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Almost sure local well-posedness for the supercritical quintic NLS

Justin T. Brereton

This paper studies the quintic nonlinear Schrödinger equation on $\mathbb{R}^d$ with randomized initial data below the critical regularity $H^{(d-1)/2}$ for $d \geq 3$. The main result is a proof of almost sure local well-posedness given a Wiener randomization of the data in H^s for $s \in \left(\frac{1}{2}(d-2), \frac{1}{2}(d-1)\right)$. The argument further develops the techniques introduced in the work of Á. Bényi, T. Oh and O. Pocovnicu on the cubic problem. The paper concludes with a condition for almost sure global well-posedness.

1. Introduction

Consider the Cauchy problem for the nonlinear Schrödinger equation. Given initial data $\phi \in H^s(\mathbb{R}^d)$, for $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ the solution $u(t, x) \in \mathbb{C}$ satisfies

$$\begin{aligned} iu_t + \Delta u &= \pm |u|^{p-1} u, \\ u|_{t=0} &= \phi, \end{aligned} \tag{1}$$

where $+$ and $-$ correspond to the defocusing and focusing cases, respectively. This equation has conserved mass and energy

$$\begin{aligned} M(t) &= \frac{1}{2} \int_{\mathbb{R}^d} |u(t,x)|^2 \, dx, \\ E(t) &= \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u(t,x)|^2 \, dx \pm \frac{1}{p+1} \int_{\mathbb{R}^d} |u(t,x)|^{p+1} \, dx. \end{aligned}$$

The NLS equation is also invariant under a dilation symmetry. Given $u(t, x)$ that solves (1), the function $u_\lambda(t, x) = \lambda^{2/(p-1)} u(\lambda^2 t, \lambda x)$ is a solution with rescaled initial data for every λ. Furthermore there is a Sobolev index $s_c = \frac{d}{2} - \frac{2}{p-1}$ such that the homogeneous Sobolev norm $\|u_\lambda\|_{\dot{H}^{s_c}}$ is constant under this scaling. This index s_c is known as the scaling critical index, and when $\frac{d}{2} - \frac{2}{p-1} = s_c = 1$ the problem is known as energy critical, since the energy scales like $\dot{H}^{s_c} = \dot{H}^1$.

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Given initial data $\phi \in H^s(\mathbb{R}^d)$, the problem is called subcritical when $s > s_c$ and supercritical when $s < s_c$.

In addition, special pairs of exponents (q, r) satisfying the bounds $2 \le q, r \le \infty$ and $(q, r, d) \neq (2, \infty, 2)$ are called Schrödinger-admissible if

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}. \tag{2}$$

For such a pair we have the well known Strichartz estimate

$$\|S(t)\phi\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \le C\|\phi\|_{L^2(\mathbb{R}^d)}, \tag{3}$$

where $S(t)$ denotes the linear Schrödinger semigroup operator $e^{it\Delta}$ that corresponds to solving the linear Schrödinger equation for time t; see [Strichartz 1977; Yajima 1987].

It is known that the NLS equation is ill-posed in the supercritical case; for such s one can construct special initial data $\phi \in H^s(\mathbb{R}^d)$ such that for every $T > 0$, (1) has no solution on $(-T, T)$ that stays in $H^s(\mathbb{R}^d)$, as demonstrated in [Alazard and Carles 2009; Christ et al. 2003]. Though local well-posedness is not guaranteed, it is important to determine if there are solutions for most supercritical initial data ϕ. This leads one to investigate the problem of almost sure well-posedness for initial data chosen for supercritical randomized initial data. Pocovnicu, Bényi and Oh have proven almost sure local well-posedness for the energy critical $\mathbb{R}^4$ problem using $X^{s,b}$ spaces in [Bényi et al. 2015a]. They then proved a separate result for the cubic equation for all $d \ge 3$ using U^p and V^p spaces and their adaptations for the Schrödinger equation in [Bényi et al. 2015b].

In this paper we adapt the techniques of [Bényi et al. 2015a; 2015b] in order to prove local well-posedness in the quintic case for dimension $d \ge 3$. Following [Bényi et al. 2015a], we apply a Wiener randomization to the initial data $\phi \in H^s(\mathbb{R}^d)$. This randomization method takes a function $\phi \in H^s(\mathbb{R}^d)$ and for each ω in a probability space Ω produces a randomized function

$$\phi^\omega = \sum_{n\in\mathbb{Z}^d} g_n(\omega)\eta(D-n)\phi \tag{4}$$

that is in $H^s(\mathbb{R}^d)$ with probability 1 but gains regularity with probability 0. The $g_n(\omega)$ are mean zero, i.i.d. complex random variables that are required to satisfy a decay condition, the Gaussian being such a random variable. The term $\eta(D-n)$ is a Fourier multiplier whose symbol approximates the characteristic function of the unit cube centered at n in frequency space.

In Section 2 we present several previously known probabilistic bounds on the Wiener randomization ϕ^ω of $\phi \in H^s(\mathbb{R}^d)$ as well as its linear Schrödinger evolution $S(t)\phi^\omega$. One of these is a probabilistic bound on $\|\langle\nabla\rangle^s S(t)\phi^\omega\|_{L^q L^r(I\times\mathbb{R}^d)}$

for arbitrarily large values of q, r. For large enough values of q, r this is a norm that scales subcritically, which means we can approach almost sure local well-posedness as if it is a subcritical problem.

Our main result is the almost sure local well-posedness of (1) with initial data ϕ^ω chosen via the Wiener randomization of any $\phi \in H^s(\mathbb{R}^d)$:

Theorem 1.1. *Fix $d \geq 3$ and $s \in \left(\frac{1}{2}(d-2), \frac{1}{2}(d-1)\right)$. Given $\phi \in H^s(\mathbb{R}^d)$ with Wiener randomization ϕ^ω, $\omega \in \Omega$, the quintic nonlinear Schrödinger equation is almost surely locally well-posed. More specifically, there exist $c_1, c_2, \theta > 0$ such that for sufficiently small $T \ll 1$, there is a set $\Omega_T \subset \Omega$ such that*

$$P(\Omega_T) \geq 1 - c_1 e^{-c_2/T^\theta \|\phi\|_{H^s}}$$

and for each $\omega \in \Omega_T$, the initial value problem

$$\begin{aligned} iu_t + \Delta u &= \pm |u|^4 u, \\ u(0) &= \phi^\omega \end{aligned}$$

has a unique solution in the function class $C((-T, T) \to H^s(\mathbb{R}^d))$.

We now provide a brief outline of the proof. In Section 3 we define the Littlewood–Paley projection operator, as well as the U^2 and V^2 spaces and their Schrödinger analogues developed by Koch, Tataru and others in [Hadac et al. 2009; Herr et al. 2011]. In Section 4 we present Strichartz estimates as well as a bilinear estimate for these spaces. The next step is to split the NLS solution u into its linear part $z(t) = S(t)\phi^\omega$ and nonlinear part

$$v(t) = \pm \int_0^t -iS(t-t')\big[|v+z|^4(v+z)\big](t')\,dt', \tag{5}$$

the integral term of Duhamel's formula. Our probabilistic bounds tells us that z almost surely has the same regularity as the initial data ϕ^ω. Therefore the linear part of the solution is almost surely in the supercritical space $H^s(\mathbb{R}^d)$, and it remains to prove existence of the nonlinear part $v(t)$. As mentioned earlier, $z(t)$ is bounded in subcritical norms, which means we can treat our linear solution $z(t)$ as a subcritical perturbative term in the Cauchy problem

$$\begin{aligned} iv_t + \Delta v &= \pm (v+z)|v+z|^4, \\ v(0) &= 0 \end{aligned} \tag{6}$$

that is satisfied by the nonlinear part v.

This means almost sure local well-posedness of $v(t)$ is essentially a subcritical problem. We prove local existence of the nonlinear part $v(t)$ using a fixed point argument based on doing a frequency decomposition of $v(t)$ and bounding it at each frequency.

Global well-posedness is a much harder problem. There is yet to be a proof of almost sure global well-posedness of any supercritical NLS problem. Pocovnicu, Bényi and Oh proved almost sure global well-posedness of $v \in H^1(\mathbb{R}^4)$ for the cubic problem under the assumption that there is a probabilistic bound on $\|v\|_{L^\infty H^1(\mathbb{R}\times\mathbb{R}^4)}$ in [Bényi et al. 2015b]. It seems difficult to prove such a bound.

One could probably prove a similar result for the 3-dimensional quintic problem, the energy critical dimension for the quintic problem. Instead we prove almost sure global well-posedness of v in the subcritical space $S^{1+c}(\mathbb{R}\times\mathbb{R}^3)$ assuming the norm $\|v\|_{L^{10}L^{10}([-T,T]\times\mathbb{R}^3)}$ does not blow up in finite time. This means that a probabilistic a priori estimate for $\|v\|_{L^{10}L^{10}([-T,T]\times\mathbb{R}^3)}$ implies almost sure global well-posedness as expressed in the following result:

Theorem 1.2. *Assume $\frac{7}{8} < s < 1$ and $0 < c < \frac{1}{8}$. Suppose we have a probabilistic a priori estimate for $\|v\|_{L^{10}L^{10}([-T,T]\times\mathbb{R}^3)}$, meaning for every $T, R > 0$ there is a function $\alpha(T, R)$ and a set $\Omega'_{T,R}$ such that*

- *for any $\omega \in \Omega'_{T,R}$, if the solution $v(t)$ to (6) exists on $(-T, T)$ then we have the bound*
$$\|v\|_{L^{10}L^{10}([-T,T]\times\mathbb{R}^3)} < R;$$
- *$P(\Omega'_{T,R}) \geq 1 - \alpha(T, R)$;*
- *for all $T > 0$, $\lim_{R\to\infty} \alpha(T, R) = 0$.*

Then given $\phi \in H^s(\mathbb{R}^3)$ with Wiener randomization ϕ^ω, the initial value problem

$$iu_t + \Delta u = \pm|u|^4 u,$$
$$u(0) = \phi^\omega$$

is almost surely globally well-posed, meaning there is a set $\Omega_{T,R} \subset \Omega$ and constants $c_1, c_2, c_3 > 0$ such that

$$P(\Omega_{T,R}) \geq 1 - c_1 e^{-c_2 R^2} - c_3\alpha(T, R)$$

and for any $\omega \in \Omega_{T,R}$ the above equation has a unique solution in the function class $C((-T, T) \to H^s(\mathbb{R}^d))$ with $v(t) \in H^{1+c}(\mathbb{R}^3)$ for any time $t \in (-T, T)$.

2. Randomization of initial data and probabilistic estimates

Our method of randomization is the Wiener decomposition of the frequency space that was used in [Bényi et al. 2015a] and first introduced in [Zhang and Fang 2012]. Consider a Schwartz class function $\psi \in \mathcal{S}(\mathbb{R}^d)$ that approximates the cube of unit length centered at the origin in $\mathbb{R}^d$, meaning that ψ is supported on $[-1, 1]^d$ and $\sum_{n\in\mathbb{Z}^d} \psi(\xi - n)$ is identically 1. Then for each n, define the Fourier multiplier η as

$$\eta(D - n)u(x) = \mathcal{F}^{-1}[\psi(\xi - n)\mathcal{F}u]. \tag{7}$$

Note that this satisfies $\sum_{n\in\mathbb{Z}^d} \eta(D-n)u(x) = u(x)$. This provides a decomposition of the function u into pieces whose frequencies are localized to cubes.

