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# $L_2$-THEORY FOR TWO INCOMPRESSIBLE FLUIDS SEPARATED BY A FREE INTERFACE

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*Dedicated to the memory of Professor Marek Burnat*

Abstract. The paper is devoted to the problem of non-stationary motion of two viscous incompressible fluids separated by a free surface and contained in a bounded vessel. It is assumed that the fluids are subject to mass forces and capillary forces at the interface. We prove the stability of a rest state under the assumption that initial velocities are small, a free interface is close to a sphere at an initial instant of time, and mass forces decay as $t \to \infty$.

## 1. Introduction

The paper deals with unsteady motion of a two-phase fluid in a container. Both phases are assumed to be viscous and incompressible; they are immiscible and separated by an unknown closed interface on which the surface tension is taken into account. The motion of a drop in a liquid medium is governed by the Navier–Stokes system including mass forces, initial and boundary conditions and, in addition, by the initial configuration of the drop.

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The first results concerning non-stationary two fluids motion with free interface were obtained in the 90s of the last century. In the case of the whole space, existence and uniqueness theorem for the problem with and without capillary forces in $L_2$-setting was proved in a finite time interval whose magnitude was determined by the norms of the data [2], [4], [6]. This result was obtained in several steps by considering model linear problems [3], [8]. Giga and Takahashi [11], [24] demonstrated the existence of global weak solutions for the Stokes and Navier–Stokes equations governing the motion of two immiscible fluids without including surface tension into consideration.

During the last years, researchers have been studying the problem on a two-phase liquid flow in the presence of the surface tension in different functional spaces and indicating various aspects of the problem. In particular, H. Abels [1] estimated the Hausdorff measure of the interface leaving open the existence of generalized solutions. Next, Shibata and Shimizu investigated the problem by operator methods in the anisotropic Sobolev spaces $W^{2,1}_{q,p}(\Omega^{\pm})$, $2 \leq n < q < \infty$, $2 < p < \infty$, $\Omega^{\pm} \subset \mathbb{R}^n$. They proved the solvability of the model diffraction problems for the Stokes system [16]. The same result for nonlinear interface problem was obtained in [17] under the assumption that the initial interface was given by the equation $x_n = \alpha(x')$, $x' \in \mathbb{R}^{n-1}$. Much attention has been paid to the problem of evolution of two fluids in a container, specifically, to the problem of the stability of a rest state (velocity vector field $\boldsymbol{v} = 0$, the pressure $p$ is constant in each fluid, the interface is a sphere with arbitrary center bounded away from the walls of the container). It was shown independently by the authors in [9] and in the series of papers of J. Prüss with collaborators (in particular, in [15], [12]), that the state is exponentially stable in the following sense: for arbitrary initial data close to an equilibrium, the problem has a unique solution defined for $t > 0$ that tends exponentially to a rest state which is different, in general, from the initial one. The proof was based on coercive (i.e. maximal regularity) estimates for the solution of a linearized problem. In all of the above mentioned papers, the interface problem was reduced to a non-linear system in two fixed domains by using the Hanzawa transformation but the arguments were quite different. It should be noticed that in [9] the problem was studied in the anisotropic Hölder spaces, while in [15], [12] the basic space was $W^{2,1}_p$, $p > n+3$. In addition, the existence of a global solution to the problem was also obtained in the Sobolev spaces for $p > n = 3$ in [23].

In [7], the global solvability of the problem was proved in the case of non-zero mass forces exponentially decaying as $t \to \infty$. We mention also papers [5], [6], where the case of the zero surface tension was considered.

