left|\sqrt{x^2(t)+5} - \sqrt{x^2(s)+5}\right| \\ &\leq \frac{|x(t) - x(s)|\, 2\,|\xi|}{2\sqrt{\xi^2+5}} \\ &\leq |x(t) - x(s)|\,, \end{aligned} \tag{4.24}$$

where $\xi \in (x(t), x(s))$. In view of (2.1) and taking into account that the function $t \exp(-2t)$ is uniformly continuous on the set $[0, L]$, we have by (4.23) and (4.24) that

$$\omega_0(T_1X) = 0 \tag{4.25}$$

and

$$\omega_0(T_2X) \leq \omega_0(X).$$

Fixing a nonempty and bounded subset X of the ball B_{r_0}, for $x, y \in X$, we get

$$|(T_1x)(t) - (T_1y)(t)| = 0. \tag{4.26}$$

Using (4.26), we have

$$\limsup_{t\to\infty} \operatorname{diam}(T_1X)(t) = 0. \tag{4.27}$$

From (4.25) and (4.27), we get

$$\mu(T_1X) = 0. \tag{4.28}$$

So, we get that the inequalities of assumption (iv) are satisfied with the constants $k_{r_0} = 0$ and $q_{r_0} = 1$.

Next we have that the inequality of assumption (v) as follows:

$$\frac{\sqrt{r_0+1}}{e} < 1. \tag{4.29}$$

By direct computation we see that the number $r_0 \in (2.43, 4.32)$ is the solution of the inequalities (4.22) and (4.29).

Further, without loss of generality, we assume that $t_1 < t_2$, for all $t_1, t_2, s \in \mathbb{R}^+$ and $x \in \mathbb{R}$ with $|x| \le r_0$, we have

$$\begin{aligned}|u(t_1,s,x) - u(t_2,s,x)| &= \left| \frac{\sqrt{1+|x|}}{\exp(t_1+s+1)} - \frac{\sqrt{1+|x|}}{\exp(t_2+s+1)} \right| \\ &\le \frac{\sqrt{1+|x|}}{\exp(s+1)} \left| \frac{1}{\exp(t_1)} - \frac{1}{\exp(t_2)} \right| \\ &\le \frac{\sqrt{1+|x|}}{\exp(s+1)} \frac{|\exp(t_2) - \exp(t_1)|}{\exp(t_1+t_2)} \\ &\le \frac{\sqrt{1+|x|}}{\exp(s+1)} \frac{|t_2 - t_1| \exp(\xi)}{\exp(t_1+t_2)} \\ &\le \frac{\sqrt{1+r_0}}{\exp(s+1)} |t_2 - t_1|,\end{aligned}$$

where $\xi \in (t_1, t_2)$. If we put $\phi_{r_0}(t) = \sqrt{1+r_0}\, t$ and $\tau(s) = 1/\exp(s+1)$, assumption (vi) is satisfied.

Finally, without loss of generality assume that $x < y$, let us observe that for all $t \in \mathbb{R}^+$ and $x, y \in \mathbb{R}$ with $|x| \le r_0$, $|y| \le r_0$ we have the inequality

$$\begin{aligned}|u(t,s,x) - u(t,s,y)| &= \left| \frac{\sqrt{1+|x|}}{\exp(t+s+1)} - \frac{\sqrt{1+|y|}}{\exp(t+s+1)} \right| \\ &\le \frac{|x-y|}{2\sqrt{1+|\xi|}\exp(t+s+1)} \\ &\le \frac{|x-y|}{2\exp(s+1)},\end{aligned}$$

where $\xi \in (x, y)$. If we take $\eta_{r_0}(t) = t/2$ and $v(s) = 1/\exp(s+1)$, the assumption (vii) is satisfied. The result follows from Theorem 3.1.

Remark 4.6. Observe that if we put $a(t) = t\exp(-2t)$, $f(t,x) = \sqrt{x^2+5}$ and $u(t,s,x) = \sqrt{1+|x|}/\exp(t+s+1)$, then (4.21) is a special case of (1.7) which is handled in [7]. It is easily seen that $a \in \mathrm{BC}(\mathbb{R}^+, \mathbb{R})$ and $\|a\| = 1/(2e)$. $f(t,0) \in \mathrm{BC}(\mathbb{R}^+, \mathbb{R})$ and f satisfies the Lipschitz condition with respect to the second variable for $k = 1$.

On the other hand, we have $g(t,s) = 1/\exp(t+s+1)$ and $h(r) = r/2$ which are imposed in the assumption (iv) of Theorem 8 in [7] for the inequality:

$$|u(t,s,x) - u(t,s,y)| \le g(t,s)h(|x-y|)$$

to be satisfied for $t, s \in \mathbb{R}^+$ and $x, y \in \mathbb{R}$.

Moreover we get $\overline{f} = \sqrt{5}$, $\overline{g} = 1/e$ and $\overline{u} = 1/e$, where

$$\overline{f} = \sup\left\{|f(t,0)| : t \in \mathbb{R}^+\right\}, \quad \overline{g} = \sup\left\{\int_0^\infty g(t,s)\, ds : t \in \mathbb{R}^+\right\}$$

and

$$\overline{u} = \sup\left\{ \int_0^\infty |u(t,s,0)|\,ds : t \in \mathbb{R}^+ \right\}.$$

Thus, assumptions (i)–(vi) of Theorem 8 in [7] are fulfilled.

Finally, let us note that the inequality of assumption (vii), given in [7]:

$$\|a\| + k\overline{g}rh(r) + k\overline{u}r + \overline{f}\overline{g}h(r) + \overline{f}\overline{u} \le r$$

takes the following form:

$$\frac{r^2 + \sqrt{5}r + 1}{2e} + \frac{r + \sqrt{5}}{e} \le r. \tag{4.30}$$

It can be checked that (4.30) does not have a positive solution.

Therefore, Theorem 8 in [7] is inapplicable to (4.21)**.**

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