The idea is then to consider a function $\phi \in H^s(\mathbb{R}^d)$ and for each ω from a probability space Ω create a randomized function $\sum_{n\in\mathbb{Z}^d} g_n(\omega)\eta(D-n)\phi$ for some random variables g_n. For each $n \in \mathbb{Z}^d$, let μ_n and ν_n be probability distributions on $\mathbb{R}$, symmetric about 0, such that for some constant c we have

$$\left|\int_{\mathbb{R}^d} e^{\lambda x}\, d\mu_n(x)\right| \le e^{c\lambda^2} \quad \text{and} \quad \left|\int_{\mathbb{R}^d} e^{\lambda x}\, d\nu_n(x)\right| \le e^{c\lambda^2} \tag{8}$$

for all $n \in \mathbb{Z}^d$, $\lambda \in \mathbb{R}$. A Gaussian random variable would be an example of a random variable with these properties. Then define each g_n to be an independent, mean zero, complex random variable on Ω such that $\mathrm{Re}(g_n)$ and $\mathrm{Im}(g_n)$ have distributions μ_n and ν_n. We define the Wiener randomization ϕ^ω of $\phi \in H^s(\mathbb{R}^d)$ to be

$$\phi^\omega = \sum_{n\in\mathbb{Z}^d} g_n(\omega)\eta(D-n)\phi. \tag{9}$$

The main advantage derived from the Wiener randomization is improved $L^p(\mathbb{R}^d)$ estimates on the randomized initial data ϕ^ω off a small set, as a result of a stronger Bernstein's inequality. Despite only requiring that ϕ be in H^s, the randomized ϕ^ω is in $L^p(\mathbb{R}^d)$ with probability 1. In addition, we have a probabilistic bound on $\|\phi^\omega\|_{H^s(\mathbb{R}^d)}$, which implies that $\phi^\omega \in H^s(\mathbb{R}^d)$ almost surely.

We have the following key bounds on ϕ^ω and its linear Schrödinger evolution with proofs from [Bényi et al. 2015a]. We prove the first estimate while omitting the proofs of the second and third estimates. For all $R > 0$, $s > 0$ and $\phi \in H^s(\mathbb{R}^d)$, we have

$$\begin{aligned} P\left(\|\phi^\omega\|_{H^s(\mathbb{R}^d)} > R\right) &\le c_1 e^{-c_2 R^2/\|\phi\|^2_{H^s(\mathbb{R}^d)}}, \\ P\left(\|S(t)\phi^\omega\|_{L^q_t L^r_x([0,T]\times\mathbb{R}^d)} > R\right) &\le c_1 e^{-c_2 R^2/T^{2/q}\|\phi\|^2_{L^2(\mathbb{R}^d)}}, \\ P\left(\|\phi^\omega\|_{L^p(\mathbb{R}^d)} > R\right) &\le c_1 e^{-c_2 R^2/\|\phi\|_{L^2(\mathbb{R}^d)}}. \end{aligned}$$

Lemma 2.1. *Given $\phi \in H^s$ with randomization ϕ^ω, for all $R > 0$ there exist positive constants c_1, c_2 such that*

$$P\left(\|\phi^\omega\|_{H^s(\mathbb{R}^d)} > R\right) \le c_1 e^{-c_2 R^2/\|\phi\|^2_{H^s(\mathbb{R}^d)}}. \tag{10}$$

Proof. The proof is taken from [Bényi et al. 2015a]. By Minkowski's inequality, we have for $p \ge 2$,

$$\begin{aligned} \mathbb{E}\left[\|\phi^\omega\|^p_{H^s(\mathbb{R}^d)}\right] &\le \left(\left\|\|\langle\nabla\rangle^s\phi^\omega\|_{L^p(\Omega)}\right\|_{L^2(\mathbb{R}^d)}\right)^p \\ &= \left(\left\|\left\|\sum_{n\in\mathbb{Z}^d} g_n(\omega)\langle\nabla\rangle^s\eta(D-n)\phi\right\|_{L^p(\Omega)}\right\|_{L^2(\mathbb{R}^d)}\right)^p. \end{aligned} \tag{11}$$

By a well known lemma on sums of random variables, stated as Lemma 2.1 in [Bényi et al. 2015a] and proven in [Burq and Tzvetkov 2008], and the fact that Fourier multipliers commute, we have

$$\begin{aligned}\mathbb{E}\big[\|\phi^\omega\|^p_{H^s(\mathbb{R}^d)}\big] &\le C\Big(\big\|\sqrt{p}\,\|g_n\langle\nabla\rangle^s\eta(D-n)\phi\|_{l^2(n\in\mathbb{Z}^d)}\big\|_{L^2(\mathbb{R}^d)}\Big)^p,\\ \mathbb{E}\big[\|\phi^\omega\|^p_{H^s(\mathbb{R}^d)}\big] &\le C\big(\sqrt{p}\,\|\phi\|_{H^s(\mathbb{R}^d)}\big)^p.\end{aligned} \tag{12}$$

So by Markov's inequality,

$$\begin{aligned}R^p\,P\big(\|\phi^\omega\|_{H^s(\mathbb{R}^d)}>R\big) &\le C\big(\sqrt{p}\,\|\phi\|_{H^s(\mathbb{R}^d)}\big)^p,\\ P\big(\|\phi^\omega\|_{H^s(\mathbb{R}^d)}>R\big) &\le \frac{\big(C_0\sqrt{p}\,\|\phi\|_{H^s(\mathbb{R}^d)}\big)^p}{R^p}.\end{aligned} \tag{13}$$

Now let $p=\big(R/\big(C_0e\|\phi\|_{H^s}\big)\big)^2$ with C_0 taken from above. There are two cases.

- $p<2$: In this case we cannot use the above work because it assumes $p\ge 2$ for Minkowski's inequality. Letting $c_2=1/(C_0^2e^2)$ we have $e^{-c_2R^2/\|\phi\|_{H^s}}\ge e^{-2}$. Now choosing $c_1\ge e^2$ we have

$$P\big(\|\phi^\omega\|_{H^s(\mathbb{R}^d)}>R\big)\le 1\le c_1e^{-2}\le c_1e^{-c_2R^2/\|\phi\|^2_{H^s}}, \tag{14}$$

  since every probabilistic outcome has probability less than 1.

- $p\ge 2$: From the definition of p above and (13), we have

$$P\big(\|\phi^\omega\|_{H^s(\mathbb{R}^d)}>R\big)\le e^{-p}\le e^{-c_2R^2/\|\phi\|^2_{H^s}}. \tag{15}$$

In both cases the lemma is proven. □

Lemma 2.2. *Given $\phi\in H^s$ with randomization ϕ^ω, for all $R>0$ there exist positive constants c_1,c_2 such that*

$$P\big(\|S(t)\phi^\omega\|_{L^q_tL^r_x([0,T]\times\mathbb{R}^d)}>R\big)\le c_1e^{-c_2R^2/T^{2/q}\|\phi\|^2_{L^2}}. \tag{16}$$

After multiplying R by a small power of T we have the following corollary.

Corollary 2.3. *For small $\theta\in\big[0,\frac{1}{q}\big)$ and $R>0$ there exist c_1,c_2 such that*

$$\begin{aligned}P\big(\|S(t)\phi^\omega\|_{L^q_tL^r_x([0,T]\times\mathbb{R}^d)}>T^\theta R\big) &\le c_1e^{-c_2R^2/T^{2/q-2\theta}\|\phi\|^2_{L^2}}\\ &\le c_1e^{-c_2R^2/\|\phi\|^2_{L^2}}.\end{aligned} \tag{17}$$

Placing derivatives inside the norm and noting that derivatives commute with Fourier multipliers such as $S(t)$ and the map $\phi\to\phi^\omega$, we have our main bound:

Theorem 2.4. *Given small $\theta \in \left[0, \frac{1}{q}\right)$ and ϕ^ω chosen according to a Wiener randomization, for all $R > 0$ there exist c_1, c_2 such that*

$$P\left(\|\langle\nabla\rangle^s S(t)\phi^\omega\|_{L_t^q L_x^r([0,T]\times\mathbb{R}^d)} > T^\theta R\right) \le c_1 e^{-c_2 R^2/T^{2/q-2\theta}\|\phi\|_{H^s}^2} \le c_1 e^{-c_2 R^2/\|\phi\|_{H^s}^2}. \tag{18}$$

This bound will be crucial in the proof of local well-posedness. This gives us as much integrability as we want in bounding a linear solution, which means the linear solution is bounded in subcritical norms, allowing us to treat local well-posedness like a subcritical problem.

Lemma 2.5. *Given $\phi \in H^s$ with randomization ϕ^ω, for all $R > 0$ there exist positive constants c_1, c_2 such that*:

$$P\left(\|\phi^\omega\|_{L^p(\mathbb{R}^d)} > R\right) \le c_1 e^{-c_2 R^2/\|\phi\|_{L^2}^2}. \tag{19}$$

Proof. The proofs of Theorem 2.4 and Lemma 2.5 can be found in [Bényi et al. 2015a]. They utilize the same basic argument as above, with some extra steps. Each proof exploits an improvement of Bernstein's inequality that results from the Wiener randomization. Note that $g_n(\omega)\eta(D-n)\phi$ has Fourier transform supported on the unit cube centered at n. Therefore, $e^{inx}g_n(\omega)\eta(D-n)\phi$ has Fourier transform supported on the unit cube centered at the origin. Bernstein's inequality implies that

$$\|e^{inx}g_n(\omega)\eta(D-n)\phi\|_{L^p} \lesssim \|e^{inx}g_n(\omega)\eta(D-n)\phi\|_{L^2} \tag{20}$$

with no loss of regularity, since multiplying by e^{inx} does not affect the L^p norm. So we obtain the bound $\|g_n(\omega)\eta(D-n)\phi\|_{L^p} \lesssim \|g_n(\omega)\eta(D-n)\phi\|_{L^2}$. This is the key ingredient in the proof that allows one to bound the higher L^p norm of ϕ^ω with high probability while only assuming that $\phi \in L^2$. □

3. Littlewood–Paley theory and function spaces

3A. ***Littlewood–Paley theory and dyadic decompositions.*** In the fixed point proof we will take the linear and nonlinear parts of our solution and dyadically decompose each into a sum of Littlewood–Paley projections. Given a smooth bump function ψ such that $\psi(\xi) = 1$ for $|\xi| \le 1$ and $\psi(\xi) = 0$ for $|\xi| \ge 2$, we have the following definition from the Littlewood–Paley theory.

Definition 3.1. Given dyadic N and a function $f \in L^2$ we define its *projection* $P_{\le N} f$ to be the Fourier multiplier such that

$$\widehat{P_{\le N} f}(\xi) = \psi\left(\frac{\xi}{N}\right)\hat{f}(\xi).$$

Of course the definition applies to a much wider range of distributions, but in this paper we need only consider functions in L^2 or H^s for some $s > 0$.

Note that $\widehat{P_{\leq N} f}$ is supported on the set $|\xi| \leq 2N$. Now we define the projection P_N that localizes to frequencies in the interval $[N/2, 2N]$.

Definition 3.2. We define $P_1 = P_{\leq 1}$ and for dyadic $N > 1$,

$$P_N f = P_{\leq N} f - P_{\leq N/2} f.$$

This defines the projection $P_N f$ with frequencies between $N/2$ and $2N$. Also, we have $\sum_N P_N f = f$, so this is indeed a decomposition.

The above info and other results on Littlewood–Paley theory can be found in the appendix of [Tao 2006].