In the present paper, the problem is treated in the Sobolev–Slobodetskiĭ spaces $W^{2+l,1+l/2}_2$, $l \in (1/2, 1)$, in the three-dimensional case. We concentrate on the proof of the stability of a rest state and construct a solution assuming that the initial data of the problem are close to this state, i.e. the velocities and mass forces are small, and the interface is close to the sphere $S_{R_0}$ of the radius $R_0$ such that the ball bounded by this sphere has the same volume as the inner fluid. We place the center of this ball to the origin which coincides with the barycenter of the drop at an initial instant, the interface being defined as a normal perturbation of $S_{R_0}(0)$. We find it reasonable to consider also the unknown interface at the time instant $t > 0$ as a normal perturbation of the sphere $S_{R_0}(h)$ of the same radius $R_0$ but with the center placed at the barycenter $h(t)$ of an inner domain. Therefore, as in our previous papers [9], [10], we introduce a term with the vector $\boldsymbol{h}(t)$ into the standard Hanzawa transformation of the two-phase domain with unknown interface into a domain with the interface $S_{R_0}(0)$. In our opinion, this term permits to take interface evolution into account in a more precise way. Next, we linearize the transformed problem. In Section 2, we study a linear problem in two domains separated by $S_{R_0}$ and prove maximal regularity estimates for a solution of the problem first on an arbitrary finite time interval in the standard spaces and then, under some additional assumptions, on the infinite interval $t > 0$ in the spaces with the exponential weight $e^{\beta t}$, $\beta > 0$. In Section 3, on the basis of these estimates and of the estimates of nonlinear terms, we construct a solution at first for $t \in (0, T_0)$ with an appropriate $T_0 > 1$, then we extend this solution with respect to $t$ into the interval $(T_0, 2T_0)$ and so forth step by step for any $t > 0$. We show that the velocities and the pressure gradient decay exponentially to zero as $t \to \infty$, and $\Gamma_t$ tends to a sphere of radius $R_0$ centered at $h(\infty)$ close to $S_{R_0}(0)$ but, in general, different from $S_{R_0}(0)$.

Moreover, we admit here a more general decay of the vector field of mass forces. The proofs are constructed in the same manner as in [20], [9], [23] but the finial estimate of a solution (see Theorem 1.1) is somewhat different from those in the preceding papers. As before, the idea of constructing a function of generalized energy [13], [19] is used for obtaining the exponential estimate instead of an analysis of the spectrum of the linear problem. It is worth noting that our technique can be generalized to the case of a multi-phase fluid and that of a dimension $n > 3$.

We pass to the statement of the problem.

Let two viscous incompressible immiscible fluids be contained in a bounded vessel $\Omega \subset \mathbb{R}^3$ and separated by a variable interface $\Gamma_t$ that is bounded away from the wall of the container $\Sigma = \partial\Omega$. It is assumed that $\Gamma_t$ is the boundary of the domain $\Omega_t^+$ filled with the fluid with the density $\rho^+$ and the dynamical viscosity $\mu^+$ that is surrounded by the other fluid with the density $\rho^-$ and the viscosity $\mu^-$ occupying the domain $\Omega_t^- = \Omega \setminus \overline{\Omega_t^+}$. It is necessary to find $\Gamma_t$, as well as the velocity vector fields $\boldsymbol{v}(x,t)$ and the pressure functions $p(x,t)$, $x \in \Omega_t^- \cup \Omega_t^+$, of both fluids satisfying the interface problem for the Navier–Stokes equations

$$
(1.1)\qquad \begin{cases} \mathcal{D}_t\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nu^{\pm}\nabla^2\boldsymbol{v} + \dfrac{1}{\rho^{\pm}}\nabla p = \boldsymbol{f}, \quad \nabla\cdot\boldsymbol{v} = 0 \quad \text{in } \Omega_t^{\pm},\ t>0, \\ \boldsymbol{v}(x,0) = \boldsymbol{v}_0(x) \quad \text{in } \Omega_0^{\pm}, \quad [\boldsymbol{v}]\big|_{\Gamma_t} \equiv \lim\limits_{\substack{x\to x_0\in\Gamma_t \\ x\in\Omega_t^+}} \boldsymbol{v}(x) - \lim\limits_{\substack{x\to x_0\in\Gamma_t \\ x\in\Omega_t^-}} \boldsymbol{v}(x) = 0, \\ [\mathbb{T}(\boldsymbol{v},p)\boldsymbol{n}]|_{\Gamma_t} = \sigma\bigg(H + \dfrac{2}{R_0}\bigg)\boldsymbol{n}, \quad \boldsymbol{v}|_{\Sigma} = 0, \quad V_n = \boldsymbol{v}\cdot\boldsymbol{n}, \end{cases}
$$

where $\mathcal{D}_t = \partial/\partial t$, $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$, $\nu^{\pm} = \mu^{\pm}/\rho^{\pm}$ is the step function of the kinematic viscosity, $\boldsymbol{v}_0$ is the initial distribution of the velocity, $\boldsymbol{f}$ is the vector field of mass forces given in $\Omega\times(0,\infty)$, $\mathbb{T}(\boldsymbol{v},p) = -p + \mu^{\pm}\mathbb{S}(\boldsymbol{v})$ is the stress tensor, $\mathbb{S}(\boldsymbol{v}) = (\nabla\boldsymbol{v}) + (\nabla\boldsymbol{v})^T$ is the doubled rate-of-strain tensor, $H$ is twice the mean curvature of $\Gamma_t (H<0$ at the points where $\Gamma_t$ is convex toward $\Omega_t^-)$, $\sigma > 0$ is the coefficient of surface tension, $\boldsymbol{n}(x,t)$ is the normal to $\Gamma_t$, exterior with respect to $\Omega_t^+$, $[\boldsymbol{v}]|_{\Gamma_t}$ is the jump of $\boldsymbol{v}$ across $\Gamma_t$, $V_n$ is the velocity of the evolution of $\Gamma_t$ in the direction $\boldsymbol{n}$, $R_0 = (3|\Omega_0^+|/4\pi)^{1/3}$, $|\Omega_0^+| = \operatorname{mes}\Omega_0^+$. We suppose that a Cartesian coordinate system $\{x\}$ is introduced in $\mathbb{R}^3$. The centered dot denotes the Cartesian scalar product.

Summation is implied over repeated indices from 1 to 3 if they are denoted by Latin letters. We mark the vectors and the vector spaces by boldface letters.

It is assumed that the surface $\Gamma_0$ is close to the sphere $S_{R_0}$ of radius $R_0$ the center of which coincides with the center of gravity of $\Omega_0^+$. Without restriction of generality, we suppose that it is placed at the origin. Then $\Gamma_0$ can be viewed as a normal perturbation of $S_{R_0}$, i.e.

$$
\Gamma_0 = \{x\in\mathbb{R}^3 \mid x = y + r_0(y)\boldsymbol{N}(y)\}, \quad y\in S_{R_0},
$$

where $\boldsymbol{N}(y) = \boldsymbol{y}/|y|$, $y\in S_{R_0}$, and $r_0$ is a given small function. We will use a similar representation formula for the unknown surface $\Gamma_t$, $t>0$:

$$
\Gamma_t = \{x\in\mathbb{R}^3 \mid x = y + \boldsymbol{h}(t) + r(y,t)\boldsymbol{N}(y)\},
$$

where $r(y,t)$ is an unknown function on $S_{R_0}$. The coordinates of the barycenter of $\Omega_t^+$ are given by

$$
(1.2)\qquad h_i(t) = \frac{1}{|\Omega_t^+|}\int_{\Omega_t^+} x_i\,dx = \frac{1}{|\Omega_t^+|}\int_0^t\int_{\Omega_t^+} v_i(x,\tau)\,dx\,d\tau, \quad i=1,2,3.
$$

We extend $\boldsymbol{N}$ in $\mathbb{R}^3$ by the formula $\boldsymbol{N}^*(y) = \omega(y)\boldsymbol{y}/|y|$, where $\omega(y)$ is a smooth function equal to 1 for $|y|\ge 2R_0/3$ and to zero for $|y|\le R_0/3$. For $r$, we introduce the extension $r^*(y,t) = \mathcal{E}r(y,t)\Phi(y)$, where $\Phi(y)$ is a smooth cut-off function equal to 1 in the neighbourhood of $S_{R_0}$ and zero near $\Sigma$, while $\mathcal{E}$ is a fixed extension operator from $S_{R_0}$ into $\mathbb{R}^3$. We also require that