3B. ***Strichartz spaces.*** In this and the following section we introduce the function spaces needed to prove well-posedness. We start with the standard Strichartz spaces: $S^s(I \times \mathbb{R}^d)$ and $N^s(I \times \mathbb{R}^d)$. Let q, r be a Schrödinger-admissible pair. Given an interval $I = [t_0, t_1]$ we define $S^s(I \times \mathbb{R}^d)$ to be the set of measurable functions bounded in the norm

$$\|u\|_{S^s(I\times\mathbb{R}^d)} = \sup_{(q,r) \text{ admissible}} \|\langle\nabla\rangle^s u\|_{L^q L^r(I\times\mathbb{R}^d)}.$$

We also define $N^{-s}(I \times \mathbb{R}^d)$ to be the dual space of $S^s(I \times \mathbb{R}^d)$, which satisfies the bound

$$\|u\|_{N^s(I\times\mathbb{R}^d)} \leq \inf_{(q,r) \text{ admissible}} \|\langle\nabla\rangle^s u\|_{L^{q'} L^{r'}(I\times\mathbb{R}^d)}.$$

The key relation between the Strichartz norms is the Strichartz estimate for solutions to the nonlinear Schrödinger equation. Suppose u is a solution to $iu_t + \Delta u = F$. Then

$$\|u\|_{S^s([t_0,t_1]\times\mathbb{R}^d)} \lesssim \|u(t_0)\|_{H^s(\mathbb{R}^d)} + \|F\|_{N^s([t_0,t_1]\times\mathbb{R}^d)}. \tag{21}$$

3C. ***U^p and V^p spaces.*** Our analysis requires a norm that measures how close a function is to a linear solution to the Schrödinger equation, so we use the U^p and V^p spaces of Koch and Tataru introduced in [Hadac et al. 2009; Herr et al. 2011]. We start by defining a U^p atom, and then the U^p and V^p spaces. Suppose $1 \leq p < \infty$ and $-\infty < t_0 < t_1 < \cdots < t_n \leq \infty$ is a partition of the real line. We denote the characteristic function of the k-th interval of this partition by $\chi_{[t_{k-1},t_k)}$.

Definition 3.3. A U^p *atom* is a step function into a Sobolev space $a(t) : \mathbb{R} \to H^s(\mathbb{R}^d)$ of the form

$$a = \sum_{k=1}^{n} \phi_k \chi_{[t_{k-1},t_k)}, \tag{22}$$

where $\sum_{k=1}^n \|\phi_k\|^p_{H^s(\mathbb{R}^d)} = 1$.

The definition applies to any Hilbert space H, but we will only need it for Sobolev spaces in this paper.

Definition 3.4. The space $U^p(\mathbb{R}; H^s)$ is the set of measurable functions bounded in the associated norm:

$$\|u\|_{U^p(\mathbb{R};H^s)} = \inf_{U^p \text{ atoms } a_j} \left\{ \sum_j |\lambda_j| : u = \sum_j \lambda_j a_j \right\}. \tag{23}$$

For the V^p spaces we continue to partition the real line, and take our norm to be the p-variation of the given function.

Definition 3.5. The space $V^p(\mathbb{R}; H^s)$ is the set of functions bounded under the V^p norm:

$$\|u\|_{V^p(\mathbb{R};H^s)} = \sup_{\text{partitions } t_k} \left(\sum_{k=1}^{n} \|u(t_k) - u(t_{k-1})\|^p_{H^s(\mathbb{R}^d)} \right)^{1/p}. \tag{24}$$

In addition, given an interval I, the norms $\|u\|_{U^p(I;H^s)}$, $\|u\|_{V^p(I;H^s)}$ and any of the following norms are defined as the restriction norms. For example,

$$\|u\|_{U^p(I;H^s)} = \inf_{\substack{w(t)=u(t),\ t\in I \\ w(\infty)=0=w(-\infty)}} \|w\|_{U^p(\mathbb{R};H^s)}. \tag{25}$$

Now we want to create a norm that measures how close our function is to a linear solution to the Schrödinger equation, much like in the definition of the $X^{s,b}$ spaces. If u is a linear solution then $S(-t)u$ is a function that is constant in time with $\|S(-t)u\|_{U^2(I;H^s)}$ and $\|S(-t)u\|_{V^2(I;H^s)}$ norms bounded by $\|u\|_{H^s}$. We define the $U^p_\Delta H^s$ and $V^p_\Delta H^s$ norms as

$$\|u\|_{U^p_\Delta H^s(\mathbb{R};H^s)} = \|S(-t)u\|_{U^p(\mathbb{R};H^s)},$$
$$\|u\|_{V^p_\Delta H^s(\mathbb{R};H^s)} = \|S(-t)u\|_{V^p(\mathbb{R};H^s)},$$

and the spaces $U^p_\Delta H^s$ and $V^p_\Delta H^s$ are defined as the set of measurable functions $u : \mathbb{R} \to H^s(\mathbb{R}^d)$ bounded in the $U^p_\Delta H^s$ and $V^p_\Delta H^s$ norms, respectively. These are useful spaces; however, in our proof we will rely on dyadic decomposition and will need to apply these norms at specific frequencies, so it is more useful to do computations in a slightly different norm adapted to dyadic decompositions.

Definition 3.6. We define the X^s and Y^s norms, and associated spaces, as follows:

$$\begin{aligned} \|u\|_{X^s(\mathbb{R})} &= \left(\sum_N N^{2s} \|P_N u\|^2_{U^2_\Delta L^2} \right)^{1/2}, \\ \|u\|_{Y^s(\mathbb{R})} &= \left(\sum_N N^{2s} \|P_N u\|^2_{V^2_\Delta L^2} \right)^{1/2}. \end{aligned} \tag{26}$$

Note that these norms are a little stronger than those above. They bound the closeness of the function u to a solution to the linear equation at each frequency, not just generally. Note that we immediately have the embedding $X^s \hookrightarrow Y^s$ as well as the bound $\|S(t)\phi\|_{X^s(\mathbb{R};H^s)} \leq \|\phi\|_{H^s(\mathbb{R}^d)}$. This bound means that these spaces are well suited to studying the linear problem.

In addition, we define the following norm for the nonhomogeneous term that will allow us to exploit duality:

$$\|F\|_{M^s(I)} = \left\| \int_{t_0}^{t} S(t-t')F(t')\,dt' \right\|_{X^s(I)}. \tag{27}$$

This is equivalent to the dual norm of Y^s, and we have the bound

$$\|F\|_{M^s(I)} \leq \sup_{\|v\|_{Y^s(I)}=1} \int_I \int_{\mathbb{R}^d} F(t,x)v(t,x)\,dx\,dt \tag{28}$$

as Lemma 3.5 in [Bényi et al. 2015b]. This is equivalent to

$$\|F\|_{M^s(I)} \leq \sup_{\|v\|_{Y^0(I)}=1} \int_I \int_{\mathbb{R}^d} \langle\nabla\rangle^s F(t,x)v(x,t)\,dx\,dt. \tag{29}$$

In addition, we have a bound analogous to the Strichartz estimate (21) for the M^s norm. Suppose $u(t,x)$ is a solution to the Cauchy problem

$$\begin{aligned} iu_t + \Delta u &= F, \\ u|_{t=0} &= u(0) \end{aligned} \tag{30}$$

on the interval I. Then we have the bound

$$\|u\|_{X^s(I)} \lesssim \|u(0)\|_{H^s(\mathbb{R}^d)} + \|F\|_{M^s(I)}. \tag{31}$$

4. Strichartz estimates

Lemma 4.1. *Let q, r be a Schrödinger-admissible pair.*

(i) *Given an interval I, for any $u \in Y^0(I)$ we have*

$$\|u\|_{L_t^q L_x^r(I\times\mathbb{R}^d)} \lesssim \|u\|_{Y^0(I)}. \tag{32}$$

(ii) *Given an interval I and $p \geq \frac{2(d+2)}{d}$, for any $u \in Y^{d/2-(d+2)/p}(I)$ we have*

$$\begin{aligned} \|u\|_{L_t^p L_x^p(I\times\mathbb{R}^d)} &\lesssim \left\| |\nabla|^{d/2-(d+2)/p} u \right\|_{Y^0(I)} \\ &\lesssim \|u\|_{Y^{d/2-(d+2)/p}(I)}. \end{aligned} \tag{33}$$

Proof. The proof of the first statement is in [Bényi et al. 2015b, Lemma 3.5]. To prove the second statement note that for $\frac{1}{p} = \frac{1}{r} - \frac{k}{d}$, Sobolev embedding implies that

$$\|u\|_{L^p(\mathbb{R}^d)} \lesssim \big\||\nabla|^k u\big\|_{L^r(\mathbb{R}^d)}. \tag{34}$$

Then taking the $L_t^p(I)$ norm of both sides we have

$$\|u\|_{L_{t,x}^p(I\times\mathbb{R}^d)} \lesssim \big\||\nabla|^k u\big\|_{L_t^p L_x^r(I\times\mathbb{R}^d)}. \tag{35}$$

Then by part (i) we have, for $\frac{2}{p}+\frac{d}{r}=\frac{d}{2}$,

$$\big\||\nabla|^k u\big\|_{L_t^p L_x^r(I\times\mathbb{R}^d)} \lesssim \big\||\nabla|^k u\big\|_{Y^0(I)}. \tag{36}$$

This proves the desired inequality in $\mathbb{R}^d$ with exponents that satisfy $\frac{2}{p}+\frac{d}{r}=\frac{d}{2}$ and $\frac{1}{p}=\frac{1}{r}-\frac{k}{d}$. Substituting, we get $k=\frac{d}{2}-\frac{d+2}{p}$. □

By selecting $q=r=\frac{2(d+2)}{d}$ and $p=2(d+2)$, we obtain the following corollaries:

Corollary 4.2. *For all $u\in Y^0(I)$ we have*

$$\|u\|_{L_{t,x}^{2(d+2)/d}(I\times\mathbb{R}^d)} \lesssim \|u\|_{Y^0(I)}. \tag{37}$$

Corollary 4.3. *For all $u\in Y^{(d-1)/2}(I)$ we have*

$$\|u\|_{L_{t,x}^{2(d+2)}(I\times\mathbb{R}^d)} \lesssim \big\||\nabla|^{(d-1)/2}u\big\|_{Y^0(I)} \lesssim \|u\|_{Y^{(d-1)/2}(I)}. \tag{38}$$

Lastly, the following is a bilinear projection lemma that gives an L^2 bound on the bilinear L^2 norm of projections at different frequencies from [Bourgain 1998; Ozawa and Tsutsumi 1998]. In addition there is a version adapted to the Schrödinger equation from [Visan 2006].

Lemma 4.4. *For dyadic $N_1\le N_2$ and $\phi_1,\phi_2\in L^2$ we have*

$$\begin{aligned}&\|P_{N_1}S(t)\phi_1\,P_{N_2}S(t)\phi_2\|_{L^2(I\times\mathbb{R}^d)}\\ &\qquad\lesssim N_1^{(d-1)/2}N_2^{-1/2}\|P_{N_1}\phi_1\|_{L^2(\mathbb{R}^d)}\|P_{N_2}\phi_2\|_{L^2(\mathbb{R}^d)}.\end{aligned} \tag{39}$$

Corollary 4.5. *For $N_1\le N_2$ and $u_1,u_2\in Y^0(I)$ we have*

$$\begin{aligned}&\|P_{N_1}u_1\,P_{N_2}u_2\|_{L^2(I\times\mathbb{R}^d)}\\ &\qquad\lesssim N_1^{(d-1)/2-}N_2^{-1/2+}\|P_{N_1}u_1\|_{Y^0(I)}\|P_{N_2}u_2\|_{Y^0(I)}.\end{aligned} \tag{40}$$

Proof. The proof is found in [Bényi et al. 2015b] as Lemma 3.5. □

This will be a key ingredient in the proof of local well-posedness because it allows us to gain half a derivative from higher frequency terms. In addition, we use the following 3-dimensional bilinear estimate that solely consists of Strichartz norms.