$$\frac{\partial r^*}{\partial N}\bigg|_{S_{R_0}} = 0,$$

$$\|r^*\|_{W_2^{l'+1/2}(\mathbb{R}^3)} \le c\|r\|_{W_2^{l'}(S_{R_0})}, \quad l' \in (0, 2+l],$$

and $r^*(y,t) = 0$ for $\big||y| - R_0\big| \ge d_0$, in particular, for $y$ close to $\Sigma$, $d_0$ is a small positive number ($W_2^m$ is a Sobolev–Slobodetskiĭ space the definition of which will be given below). It follows that

$$\|\mathcal{D}_t r^*\|_{W_2^{l'+1/2}(\mathbb{R}^3)} \le c\|\mathcal{D}_t r\|_{W_2^{l'}(S_{R_0})}, \quad l' \in (0, 2+l]. \tag{1.3}$$

We define the modified Hanzawa transformation

$$x = y + r^*(y,t)\boldsymbol{N}^*(y) + \chi(y)\boldsymbol{h}(t) \equiv e_{r,\boldsymbol{h}}(y,t), \tag{1.4}$$

where $\chi(y)$ is a smooth cutoff function, equal to one for $\big||y| - R_0\big| \le d_0/2$ and to zero for $\big||y| - R_0\big| \ge d_0$. If $r$ and $\boldsymbol{h}(t)$ are sufficiently small and $d_0$ is chosen in a proper way, then this mapping is invertible and it establishes one-to-one correspondences between the ball $B^+ \equiv \{|y| < R_0\}$ and $\Omega_t^+$, $S_{R_0}$ and $\Gamma_t$, $B^- \equiv \Omega \setminus \overline{B^+}$ and $\Omega_t^-$ (this is obvious for $t = 0$ when $\boldsymbol{h}(0) = 0$, and it remains true for small $\boldsymbol{h}(t)$).

We denote by $\mathbb{L}$ the Jacobi matrix of transformation (1.4), and we set $L = \det \mathbb{L}$, $\widehat{\mathbb{L}} = L\mathbb{L}^{-1}$. Clearly,

$$\mathbb{L}(r,\boldsymbol{h}) = \left\{\delta_j^i + \frac{\partial\big(r^*(y,t)N_i^*(y)\big)}{\partial y_j} + h_i(t)\frac{\partial\chi(y)}{\partial y_j}\right\}_{i,j=1}^3.$$

For $y$ located on $S_{R_0}$, we have $\nabla\chi = 0$ and

$$\mathbb{L} = \mathbb{L}(r,\boldsymbol{0}) = \left\{\delta_j^i + \frac{\partial\big(r(y,t)N_i(y)\big)}{\partial y_j}\right\}_{i,j=1}^3.$$

Mapping (1.4) converts (1.1) into

$$\begin{cases} \mathcal{D}_t\boldsymbol{u} - \nu^\pm\widetilde{\nabla}^2\boldsymbol{u} - (\mathbb{L}^{-1}(\mathcal{D}_t r^*\boldsymbol{N}^* + \chi\dot{\boldsymbol{h}})\cdot\nabla)\boldsymbol{u} & \\ \qquad\qquad +(\mathbb{L}^{-1}\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \frac{1}{\rho^\pm}\widetilde{\nabla}q = \widehat{\boldsymbol{f}}, \quad \widetilde{\nabla}\cdot\boldsymbol{u} = 0 & \text{in } B^\pm,\ t>0, \\ \boldsymbol{u}(y,0) = \boldsymbol{u}_0(y) & \text{in } B^\pm, \\ r(y,0) = r_0(y) & \text{on } S_{R_0}, \\ [\boldsymbol{u}]\big|_{S_{R_0}} = 0, \quad [\mu^\pm\Pi\widetilde{\mathbb{S}}(\boldsymbol{u})\boldsymbol{n}]\big|_{S_{R_0}} = 0, \quad \boldsymbol{u}|_\Sigma = 0, & \\ [-q + \mu^\pm\boldsymbol{n}\cdot\widetilde{\mathbb{S}}(\boldsymbol{u})\boldsymbol{n}]\big|_{S_{R_0}} = \sigma\left(H\big(e_{r,\boldsymbol{0}}(y,t),t\big) + \dfrac{2}{R_0}\right), & \\ \displaystyle\int_\Omega q(y,t)\,dy = 0, & \\ \mathcal{D}_t r - \left(\boldsymbol{u} - \dfrac{1}{|B^+|}\displaystyle\int_{B^+}\boldsymbol{u}L(r,\boldsymbol{0})\,dy\right)\cdot\dfrac{\boldsymbol{n}}{\boldsymbol{N}\cdot\boldsymbol{n}} = 0 & \text{on } S_{R_0}, \end{cases} \tag{1.5}$$

where

- $\boldsymbol{u} = \boldsymbol{v}(e_{r,\boldsymbol{h}}, t)$, $q = p(e_{r,\boldsymbol{h}}, t)$,
- $\widetilde{\nabla} = \mathbb{L}^{-T}\nabla$ is the transformed gradient $\nabla_x$ ("$T$" means transposition),
- $\widetilde{\mathbb{S}}(\boldsymbol{u}) = \widetilde{\nabla}\boldsymbol{u} + (\widetilde{\nabla}\boldsymbol{u})^T$ is the transformed doubled rate-of-strain tensor,
- $\widehat{\boldsymbol{f}}(y,t) = \boldsymbol{f}(e_{r,\boldsymbol{h}}(y,t), t)$,
- $\boldsymbol{u}_0(y) = \boldsymbol{v}_0(e_{r_0,\boldsymbol{0}}(y))$,
- $\Pi\boldsymbol{g} = \boldsymbol{g} - \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{g})$.

The equation for $r$ on $S_{R_0}$ arises from the condition $V_n = \boldsymbol{v}\cdot\boldsymbol{n}$ on the interface in view of (1.4) and (1.2), since $V_n \equiv \mathcal{D}_t\boldsymbol{x}\cdot\boldsymbol{n} = \mathcal{D}_t r(\boldsymbol{N}\cdot\boldsymbol{n}) + \dot{\boldsymbol{h}}\cdot\boldsymbol{n}$, $\dot{\boldsymbol{h}} \equiv d\boldsymbol{h}/\,dt$. System (1.5) can be written in the form

$$\begin{cases}
\mathcal{D}_t\boldsymbol{u} - \nu^{\pm}\nabla^2\boldsymbol{u} + \dfrac{1}{\rho^{\pm}}\nabla q = \boldsymbol{l}_1(\boldsymbol{u},q,r) + \widehat{\boldsymbol{f}}, \quad \nabla\cdot\boldsymbol{u} = l_2(\boldsymbol{u},r) & \text{in } B^{\pm},\ t>0,\\
\boldsymbol{u}(y,0) = \boldsymbol{u}_0(y) & \text{in } B^{\pm},\\
r(y,0) = r_0(y) & \text{on } S_{R_0},\\
[\boldsymbol{u}]\big|_{S_{R_0}} = 0, \quad [\mu^{\pm}\Pi_0\mathbb{S}(\boldsymbol{u})\boldsymbol{N}]\big|_{S_{R_0}} = \boldsymbol{l}_3(\boldsymbol{u},r), \quad \boldsymbol{u}|_{\Sigma} = 0, &\\
[-q + \mu^{\pm}\boldsymbol{N}\cdot\mathbb{S}(\boldsymbol{u})\boldsymbol{N}]\big|_{S_{R_0}} - \sigma\mathcal{B}_0 r = l_4(\boldsymbol{u},r) + \sigma l_5(r), &\\
\mathcal{D}_t r - \left(\boldsymbol{u} - \dfrac{1}{|B^+|}\displaystyle\int_{B^+}\boldsymbol{u}\,dy\right)\cdot\boldsymbol{N} = l_6(\boldsymbol{u},r) & \text{on } S_{R_0},\\
\displaystyle\int_{\Omega} q(y,t)\,dy = 0, &
\end{cases}$$