Theorem 4.6. *For dyadic $N_1 \leq N_2$ and any small $\delta > 0$ we have*

$$\begin{aligned}\|P_{N_1}u_1 P_{N_2}u_2\|_{L^2(I\times\mathbb{R}^3)} &\lesssim N_1^{(d-1)/2-\delta} N_2^{-1/2+\delta} \big(\|P_{N_1}u_1(0)\|_{L^2(\mathbb{R}^3)} \\ &\qquad + \|(i\partial_t+\Delta)P_{N_2}u_2\|_{L^{3/2}L^{18/13}(I\times\mathbb{R}^3)}\big) \\ &\quad \times \big(\|P_{N_2}u_2(0)\|_{L^2(\mathbb{R}^3)} \\ &\qquad + \|(i\partial_t+\Delta)P_{N_2}u_2\|_{L^{3/2}L^{18/13}(I\times\mathbb{R}^3)}\big). \end{aligned} \tag{41}$$

Proof. The proof is found in [Visan 2006] as Lemma 2.5. □

This will be a key ingredient in the proof of Theorem 1.2 in Section 6.

5. Almost sure local well-posedness

We now begin the proof of Theorem 1.1. Given some $\phi \in H^s(\mathbb{R}^d)$, let ϕ^ω be its Wiener randomization, and recall that $z(t) = S(t)\phi^\omega$ denotes the linear part of the NLS solution and $v(t)$ is the solution to (6).

Even though we do not have long term bounds on the H^s norm of v, we know that $v(0) = 0$. Exploiting our probabilistic bound on $z(t)$ in subcritical norms, we show that for $\rho \in \big(\frac{d-1}{2}, s+\frac{1}{2}\big)$ the norm $\|v\|_{X^\rho((-T,T))}$ is bounded for small enough time T.

Our method is a fixed point argument. We define

$$\Gamma v(t) = \pm \int_0^t -iS(t-t')\big[|v+z|^4(v+z)\big](t')\,dt' \tag{42}$$

and note that v is a solution if and only if $\Gamma v = v$. We now prove the following proposition, which is the bulk of our fixed point argument.

Proposition 5.1. *Assume s and ρ satisfy the bounds*

$$\frac{d}{2} > s + \frac{1}{2} > \rho > \frac{d-1}{2}. \tag{43}$$

Given $\phi \in H^s(\mathbb{R}^d)$ with randomization ϕ^ω, there exists small $\theta > 0$ such that for every $R > 0$ and sufficiently small $T \ll 1$ depending on R, we have

- $\|\Gamma v\|_{X^\rho([0,T))} \lesssim T^\theta\big(\|v\|^5_{X^\rho([0,T))} + R^5\big)$ *off a set of measure* $c_1 e^{-c_2 R^2/\|\phi\|^2_{H^s}}$,
- $\|\Gamma v_1 - \Gamma v_2\|_{X^\rho([0,T))}$
$$\lesssim T^\theta\big(R^4 + \|v_1\|^4_{X^\rho([0,T))} + \|v_2\|^4_{X^\rho([0,T))}\big)\|v_1 - v_2\|_{X^\rho([0,T))}$$
off a set of measure $c_1 e^{-c_2 R^2/\|\phi\|^2_{H^s}}$.

This stems from Theorem 2.4, which tells us that for $\theta < \frac{1}{q}$ we have

$$P\big(\|\langle\nabla\rangle^s z\|_{L_t^q L_x^r([0,T)\times\mathbb{R}^d)} \leq T^\theta R\big) \geq 1 - c_1 e^{-c_2 R^2/\|\phi\|^2_{H^s}}, \tag{44}$$

allowing us to gain a factor of T.

Proof. We only prove the bound on $\|\Gamma v\|_{X^\rho([0,T))}$, as the proof of the bound on $\|\Gamma v_1 - \Gamma v_2\|_{X^\rho([0,T))}$ is similar. For dyadic $N \geq 1$ define

$$\begin{aligned}\Gamma_N(v) &= P_{\leq N}\Gamma(v)\\ &= P_{\leq N}\left(\pm\int_0^t -iS(t-t')\big[|v+z|^4(v+z)\big](t')\,dt'\right)\\ &= \pm\int_0^t -iS(t-t')P_{\leq N}\big[|v+z|^4(v+z)(t')\big]\,dt'.\end{aligned}$$

By (29) we have

$$\begin{aligned}\|\Gamma_N v\|_{X^\rho} &= \big\|P_{\leq N}\big[(v+z)|v+z|^4\big]\big\|_{M^\rho}\\ &\leq \sup_{\|v_6\|_{Y^0}\leq 1}\int_0^T\int_{\mathbb{R}^d}\langle\nabla^\rho\rangle|v+z|^4(v+z)\,\overline{P_{\leq N}v_6}\,dx\,dt.\end{aligned}$$

Now, noting that

$$\begin{aligned}\|\Gamma v\|_{X^\rho} &= \lim_{N\to\infty}\|\Gamma_N v\|_{X^\rho}\\ &= \sup_{\|v_6\|_{Y^0}\leq 1}\int_0^T\int_{\mathbb{R}^d}\langle\nabla^\rho\rangle|v+z|^4(v+z)\overline{v_6}\,dx\,dt,\end{aligned}$$

it suffices to show that for small $\theta > 0$ this integral is $\leq CT^\theta(R^5 + \|v\|_{X^\rho}^5)\|v_6\|_{Y^0}$ off a set of measure $c_1 e^{-c_2R^2/\|\phi\|_{H^s}^2}$. We do this by proving the bound

$$\int_0^T\int_{\mathbb{R}^d}\langle\nabla\rangle^\rho|v+z|^4(v+z)\overline{v_6}\,dx\,dt \leq CT^\theta(R^5 + \|v\|_{X^\rho}^5)\|v_6\|_{Y^0} \tag{45}$$

via case by case analysis of terms of the form $\langle\nabla\rangle^\rho[w_1w_2w_3w_4w_5]v_6$, where each w_i is either $v_i = v$ or $z_i = z$ (or its complex conjugate), and each is dyadically decomposed into $\sum_{N_i\geq 1,\,\text{dyadic}} P_{N_i}v_i$ and $\sum_{N_j\geq 1,\,\text{dyadic}} P_{N_j}z_j$. Dyadic decomposition allows us to assume the derivatives are placed on the highest frequency term, or split them between two comparably high frequency terms. Also, we just write w_i instead of $P_{N_i}w_i$ as we sum over dyadic integers $N_i \geq 1$.

We consider four main cases based on whether each w_i is a v_i or z_i, and which two terms have the highest frequencies:

- Case 1: All five terms are v.
- Case 2: At least one term is a v and it has one of the two highest frequencies.
- Case 3: The two highest frequencies are on z terms.
- Case 4: The two highest frequencies are on a z term and the v_6 term.

These four cases are then divided into smaller subcases:

• Case 1: $v_1 v_2 v_3 v_4 v_5 v_6$.

In this case all terms are v. We do not do dyadic decompositions; instead we cut the frequency space into 5 pieces based on which frequency is largest, and assume without loss of generality that ξ_1 is. We split into two cases, based on the value of ρ, which determines which exponents we can use in Hölder's inequality.

(1a) $\rho < \frac{d}{2} - \frac{1}{4}$.

Noting that $\rho < \frac{d}{2} - \frac{1}{4}$, we apply Hölder's inequality with t exponents

$$\tau_1 = \frac{2(d+2)}{d(2d-4\rho-1)}, \quad \tau_2 = \cdots = \tau_5 = \frac{4(d+2)}{d+2-d(d-2\rho)}, \quad \tau_6 = \frac{2(d+2)}{d}$$

and x exponents

$$\sigma_1 = \frac{2(d+2)}{8\rho+4-3d}, \qquad \sigma_2 = \cdots = \sigma_5 = \frac{2(d+2)}{d-2\rho}, \qquad \sigma_6 = \frac{2(d+2)}{d}$$

and Lemma 4.1:

$$\begin{aligned} I &= \int_0^T \int_{\mathbb{R}^d} \langle\nabla\rangle^\rho v_1 v_2 v_3 v_4 v_5 v_6 \, dx \, dt \\ &\leq \|\langle\nabla\rangle^\rho v_1\|_{L^{\tau_1} L^{\sigma_1}} \|v_2\|_{L^{\tau_2} L^{\sigma_2}} \|v_3\|_{L^{\tau_3} L^{\sigma_3}} \|v_4\|_{L^{\tau_4} L^{\sigma_4}} \|v_5\|_{L^{\tau_5} L^{\sigma_5}} \|v_6\|_{L^{\tau_6}} \\ &\leq \|v_1\|_{Y^\rho} T^\theta \|v_2\|_{L^{\sigma_2} L^{\sigma_2}} \|v_3\|_{L^{\sigma_3} L^{\sigma_3}} \|v_4\|_{L^{\sigma_4} L^{\sigma_4}} \|v_5\|_{L^{\sigma_5} L^{\sigma_5}} \|v_6\|_{Y^0} \\ &\leq T^\theta \prod_{i=1}^5 \|v_i\|_{Y^\rho} \|v_6\|_{Y^0} \end{aligned}$$

for some $\theta > 0$.

(1b) $\frac{d}{2} - \frac{1}{4} \leq \rho < s + \frac{1}{2}$.

Noting that $\rho \geq \frac{d}{2} - \frac{1}{4}$ we apply Hölder's inequality with t exponents $\tau_1 = \infty$, $\tau_2 = \cdots = \tau_5 = 8(d+2)/(d+4)$, $\tau_6 = 2(d+2)/d$ and x exponents $\sigma_1 = 2$, $\sigma_2 = \cdots = \sigma_5 = 4(d+2)$, $\sigma_6 = 2(d+2)/d$ and Lemma 4.1:

$$\begin{aligned} I &= \int_0^T \int_{\mathbb{R}^d} \langle\nabla\rangle^\rho v_1 v_2 v_3 v_4 v_5 v_6 \, dx \, dt \\ &\leq \|\langle\nabla\rangle^\rho v_1\|_{L^{\tau_1} L^{\sigma_1}} \|v_2\|_{L^{\tau_2} L^{\sigma_2}} \|v_3\|_{L^{\tau_3} L^{\sigma_3}} \|v_4\|_{L^{\tau_4} L^{\sigma_4}} \|v_5\|_{L^{\tau_5} L^{\sigma_5}} \|v_6\|_{L^{\tau_6}} \\ &\leq \|v_1\|_{Y^\rho} T^\theta \|v_2\|_{L^{\sigma_2}} \|v_3\|_{L^{\sigma_3}} \|v_4\|_{L^{\sigma_4}} \|v_5\|_{L^{\sigma_5}} \|v_6\|_{Y^0} \\ &\leq T^\theta \|v_1\|_{Y^\rho} \prod_{i=2}^5 \|v_i\|_{Y^{(d/2)-(1/4)}} \|v_6\|_{Y^0} \end{aligned}$$

for some $\theta > 0$.

• Case 2: $v_1 w_2 w_3 w_4 z_5 v_6$, $N_1 \gtrsim N_2, N_3, N_4, N_5$.

In this case there is at least one v term and the highest frequency term is a v. Therefore we can assume the derivatives fall on the v_1 term with the highest frequency.

(2a) w_2, w_3, w_4 are all z terms, $N_5 \geq N_4 \geq N_3 \geq N_2 \geq N_1^{1/2(d-1)}$.