where

$$\begin{aligned}
\mathcal{B}_0 r &= \Delta_{S_{R_0}} r + 2R_0^{-2} r,\\
\boldsymbol{l}_1(\boldsymbol{u},q,r) &= \nu^{\pm}\big(\widetilde{\nabla}^2 - \nabla^2\big)\boldsymbol{u} + \frac{1}{\rho^{\pm}}(\nabla - \widetilde{\nabla})q\\
&\quad + \big(\mathbb{L}^{-1}\big(\mathcal{D}_t r^*\boldsymbol{N}^* + \chi\dot{\boldsymbol{h}}(t)\big)\cdot\nabla\big)\boldsymbol{u} - (\mathbb{L}^{-1}\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\\
l_2(\boldsymbol{u},r) &= (\mathbb{I} - \widehat{\mathbb{L}}^T)\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{L}(\boldsymbol{u},r), \quad \boldsymbol{L}(\boldsymbol{u},r) = (\mathbb{I} - \widehat{\mathbb{L}})\boldsymbol{u},\\
\boldsymbol{l}_3(\boldsymbol{u},r) &= \big[\mu^{\pm}\Pi_0\big(\Pi_0\mathbb{S}(\boldsymbol{u})\boldsymbol{N} - \Pi\widetilde{\mathbb{S}}(\boldsymbol{u})\boldsymbol{n}\big)\big]\big|_{S_{R_0}},\\
l_4(\boldsymbol{u},r) &= \big[\mu^{\pm}\big(\boldsymbol{N}\cdot\mathbb{S}(\boldsymbol{u})\boldsymbol{N} - \boldsymbol{n}\cdot\widetilde{\mathbb{S}}(\boldsymbol{u})\boldsymbol{n}\big)\big]\big|_{S_{R_0}},\\
l_5(r) &= -\int_0^1 (1-s)\frac{d^2}{ds^2}\widehat{\mathbb{L}}^T(sr,\boldsymbol{0})\nabla\cdot\boldsymbol{n}_s\,ds, \quad \boldsymbol{n}_s = \frac{\widehat{\mathbb{L}}^T(sr,\boldsymbol{0})\boldsymbol{N}}{|\widehat{\mathbb{L}}^T(sr,\boldsymbol{0})\boldsymbol{N}|},\\
l_6(\boldsymbol{u},r) &= \left(\boldsymbol{u} - \frac{1}{|B^+|}\int_{B^+}\boldsymbol{u}(y',t)\,dy'\right)\left(\frac{\widehat{\mathbb{L}}^T(r,\boldsymbol{0})\boldsymbol{N}}{\boldsymbol{N}\cdot\widehat{\mathbb{L}}^T(r,\boldsymbol{0})\boldsymbol{N}} - \boldsymbol{N}\right)\\
&\quad - \frac{1}{|B^+|}\int_{B^+}\big(L(r,\boldsymbol{0}) - 1\big)\boldsymbol{u}\,dy\frac{\widehat{\mathbb{L}}^T(r,\boldsymbol{0})\boldsymbol{N}}{\boldsymbol{N}\cdot\widehat{\mathbb{L}}^T(r,\boldsymbol{0})\boldsymbol{N}},\\
\Pi_0\boldsymbol{g} &= \boldsymbol{g} - \boldsymbol{N}(\boldsymbol{N}\cdot\boldsymbol{g}), \qquad \boldsymbol{N}(y)\cdot\widehat{\mathbb{L}}^T(y,t)\boldsymbol{N}(y) = \boldsymbol{y}\cdot\widehat{\mathbb{L}}^T\boldsymbol{y}/|y|^2.
\end{aligned} \tag{1.6}$$

The vectors $\boldsymbol{n}(x,t)$ and $\boldsymbol{N}(y)$ are connected by

$$\boldsymbol{n}(x,t)|_{x=e_{r,\mathbf{0}}(y,t)} = \frac{\widehat{\mathbb{L}}^T(r,\mathbf{0})\boldsymbol{N}(y)}{|\widehat{\mathbb{L}}^T(r,\mathbf{0})\boldsymbol{N}(y)|}\bigg|_{S_{R_0}},$$

moreover, $H(e_{r,\mathbf{0}},t)+2/R_0 = \mathcal{B}_0 r + l_5$, where $\mathcal{B}_0 r$ is the first variation of $H+2/R_0$ with respect to $r$ and $l_5$ is a nonlinear remainder. By $\Delta_{S_{R_0}}$ we denote the Laplace–Beltrami operator on $S_{R_0}$, while $\boldsymbol{n}_s$ is the normal to the surface

$$\Gamma_{t,s} = \{x\in\mathbb{R}^3 \mid x = y + sr(y,t)\boldsymbol{N}(y),\ y\in S_{R_0}\},\quad s\in(0,1).$$