We have assumed that v_1 has the highest frequency: $N_1 \geq N_2, N_3, N_4, N_5$. Now we apply Hölder's inequality, Lemma 4.1 and our probabilistic bound on the linear term, Theorem 2.4, and note that $s > \frac{1}{2}$. Setting $\tau = 2(d+2)/d$ and $\sigma = 2(d+2)$, we have

$$\begin{aligned}
I &= \int_0^T \int_{\mathbb{R}^d} \langle\nabla\rangle^\rho v_1 z_2 z_3 z_4 z_5 v_6 \, dx \, dt \\
&\leq \|\langle\nabla\rangle^\rho v_1\|_{L^\tau} \|z_2\|_{L^\sigma} \|z_3\|_{L^\sigma} \|z_4\|_{L^\sigma} \|z_5\|_{L^\sigma} \|v_6\|_{L^\tau} \\
&\leq \|v_1\|_{Y^\rho} (N_2 N_3 N_4 N_5)^{-s} \prod_{i=2}^5 \|\langle\nabla\rangle^s z_i\|_{L^\sigma} \|v_6\|_{Y^0} \\
&\leq \|v_1\|_{Y^\rho} (N_2 N_3 N_4 N_5)^{-1/2} \prod_{i=2}^5 \|\langle\nabla\rangle^s z_i\|_{L^\sigma} \|v_6\|_{Y^0} \\
&\leq \|v_1\|_{Y^\rho} (N_2)^{-2} \prod_{i=2}^5 \|\langle\nabla\rangle^s z_i\|_{L^\sigma} \|v_6\|_{Y^0} \\
&\leq \|v_1\|_{Y^\rho} (N_1)^{-1/(d-1)} \prod_{i=2}^5 \|\langle\nabla\rangle^s z_i\|_{L^\sigma} \|v_6\|_{Y^0}.
\end{aligned}$$

Noting that N_1 is the highest frequency, the sum over all frequencies is bounded by $\|v\|_{Y^\rho} T^\theta R^4 \|v_6\|_{Y^0}$ off a set of measure $c_1 e^{-c_2 R^2/\|\phi\|_{H^s}^2}$.

(2b) w_2, w_3, w_4 are all z terms, $N_2 \leq N_1^{1/2(d-1)}$.

We apply Hölder's inequality, Theorem 2.4, Lemma 4.1 and our bilinear estimate Corollary 4.5, utilizing the assumption that $N_2 \leq N_1^{1/2(d-1)}$. This time with $\tau = 2(d+2)/d$ and $\sigma = 3(d+2)$, we have

$$\begin{aligned}
I &= \int_0^T \int_{\mathbb{R}^d} \langle\nabla\rangle^\rho v_1 z_2 z_3 z_4 z_5 v_6 \, dx \, dt \\
&\leq \|\langle\nabla\rangle^\rho v_1 z_2\|_{L^2} \|z_3\|_{L^\sigma} \|z_4\|_{L^\sigma} \|z_5\|_{L^\sigma} \|v_6\|_{L^\tau} \\
&\leq N_1^{\frac{-1}{2}+} \|v_1\|_{Y^\rho} N_2^{\frac{1}{2}(d-1)-} \|z_2\|_{Y^0} \|z_3\|_{L^\sigma} \|z_4\|_{L^\sigma} \|z_5\|_{L^\sigma} \|v_6\|_{L^\tau} \\
&\leq N_1^{\frac{-1}{2}+} \|v_1\|_{Y^\rho} N_1^{\frac{1}{4}} \|z_2\|_{Y^0} T^{0+} R^3 \|v_6\|_{Y^0} \\
&\leq N_1^{\frac{-1}{4}+} \|v_1\|_{Y^\rho} T^{0+} R^4 \|v_6\|_{Y^0},
\end{aligned}$$

which is $\leq \|v\|_{Y^\rho} T^{0+} R^4$ off a set of small measure.

(2c) $w_2 = v_2$ is a v term, and the others can be anything.

In this case we still have $N_1 \geq N_i, i = 2, \dots, 6$. Applying Hölder's inequality, Lemma 4.1, Theorem 2.4 and Corollary 4.5 with $\tau = 2(d+2)$ and $\sigma = 4(d+2)$, we have

$$\begin{aligned} I &= \int_0^T \int_{\mathbb{R}^d} \langle\nabla\rangle^\rho v_1 w_2 w_3 w_4 z_5 v_6 \, dx \, dt \\ &\leq \|\langle\nabla\rangle^\rho v_1 v_2\|_{L^2} \|w_3\|_{L^\tau} \|w_4\|_{L^\sigma} \|z_5\|_{L^\sigma} \|v_6\|_{L^{\tau/d}} \\ &\leq N_1^{\frac{-1}{2}+} \|v_1\|_{Y^\rho} N_2^{\frac{1}{2}(d-1)-} \|v_2\|_{Y^0} \|w_3\|_{L^\tau} \|w_4\|_{L^\sigma} \|z_5\|_{L^\sigma} \|v_6\|_{Y^0} \\ &\leq N_1^{\frac{-1}{2}+} \|v_1\|_{Y^\rho} \|v_2\|_{Y^{\frac{1}{2}(d-1)}} \|w_3\|_{L^\tau} \|w_4\|_{L^\sigma} \|z_5\|_{L^\sigma} \|v_6\|_{Y^0}. \end{aligned}$$

Now if w_3 is a v term, then $\|w_3\|_{L^\tau} \lesssim \|v_3\|_{Y^{(d-1)/2}}$ as required. If w_3 is a z term, then $\|w_3\|_{L^\tau} \leq T^{0+} R$ off a set of small measure. So either way this term is bounded.

If w_4 is a z term then, again, $\|w_4\|_{L^\sigma} \leq T^{0+} R$ off a set of small measure. The only trouble is if w_4 is a v term, in which case our inequality only gives us

$$\begin{aligned} \|w_4\|_{L^\sigma} &\lesssim \big\||\nabla|^{(d/2)-(1/4)} v_4\big\|_{Y^0} \\ &\lesssim N_4^{1/4} \|v_4\|_{Y^{(d-1)/2}}. \end{aligned}$$

There is an extra quarter derivative; however, since N_1 is the biggest frequency we have $N_4^{1/4} \leq N_1^{1/4}$, which is absorbed by the $N_1^{-1/2+}$ term.

Therefore each term in this case is bounded by $T^{0+}(\|v\|_{X^\rho}^5 + R^5)\|v_6\|_{Y^0}$ off a set of measure $c_1 e^{-c_2 R^2/\|\phi\|_{H^s}^2}$.

• Case 3: $w_1 w_2 w_3 z_4 z_5 v_6$, $N_4 \sim N_5 \gtrsim N_1, N_2, N_3, N_6$.

In this case the two biggest frequencies are on z terms, z_4 and z_5. The first three terms are denoted $w_i, i = 1, 2, 3$ and represent either v or z. Assume without loss of generality that $N_1 \leq N_2 \leq \cdots \leq N_4 \sim N_5 \geq N_6$. Applying Hölder's inequality for exponents

$$\tau_1 = \cdots = \tau_3 = 2(d+2), \qquad \tau_4 = \tau_5 = \frac{4(d+2)}{d+1}, \qquad \tau_6 = \frac{2(d+2)}{d},$$

Lemma 4.1 and Theorem 2.4 we have

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^d} w_1 w_2 w_3 z_4 \langle\nabla\rangle^\rho z_5 v_6 \, dx \, dt \\ &\qquad \leq \|w_1\|_{L^{\tau_1}} \|w_2\|_{L^{\tau_2}} \|w_3\|_{L^{\tau_3}} \|\langle\nabla\rangle^{\rho/2} z_4\|_{L^{\tau_4}} \|\langle\nabla\rangle^{\rho/2} z_5\|_{L^{\tau_5}} \|v_6\|_{L^{\tau_6}}. \end{aligned}$$

For $\frac{1}{2}\rho < s$ this term is bounded by

$$T^{0+} R^2 (\|v\|_{Y^{(d-1)/2}}^3 + T^{0+} R^3) \|v_6\|_{Y^0}$$

off a set of measure $c_1 e^{-c_2 R^2/\|\phi\|_{H^s}^2}$. Note that $\rho < s + \frac{1}{2} < 2s$ so ρ satisfies the requirements.

• Case 4: $w_1 w_2 w_3 w_4 z_5 v_6$, $N_5 \sim N_6 \gtrsim N_1, N_2, N_3, N_4$.

This is the biggest case by far and we divide it into several subcases based on how many v terms there are.

(4a) $z_1 z_2 z_3 z_4 z_5 v_6$, $N_5 \sim N_6 \gtrsim N_1, N_2, N_3, N_4$.

Assume without loss of generality $N_1 \le N_2 \le N_3 \le N_4 \lesssim N_5 \sim N_6$. By Hölder's inequality with exponents $(2, 8, 8, 8, 8)$, Corollary 4.5 and Theorem 2.4, we have

$$\begin{aligned} I &= \int_0^T \int_{\mathbb{R}^d} z_1 z_2 z_3 z_4 \langle\nabla\rangle^\rho z_5 v_6 \, dx \, dt \\ &\le \|z_1 v_6\|_{L^2} \|z_2\|_{L^8} \|z_3\|_{L^8} \|z_4\|_{L^8} \|\langle\nabla\rangle^\rho z_5\|_{L^8} \\ &\lesssim N_1^{\frac{1}{2}(d-1)-s+} N_5^{\frac{-1}{2}+} \|\langle\nabla\rangle^s z_1\|_{Y^0} \|v_6\|_{Y^0} \|z_2\|_{L^8} \|z_3\|_{L^8} \|z_4\|_{L^8} \|\langle\nabla\rangle^\rho z_5\|_{L^8} \\ &\lesssim N_1^{\frac{1}{2}(d-1)+} (N_1 N_2 N_3 N_4)^{-s} N_5^{\rho+\frac{-1}{2}-s+} \prod_{i=2}^{5} \|\langle\nabla\rangle^s z_i\|_{Y^0} \|\langle\nabla\rangle^s z_i\|_{L^8} \|v_6\|_{Y^0}. \end{aligned}$$

When $s + \frac{1}{2} > \rho$ and $s > \frac{1}{8}(d-1)$ the powers of the frequencies are negative and the sum is bounded by $T^{0+} R^5 \|v_6\|_{Y^0}$. We have assumed $s + \frac{1}{2} > \rho \ge \frac{1}{2}(d-1)$ in the statement of the theorem, and note that for $d \ge 3$, $\frac{1}{2}(d-2) > \frac{1}{8}(d-1)$ and therefore we only require $s > \frac{1}{2}(d-2)$; however,

$$s > \rho - \tfrac{1}{2} \ge \tfrac{1}{2}(d-2).$$

Thus, this term is bounded by $T^{0+} R^5 \|v_6\|_{Y^0}$ off a set of measure $c_1 e^{-c_2 R^2/\|\phi\|_{H^s}^2}$.

In all following cases, we can assume there is at least one v, at least one z and that N_5 and N_6 are the highest frequencies.

(4b) $v_1 z_2 z_3 z_4 z_5 v_6$, $N_5 \sim N_6 \gtrsim N_1, N_2, N_3, N_4$.

Assume without loss of generality that $N_2 \le N_3 \le N_4 \le N_5 \sim N_6 \ge N_1$.

Noting that $N_2 \le N_3, N_4$, we set $\tau = 6(d+2)/(d+1)$ and apply Hölder's inequality, Corollary 4.5 and Theorem 2.4 to obtain

$$\begin{aligned} I &= \int_0^T \int_{\mathbb{R}^d} v_1 z_2 z_3 z_4 \langle\nabla\rangle^\rho z_5 v_6 \, dx \, dt \\ &\le \|z_2 v_6\|_{L^2} \|v_1\|_{L^{2(d+2)}} \|z_3\|_{L^\tau} \|z_4\|_{L^\tau} \|\langle\nabla\rangle^\rho z_5\|_{L^\tau} \\ &\lesssim N_6^{\frac{-1}{2}+} N_2^{\frac{1}{2}(d-1)-} \|z_2\|_{Y^0} \|v_6\|_{Y^0} \|v_1\|_{Y^{\frac{1}{2}(d-1)}} (N_3 N_4)^{-s} N_5^{\rho-s} R^3 \\ &\lesssim N_6^{\rho-s-\frac{1}{2}+} N_2^{\frac{1}{2}(d-1)-s-} (N_3 N_4)^{-s} T^{0+} R^4 \|v_1\|_{Y^{\frac{1}{2}(d-1)}} \|v_6\|_{Y^0} \\ &\lesssim N_6^{\rho-s-\frac{1}{2}+} (N_2 N_3 N_4)^{\frac{1}{6}(d-1)-s} \|v_1\|_{Y^{\frac{1}{2}(d-1)}} T^{0+} R^4 \|v_6\|_{Y^0}. \end{aligned}$$

When $s > \frac{1}{6}(d-1)$ the powers of the frequencies are negative, and the sum is bounded by $\|v\|_{Y^{(d-1)/2}} T^{0+} R^4 \|v_6\|_{Y^0}$. And indeed, $s > \frac{1}{6}(d-1)$, since we have $s > \frac{1}{2}(d-2)$ and $d \geq 3$.

(4c) $v_1 v_2 z_3 z_4 z_5 v_6$, $N_5 \sim N_6 \gtrsim N_1, N_2, N_3, N_4$.

Assume without loss of generality that $N_1 \geq N_2$, $N_3 \leq N_4 \leq N_5 \sim N_6$. With $\tau = 6(d+2)/(d+1)$, by Hölder's inequality, Corollary 4.5 and Theorem 2.4 we have

$$\begin{aligned}
I &= \int_0^T \int_{\mathbb{R}^d} v_1 v_2 z_3 z_4 \langle \nabla \rangle^\rho z_5 v_6 \, dx \, dt, \\
&\leq \|v_1 v_6\|_{L^2} \|v_2\|_{L^{2(d+2)}} \|z_3\|_{L^\tau} \|z_4\|_{L^\tau} \|\langle \nabla \rangle^\rho z_5\|_{L^\tau} \\
&\lesssim N_6^{\frac{-1}{2}+} N_1^{\frac{1}{2}(d-1)-} \|v_1\|_{Y^0} \|v_6\|_{Y^0} \|v_2\|_{Y^{\frac{1}{2}(d-1)}} (N_3 N_4)^{-s} (N_5)^\rho T^{0+} R^3 \\
&\lesssim N_1^{0-} (N_3 N_4)^{-s} N_5^{\rho - s - \frac{1}{2}+} \|v_1\|_{Y^{\frac{1}{2}(d-1)}} \|v_2\|_{Y^{\frac{1}{2}(d-1)}} T^{0+} R^3 \|v_6\|_{Y^0} \\
&\lesssim N_1^{0-} N_2^{0-} (N_3 N_4)^{-s} N_5^{\rho - s - \frac{1}{2}+} \|v_1\|_{Y^{\frac{1}{2}(d-1)}} \|v_2\|_{Y^{\frac{1}{2}(d-1)}} T^{0+} R^3 \|v_6\|_{Y^0}.
\end{aligned}$$

Since $s + \frac{1}{2} > \rho$, all powers of the frequencies are negative and the sum is bounded by $\|v\|^2_{Y^{(d-1)/2}} T^{0+} R^3 \|v_6\|_{Y^0}$.

(4d) $v_1 v_2 v_3 z_4 z_5 v_6$, $N_5 \sim N_6 \gtrsim N_1, N_2, N_3, N_4$.

Assume without loss of generality that $N_1 \geq N_2 \geq N_3, N_4 \leq N_5$. Setting $\tau = 2(d+2)$ and $\sigma = 4(d+2)/d$, by Hölder's inequality, Corollary 4.5 and Theorem 2.4 we have

$$\begin{aligned}
I &= \int_0^T \int_{\mathbb{R}^d} v_1 v_2 v_3 z_4 \langle \nabla \rangle^\rho z_5 v_6 \, dx \, dt, \\
&\leq \|v_1 v_6\|_{L^2} \|v_2\|_{L^\tau} \|v_3\|_{L^\tau} \|z_4\|_{L^\sigma} \|\langle \nabla \rangle^\rho z_5\|_{L^\sigma} \\
&\lesssim N_6^{\frac{-1}{2}+} N_1^{\frac{1}{2}(d-1)-} \|v_1\|_{Y^0} \|v_6\|_{Y^0} \|v_2\|_{Y^{\frac{1}{2}(d-1)}} \|v_3\|_{Y^{\frac{1}{2}(d-1)}} N_4^{-s} N_5^{\rho - s} T^{0+} R^2 \\
&\lesssim N_1^{0-} N_4^{-s} N_5^{\rho - s - \frac{1}{2}+} \|v_1\|_{Y^{\frac{1}{2}(d-1)}} \|v_2\|_{Y^{\frac{1}{2}(d-1)}} \|v_3\|_{Y^{\frac{1}{2}(d-1)}} T^{0+} R^2 \|v_6\|_{Y^0}.
\end{aligned}$$

Since $s + \frac{1}{2} > \rho$, all powers of the frequencies are negative and the sum is bounded by $\|v\|^3_{Y^{(d-1)/2}} T^{0+} R^2 \|v_6\|_{Y^0}$.

(4e) $v_1 v_2 v_3 v_4 z_5 v_6$, $N_5 \sim N_6 \gtrsim N_1, N_2, N_3, N_4$.

Assume without loss of generality that $N_1 \leq N_2 \leq N_3 \leq N_4 \leq N_5$. Again with $\tau = 2(d+2)$ and $\sigma = 4(d+2)/d$, by Hölder's inequality, Corollary 4.5 and Theorem 2.4 we have

$$
\begin{aligned}
I &= \int_0^T \int_{\mathbb{R}^d} v_1 v_2 v_3 v_4 \langle\nabla\rangle^\rho z_5 v_6 \, dx \, dt, \\
&\le \|v_1 v_6\|_{L^2} \|v_2\|_{L^\tau} \|v_3\|_{L^\tau} \|v_4\|_{L^\sigma} \|\langle\nabla\rangle^\rho z_5\|_{L^\sigma} \\
&\lesssim N_6^{\frac{-1}{2}+} N_1^{\frac{1}{2}(d-1)-} \\
&\qquad \times \|v_1\|_{Y^0} \|v_6\|_{Y^0} \|v_2\|_{Y^{\frac{1}{2}(d-1)}} \|v_3\|_{Y^{\frac{1}{2}(d-1)}} \|v_4\|_{Y^{\frac{1}{4}d}} N_5^{\rho-s} T^{0+} R \\
&\lesssim N_1^{0-} N_4^{-\frac{1}{4}(d-2)} N_5^{\rho-s-\frac{1}{2}+} \\
&\qquad \times \|v_1\|_{Y^{\frac{1}{2}(d-1)}} \|v_2\|_{Y^{\frac{1}{2}(d-1)}} \|v_3\|_{Y^{\frac{1}{2}(d-1)}} \|v_4\|_{Y^{\frac{1}{2}(d-1)}} T^{0+} R \|v_6\|_{Y^0}.
\end{aligned}
$$

Since $s + \frac{1}{2} > \rho$, all powers of the frequencies are negative and the sum is bounded by $\|v\|^4_{Y^{(d-1)/2}} T^{0+} R \|v_6\|_{Y^0}$ off a set of small measure.

In each case the term is bounded by $CT^\theta(R^5 + \|v\|^5_{Y^\rho})\|v_6\|_{Y^0}$, for some $\theta > 0$. This completes the proof of the first part of the proposition. The proof of

$$\|\Gamma v_1 - \Gamma v_2\|_{X^\rho} \le CT^\theta(R^4 + \|v_1\|^4_{X^\rho} + \|v_2\|^4_{X^\rho})\|v_1 - v_2\|_{X^\rho}$$

off a set of measure $c_1 e^{-c_2 R^2/\|\phi\|^2_{H^s}}$ is similar and is omitted. □

Using this key proposition we can close the fixed point argument in the final theorem.

Proof of Theorem 1.1. Let B_r denote the ball of radius r in $X^\rho([0,T))$ with $\frac{1}{2}d > s + \frac{1}{2} > \rho \ge \frac{1}{2}(d-1)$ as in the previous proposition. We claim that for small enough T and small but fixed r the map Γ is a contraction on B_r outside a set of measure $c_1 e^{-c_2 R^2/\|\phi\|^2_{H^s}}$. See Section 1.6 of [Tao 2006] for an overview of contraction based fixed point arguments.

To apply the theory for fix point arguments we require, off a small set, the contraction conditions

- $\|\Gamma v\|_{X^\rho([0,T))} \le r$ for $v \in B_r$,
- $\|\Gamma v_1 - \Gamma v_2\|_{X^\rho([0,T))} \le \frac{1}{2}\|v_1 - v_2\|_{X^\rho([0,T))}$.

By the bounds from the proposition, we have, for all R and some fixed constant C, $\|\Gamma v\|_{X^\rho} \le CT^\theta(R^5 + r^5)$ and $\|\Gamma v_1 - \Gamma v_2\|_{X^\rho} \le C\|v_1 - v_2\|_{X^\rho} T^\theta(2r^4 + R^4)$ off a set of measure $c_1 e^{-c_2 R^2/\|\phi\|_{H^s}}$.

The contraction conditions are satisfied if we select r, R, T such that

$$
\begin{aligned}
r &\le R, \\
CT^\theta R^5 &\le \frac{r}{8}.
\end{aligned}
\tag{46}
$$

We can fix a value of r to satisfy the first bound. Selecting T such that $T \sim R^{-5/\theta}$

the second bound of a contraction is satisfied, and we conclude that the map Γ has a fixed point in B_r.

Therefore, for sufficiently small T, the equation $\Gamma v = v$ has a solution in B for every ϕ^ω off this set of measure $c_1 e^{-c_2 R^2/\|\phi\|_{H^s}^2}$. Setting $\alpha = -2\theta/5$, there exists a set $\Omega_T \subset \Omega$ of measure $\geq 1 - c_1 e^{-c_2/T^\alpha \|\phi\|_{H^s}^2}$ such that for $t \in [0, T)$ the Duhamel equation

$$v(t) = \pm \int_0^t -iS(t-t')\big[|v+z|^4(v+z)\big](t')\,dt' \tag{47}$$

has a unique solution in $X^\rho([0,T))$. The same argument proves the existence of a solution in $X^\rho((-T,0])$ on a set of the same measure. Taking $u(t) = S(t)\phi + v(t)$ we have a solution on the interval $(-T,T)$ in the class

$$H^s(\mathbb{R}^d) + C((-T,T) \to H^\rho(\mathbb{R}^3)) \subset H^s(\mathbb{R}^d). \qquad \square$$

6. A condition for global well-posedness

We now present the proof of Theorem 1.2. The proof relies upon the following proposition.

Proposition 6.1. *Suppose* $0 < c < \frac{1}{8}$, $\frac{7}{8} < s < 1$ *and* $\|\phi^\omega\|_{H^s(\mathbb{R}^3)} < R$. *There exists a small positive constant* $\epsilon \ll \frac{1}{R}$ *such that for any interval* $[t_1, t_2]$ *satisfying*

$$|t_1 - t_2| \leq 1, \quad \|v\|_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)} < \epsilon \quad \text{and} \quad \|\langle\nabla\rangle^s z\|_{L^qL^r([t_1,t_2]\times\mathbb{R}^3)} < \epsilon$$

for the pairs $(q,r) \in \{(10,10), (\frac{15}{2}, \frac{15}{7}), (\frac{30}{7}, 15)\}$, *we have*

$$\|v\|_{S^{1+c}([t_1,t_2]\times\mathbb{R}^3)} \lesssim \|v(t_1)\|_{H^{1+c}(\mathbb{R}^3)} + C(\epsilon).$$

We first give the proof of Theorem 1.2 given that Proposition 6.1 is true. The rest of the paper is devoted to proving Proposition 6.1.

Proof of Theorem 1.2. Assume Proposition 6.1 and the hypothesis of Theorem 1.2, that there exists such a function α. Fix values T, R and a set $\Omega'_{T,R}$ satisfying the properties outlined in Theorem 1.2. By Theorem 2.4 and Lemma 2.1 there is a set $\Omega_{T,R} \subset \Omega'_{T,R}$ of measure at least $1 - c_1 e^{-c_2 R^2/T\|\phi\|_{H^s}} - \alpha(T,R)$ such that for any $\omega \in \Omega_{T,R}$ and $(q,r) \in \{(\infty,2), (10,10), (\frac{15}{2}, \frac{15}{7}), (\frac{30}{7}, 15)\}$ we have

$$\|\langle\nabla\rangle^s z\|_{L^qL^r([-T,T]\times\mathbb{R}^3)} < R, \tag{48}$$

and for any solution v to (6) we have

$$\|v\|_{L^{10}L^{10}([-T,T]\times\mathbb{R}^3)} < R. \tag{49}$$

Now assume that ω is indeed in the set $\Omega_{T,R}$. Note that by the local well-posedness theory a solution exists on some short time interval $(-t,t)$. Suppose for

sake of contradiction there is a pair of times $-T < T_{\min} < 0 < T_{\max} < T$ such that the solution $v(t)$ cannot be extended in H^{1+c} past $(T_{\min}, T_{\max})$.

We know that $\|v\|_{L^{10}L^{10}((T_{\min},T_{\max})\times\mathbb{R}^3)} < R$ and we have

$$\|\langle\nabla\rangle^s z\|_{L^qL^r([-T,T]\times\mathbb{R}^3)} < R$$

for each necessary pair (q, r). Therefore, we can split $[T_{\min}, T_{\max}]$ into a finite number of subintervals I on which $\|v\|_{L^{10}L^{10}(I\times\mathbb{R}^3)} < \epsilon$ and $\|\langle\nabla\rangle^s z\|_{L^qL^r(I\times\mathbb{R}^3)} < \epsilon$ for $(q,r) \in \{(10,10), (\frac{15}{2}, \frac{15}{7}), (\frac{30}{7}, 15)\}$.

This means that on each subinterval $[t_i, t_{i+1}]$, the conditions of Proposition 6.1 are met, and therefore the $\|v\|_{S^{1+c}([t_i,t_{i+1}]\times\mathbb{R}^3)}$ norm is finite. Therefore, there exists a solution in the space $S^{1+c}([t_i, t_{i+1}]\times\mathbb{R}^3)$ on each successive interval $[t_i, t_{i+1}]$, which implies that the $\|v\|_{L^\infty H^{1+c}}$ norm is bounded at each endpoint. This means the S^{1+c} norm is bounded on the next interval. Iterating this argument over each subinterval, this implies the S^{1+c} norm of the nonlinear solution $v(t)$ is bounded on the whole interval $[T_{\min}, T_{\max}]$. In addition, $\|v(T_{\min})\|_{H^{1+c}}$ and $\|v(T_{\max})\|_{H^{1+c}}$ are both finite. Therefore, one can apply the local well-posedness theory to extend the solution beyond $[T_{\min}, T_{\max}]$, which is a contradiction. □

This concludes the proof of Theorem 1.2. It remains to prove Proposition 6.1.

Proof of Proposition 6.1. The nonlinear part of the solution v satisfies the differential equation

$$iv_t + \Delta v = (v+z)|v+z|^4 = v|v|^4 + f(v,z) \tag{50}$$

for the function $f(v,z) = (v+z)|v+z|^4 - v|v|^4 \lesssim |z|^5 + |z|\cdot|v|^4$.

By the Strichartz estimates (21), we have the bound

$$\begin{aligned}\|v\|_{S^{1+c}([t_1,t_2]\times\mathbb{R}^3)} &\\ \lesssim \|v(t_1)\|_{H^{1+c}} + \big\||v|v|^4\big\|_{N^{1+c}([t_1,t_2]\times\mathbb{R}^3)} &+ \|f(v,z)\|_{N^{1+c}([t_1,t_2]\times\mathbb{R}^3)}.\end{aligned} \tag{51}$$

So we need to bound the two remaining terms.

Lemma 6.2. *If v is a solution to (50), then*

$$\big\||v|v|^4\big\|_{N^{1+c}([t_1,t_2]\times\mathbb{R}^3)} \lesssim \epsilon^4\|v\|_{S^{1+c}([t_1,t_2]\times\mathbb{R}^3)}.$$

Proof. Note that the pair $(\frac{10}{3}, \frac{10}{3})$ is Schrödinger-admissible and has Hölder conjugate $(\frac{10}{7}, \frac{10}{7})$. Therefore, by (21) we have

$$\begin{aligned}\big\|v\cdot|v|^4\big\|_{N^{1+c}([t_1,t_2]\times\mathbb{R}^3)} &\lesssim \big\|\langle\nabla\rangle^{1+c} v\cdot|v|^4\big\|_{L^{10/7}L^{10/7}([t_1,t_2]\times\mathbb{R}^3)}\\ &\lesssim \|\langle\nabla\rangle^{1+c} v\|_{L^{10/3}L^{10/3}([t_1,t_2]\times\mathbb{R}^3)}\|v\|^4_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)}\\ &\lesssim \|v\|_{S^{1+c}([t_1,t_2]\times\mathbb{R}^3)}\|v\|^4_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)}\\ &\lesssim \epsilon^4\|v\|_{S^{1+c}([t_1,t_2]\times\mathbb{R}^3)}.\end{aligned}$$

□

Proposition 6.3. *Assume* $0 < c < \frac{1}{8}$, $f(v,z) = (v+z)|v+z|^4 - v|v|^4$ *and that* z *and* v *satisfy the* R *and* ϵ *bounds in the proposition, where* z *is the linear solution and* v *is the solution to (6). Then we have*

$$\begin{aligned}\|f\|_{N^{1+c}([t_1,t_2]\times\mathbb{R}^3)} &\lesssim \|\langle\nabla\rangle^{1+c} f\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)} \\ &\lesssim \sqrt{\epsilon^7 R}\big(\sqrt{\epsilon R} + \|v_1(t_1)\|_{H^1(\mathbb{R}^3)} \\ &\qquad + \|\langle\nabla\rangle u^5\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)}\big). \quad (52)\end{aligned}$$

Proof. Observing that $\big(3, \frac{18}{5}\big)$ is Schrödinger-admissible, we have

$$\begin{aligned}\|f\|_{N^{1+c}([t_1,t_2]\times\mathbb{R}^3)} &\leq \|f\|_{L^{3/2}W^{1+c,18/13}([t_1,t_2]\times\mathbb{R}^3)} \\ &\lesssim \sup_{\|w_6\|_{L^3L^{18/5}([t_1,t_2]\times\mathbb{R}^3)}\leq 1} \int_{t_1}^{t_2}\int_x \langle\nabla\rangle^{1+c}[f]w_6\,dw. \quad (53)\end{aligned}$$

The function $f(v,z)$ is a sum of terms of the form $w_1w_2w_3w_4z_5$, where each w_i is either a v or z term. We dyadically decompose these first five terms (not w_6), refer to $P_{N_i}w_i$ as w_i, and sum over all frequencies N_1–N_5 and combinations of v, z in integrals of the form

$$\begin{aligned}&\|\langle\nabla\rangle^{1+c} f\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)} \\ &\qquad\lesssim \sup_{\|w_6\|_{L^3L^{18/5}([t_1,t_2]\times\mathbb{R}^3)}\leq 1} \int_{t_1}^{t_2}\int_x \langle\nabla\rangle^{1+c}[w_1w_2w_3w_4v_5]w_6\,dw. \quad (54)\end{aligned}$$

We can assume that the $1+c$ derivatives fall on the term with highest frequency. Before going through cases, we prove the following lemmas that combine interpolation with the bilinear estimate, Theorem 4.6.

Lemma 6.4. *If* $N_1 \leq N_2$, *then for any pair of dyadic components* $v_1 = P_{N_1}v$, $z_2 = P_{N_2}z$ *we have the bound*

$$\begin{aligned}&\|v_1z_5\|_{L^{30/11}L^{15/8}([t_1,t_2]\times\mathbb{R}^3)} \\ &\quad\lesssim N_2^{-1/4-s+\delta/2}\|v_1\|^{1/2}_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)} \\ &\qquad\times\big(\|v_1(t_1)\|_{H^1(\mathbb{R}^3)} + \|\langle\nabla\rangle u^5\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)}\big)^{1/2}\sqrt{\epsilon R}. \quad (55)\end{aligned}$$

Proof. First note that for $N_1 \leq N_2$ we have the bilinear estimate Theorem 4.6:

$$\begin{aligned}&\|v_1z_2\|_{L^2L^2([t_1,t_2]\times\mathbb{R}^3)} \\ &\quad\lesssim N_2^{-1/2+\delta}N_1^{1-\delta}\big(\|v_1(t_1)\|_{L^2(\mathbb{R}^3)} + \|u^5\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)}\big)\|z(t_1)\|_{L^2(\mathbb{R}^3)},\end{aligned}$$

$$\|v_1 z_2\|_{L^2L^2([t_1,t_2]\times\mathbb{R}^3)} \lesssim N_2^{-1/2-s+\delta}\big(\|v_1(t_1)\|_{H^1(\mathbb{R}^3)} + \|\langle\nabla\rangle u^5\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)}\big)R. \quad (56)$$

Also, by Hölder's inequality

$$\|v_1 z_2\|_{L^{30/7}L^{30/17}([t_1,t_2]\times\mathbb{R}^3)} \le \|v_1\|_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)}\|z_2\|_{L^{30/4}L^{30/14}([t_1,t_2]\times\mathbb{R}^3)} \le N_2^{-s}\|v_1\|_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)}\epsilon. \quad (57)$$

Now note that

$$\frac{1}{30/11} = \frac{1/2}{2} + \frac{1/2}{30/7}, \qquad \frac{1}{15/8} = \frac{1/2}{2} + \frac{1/2}{30/17}. \quad (58)$$

Interpolating with exponents $\frac{1}{2}, \frac{1}{2}$ yields

$$\|v_1 z_5\|_{L^{30/11}L^{15/8}([t_1,t_2]\times\mathbb{R}^3)} \lesssim N_2^{-1/4-s+\delta/2}\|v_1\|^{1/2}_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)} \times \big(\|v_1(t_1)\|_{H^1(\mathbb{R}^3)} + \|\langle\nabla\rangle u^5\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)}\big)^{1/2}\sqrt{\epsilon R}. \quad (59)$$

This completes the proof. □

Lemma 6.5. *If $N_2 \le N_1$, then for any pair of dyadic components $v_1 = P_{N_1}v$, $z_2 = P_{n_2}z$ we have the bound*

$$\|v_1 z_2\|_{L^{30/11}L^{15/8}([t_1,t_2]\times\mathbb{R}^3)} \lesssim N_1^{-3/4+\delta/2}N_2^{1/2-s} \times \big(\|v_1(t_1)\|_{H^1(\mathbb{R}^3)} + \|\langle\nabla\rangle u^5\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)}\big)\sqrt{\epsilon R}. \quad (60)$$

Proof. The bilinear estimate Theorem 4.6 tells us that

$$\|v_1 z_2\|_{L^2L^2([t_1,t_2]\times\mathbb{R}^3)} \lesssim N_1^{-1/2+\delta}N_2^{1-\delta} \times \big(\|v_1(t_1)\|_{L^2(\mathbb{R}^3)} + \|\langle\nabla\rangle u^5\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)}\big)\|z(t_1)\|_{L^2(\mathbb{R}^3)}$$

$$\|v_1 z_2\|_{L^2L^2([t_1,t_2]\times\mathbb{R}^3)} \le N_1^{-3/2+\delta}N_2^{1-s}\big(\|v_1\|_{H^1(\mathbb{R}^3)} + \|\langle\nabla\rangle u^5\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)}\big)R. \quad (61)$$

Also, by Hölder's inequality we have

$$\|v_1 z_2\|_{L^{30/7}L^{30/17}([t_1,t_2]\times\mathbb{R}^3)} \lesssim \|v_1\|_{L^\infty L^2([t_1,t_2]\times\mathbb{R}^3)}\|z_2\|_{L^{30/7}L^{15}([t_1,t_2]\times\mathbb{R}^3)} \lesssim N_1^{-1}N_2^{-s}\|v_1\|_{S^1([t_1,t_2]\times\mathbb{R}^3)}\epsilon. \quad (62)$$

So with exponents $\frac{1}{2}, \frac{1}{2}$ we interpolate between the L^2L^2 and $L^{30/7}L^{30/17}$ bounds, and apply the Strichartz estimate to get

$$\begin{aligned}\|v_1 z_2\|_{L^{30/11}L^{15/8}([t_1,t_2]\times\mathbb{R}^3)} &\lesssim N_1^{-3/4+\delta/2} N_2^{1/2-s} \\ &\quad\times \left(\|v_1(t_1)\|_{H^1(\mathbb{R}^3)} + \|\langle\nabla\rangle u^5\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)}\right)\sqrt{\epsilon R}, \quad (63)\end{aligned}$$

completing the proof. □

Lemma 6.6. *If $N_1 \le N_2$ then for $z_1 = P_{N_1} z$, $z_2 = P_{n_2} z$ we have*

$$\|z_1 z_2\|_{L^{30/11}L^{15/8}([t_1,t_2]\times\mathbb{R}^3)} \le N_2^{-1/4-s+\delta/2} N_1^{1/2-s} R\epsilon. \quad (64)$$

Proof. The proof is identical to that of Lemma 6.4 except that v_1 has been replaced with z_1, which is put in an $L^{10}L^{10}$ norm. □

In analyzing terms of the form $w_1 w_2 w_3 w_4 v_5$, there are two cases for where the highest frequencies occur:

- Case 1: The highest frequency is on a z term.
- Case 2: The highest frequency is on a v term.

Throughout these cases we utilize the facts that $\|v\|_{L^qL^r} \lesssim \|v\|_{S^1}$ for $\frac{2}{q} + \frac{3}{r} = \frac{1}{2}$ and $\|v\|_{L^qL^r} \lesssim \|v\|_{S^0}$ for (q, r) Schrödinger admissible. We also use the above three lemmas. Now we begin the analysis of cases.

Case 1: In this case the highest frequency is on z_5. We have all the derivatives falling on z_5.

(1a) $v_1 w_2 w_3 w_4 z_5$ case.

Applying Hölder's inequality, Lemma 6.4 and our assumptions about ϵ we have

$$\begin{aligned} I &= \int_{t_1}^{t_2}\int_x v_1 w_2 w_3 w_4 \langle\nabla\rangle^{1+c} z_5 w_6 \, dw \\ &\le N_5^{1+c} \|w_2\|_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)} \|w_3\|_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)} \|w_4\|_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)} \\ &\qquad\times \|v_1 z_5\|_{L^{30/11}L^{15/8}([t_1,t_2]\times\mathbb{R}^3)} \|w_6\|_{L^3L^{18/5}} \\ &\le N_5^{3/4+c-s+\delta/2} \|w_2\|_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)} \|w_3\|_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)} \\ &\qquad\times \|w_4\|_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)} \|v_1\|^{1/2}_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)} \\ &\qquad\times \left(\|v_1(t_1)\|_{H^1(\mathbb{R}^3)} + \|\langle\nabla\rangle u^5\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)}\right)^{1/2} \\ &\qquad\times \sqrt{\epsilon R} \|w_6\|_{L^3L^{18/5}}. \quad (65)\end{aligned}$$

For $c < \frac{1}{8}$, $s > \frac{7}{8}$ and $\delta = 0+$, the power of N_5 is negative, and the sum converges. The w_i terms are all bounded by $\|v_i\|_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)}$ or ϵ. So this is bounded by $\epsilon^4 R^{1/2}\big(\|v_1(t_1)\|_{H^1(\mathbb{R}^3)} + \|\langle\nabla\rangle u^5\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)}\big)^{1/2}$.

(1b) $z_1z_2z_3z_4z_5$ case.

Applying Hölder's inequality, Lemma 6.6, and our ϵ bounds we have

$$\begin{aligned} I &= \int_{t_1}^{t_2}\int_x z_1z_2z_3z_4\langle\nabla\rangle^{1+c} z_5 w_6\,dw \\ &\le N_5^{1+c}\|z_2\|_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)}\|z_3\|_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)} \\ &\qquad\times\|z_4\|_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)}\|z_1z_5\|_{L^{30/11}L^{15/8}([t_1,t_2]\times\mathbb{R}^3)}\|w_6\|_{L^3L^{18/5}} \\ &\le N_1^{1/2-s}N_5^{3/4-s+c+\delta}\|z_2\|_{L^{10}L^{10}}\|z_3\|_{L^{10}L^{10}} \\ &\qquad\times\|z_4\|_{L^{10}L^{10}}\epsilon R\|w_6\|_{L^3L^{18/5}}. \end{aligned} \tag{66}$$

For $s > \frac{7}{8}$ and $c < \frac{1}{8}$ and $\delta = 0+$, both powers are negative and this is bounded by $\epsilon^4 R$.

Case 2: In this case the highest frequency falls on v, meaning $N_1 \ge N_2, \dots, N_5$. Applying Hölder's inequality, Lemma 6.5, and our ϵ bounds, for $N_5 \le N_1$ we have

$$\begin{aligned} I &= \int_{t_1}^{t_2}\int_x \langle\nabla\rangle^{1+c} v_1w_2w_3w_4z_5w_6\,dw \\ &\le N_1^{1+c}\|w_2\|_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)}\|w_3\|_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)} \\ &\qquad\times\|w_4\|_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)}\|v_1z_5\|_{L^{30/11}L^{15/8}([t_1,t_2]\times\mathbb{R}^3)}\|w_6\|_{L^3L^{18/5}} \\ &\le N_1^{-1/4+c+\delta/2}N_5^{1/2-s}\|w_2\|_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)}\|w_3\|_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)} \\ &\qquad\times\|w_4\|_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)}\big(\|v_1(t_1)\|_{H^1(\mathbb{R}^3)} + \|\langle\nabla\rangle u^5\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)}\big) \\ &\qquad\times\sqrt{\epsilon R}\|w_6\|_{L^3L^{18/5}}. \end{aligned} \tag{67}$$

As in case (1a), the $\|w_i\|_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)}$ terms are all bounded by either ϵ or $\|v\|_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)}$. Thus, for $c < \frac{1}{8}$ the sum over frequencies is bounded by

$$\sqrt{\epsilon^7 R}\big(\|v_1(t_1)\|_{H^1(\mathbb{R}^3)} + \|\langle\nabla\rangle u^5\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)}\big).$$

So in all cases the integral is bounded by

$$\sqrt{\epsilon^7 R}\big(\sqrt{\epsilon R} + \|v_1(t_1)\|_{H^1(\mathbb{R}^3)} + \|\langle\nabla\rangle u^5\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)}\big).$$

This completes the proof of Proposition 6.3. □

So combining Lemma 6.2, Proposition 6.3 and the fact that $\epsilon \ll 1$ we arrive at the following pair of inequalities:

$$\begin{aligned}
&\|v\|_{S^{1+c}([t_1,t_2]\times\mathbb{R}^3)}\\
&\qquad\lesssim \|v(t_1)\|_{H^{1+c}} + \|\langle\nabla\rangle^{1+c} f\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)},\\
&\|\langle\nabla\rangle^{1+c} f\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)}\\
&\qquad\lesssim \sqrt{\epsilon^7 R}\big(\sqrt{\epsilon R} + \|v_1(t_1)\|_{H^1(\mathbb{R}^3)} + \|\langle\nabla\rangle u^5\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)}\big).
\end{aligned}$$

So all that remains is to bound

$$\begin{aligned}
&\|\langle\nabla\rangle u^5\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)}\\
&\qquad\le \|\langle\nabla\rangle v^5\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)} + \|\langle\nabla\rangle f\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)}. \qquad (68)
\end{aligned}$$

First observe that

$$\begin{aligned}
&\|\langle\nabla\rangle v^5\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)}\\
&\qquad\le \|\langle\nabla\rangle v\cdot v^4\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)}\\
&\qquad\le \|\langle\nabla\rangle v\|_{L^{15/4}L^{90/29}([t_1,t_2]\times\mathbb{R}^3)}\|v\|^4_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)}\\
&\qquad\le \|v\|_{S^1([t_1,t_2]\times\mathbb{R}^3)}\|v\|^4_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)}\\
&\qquad\lesssim \|\langle\nabla\rangle v^5\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)}\|v\|^4_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)}\\
&\qquad\qquad + \big(\|v(t_1)\|_{H^1(\mathbb{R}^3)} + \|f\|_{N^1([t_1,t_2]\times\mathbb{R}^3)}\big)\|v\|^4_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)}
\end{aligned}$$

and for $\|v\|_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)}$ less than ϵ we have

$$\|\langle\nabla\rangle v^5\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)} \lesssim \epsilon^4\big(\|v(t_1)\|_{H^1(\mathbb{R}^3)} + \|f\|_{N^1([t_1,t_2]\times\mathbb{R}^3)}\big). \qquad (69)$$

Noting that $\|v\|_{L^{10}L^{10}([t_1,t_2]\times\mathbb{R}^3)}$ is small and combining Proposition 6.3, (68) and (69), we have

$$\begin{aligned}
&\|\langle\nabla\rangle^{1+c} f\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)}\\
&\quad\lesssim \sqrt{\epsilon^7 R}\big(\sqrt{\epsilon R} + \|v(t_1)\|_{H^1(\mathbb{R}^3)} + \|\langle\nabla\rangle f\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)}\big)\\
&\quad\lesssim \sqrt{\epsilon^7 R}\big(\sqrt{\epsilon R} + \|v(t_1)\|_{H^1(\mathbb{R}^3)} + \|\langle\nabla\rangle^{1+c} f\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)}\big). \qquad (70)
\end{aligned}$$

For $\epsilon \ll \frac{1}{R}$ this implies that

$$\|\langle\nabla\rangle^{1+c} f\|_{L^{3/2}L^{18/13}([t_1,t_2]\times\mathbb{R}^3)} \lesssim \sqrt{\epsilon^7 R}\big(\sqrt{\epsilon R} + \|v(t_1)\|_{H^1(\mathbb{R}^3)}\big).$$

This gives us the necessary bound on f.

Combining this result with Lemma 6.2, we have

$$\|v\|_{S^{1+c}([t_1,t_2]\times\mathbb{R}^3)} \lesssim \|v(t_1)\|_{H^{1+c}(\mathbb{R}^3)} + C(\epsilon) \qquad (71)$$

for sufficiently small $\epsilon \ll \frac{1}{R}$, which completes the proof of Proposition 6.1. □

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