Remark 1.1. Equation $z = y + r(y,t)\boldsymbol{N}(y)$ defines the surface $\Gamma_t$ shifted by the vector $-\boldsymbol{h}(t)$.

We have added the normalization condition $\int_\Omega q\,dy = 0$ for $q$. It can be taken also in another form, for instance,

$$\int_{B^\pm} q^\pm(y,t)\,dy = 0 \tag{1.7}$$

or

$$\int_\Sigma q(y,t)\,dS = 0.$$

Pressure functions satisfying different conditions differ from each other by certain functions of time. If $\int_\Omega q\,dy = 0$, $\widehat{q}^-$ satisfies (1.7), say, in the domain $B^-$ and $\widetilde{q}$ does it on $\Sigma$, respectively, then $q(y,t) = \widehat{q}^-(y,t) + \widehat{c}(t) = \widetilde{q}(y,t) + \widetilde{c}(t)$ with

$$\widehat{c}(t) = |B^-|^{-1}\int_{B^-} q(\xi,t)\,d\xi,\qquad \widetilde{c}(t) = |\Sigma|^{-1}\int_\Sigma q(\xi,t)\,d\Sigma.$$

It is easily seen that $[q]|_{S_{R_0}} = [\widehat{q}^-]|_{S_{R_0}} = [\widetilde{q}]|_{S_{R_0}}$.

We notice that the condition $|\Omega_t^+| = 4\pi R_0^3/3$ and the fact that the barycenter of $\Omega_t^+$ is placed at the origin $\{y=0\}$ can be expressed in terms of $r$ as follows:

$$\begin{gathered}\int_{S_{R_0}} \big((R_0+r)^3 - R_0^3\big)\,dS = 0,\\ \int_{S_{R_0}} y_j\big((R_0+r)^4 - R_0^4\big)\,dS = 0,\quad j=1,2,3.\end{gathered} \tag{1.8}$$

We define the Sobolev–Slobodetskiĭ spaces which we use in the present paper. The isotropic space $W_2^l(\Omega)$, $\Omega\subset\mathbb{R}^n$, is the space with the norm

$$\|u\|^2_{W_2^l(\Omega)} = \sum_{0\le|\boldsymbol{j}|\le l}\|\mathcal{D}_x^{\boldsymbol{j}}u\|^2_\Omega \equiv \sum_{0\le|\boldsymbol{j}|\le l}\int_\Omega |\mathcal{D}_x^{\boldsymbol{j}}u(x)|^2\,dx$$

if $l=[l]$, i.e. $l$ is an integral number, and

$$\|u\|^2_{W_2^l(\Omega)} = \|u\|^2_{W_2^{[l]}(\Omega)} + \sum_{|\boldsymbol{j}|=[l]}\int_\Omega\int_\Omega |\mathcal{D}_x^{\boldsymbol{j}}u(x) - \mathcal{D}_y^{\boldsymbol{j}}u(y)|^2\frac{dx\,dy}{|x-y|^{n+2\lambda}}$$

if $l=[l]+\lambda$, $\lambda\in(0,1)$. As usual, $\mathcal{D}_x^{\boldsymbol{j}}u$ denotes a (generalized) partial derivative $\partial^{|\boldsymbol{j}|}u/\partial x_1^{j_1}\dots\partial x_n^{j_n}$, where $\boldsymbol{j}=(j_1,\dots,j_n)$ and $|\boldsymbol{j}| = j_1+\dots+j_n$.

We introduce the anisotropic spaces

$$W_2^{l,0}(Q_T) = L_2\big((0,T), W_2^l(\Omega)\big), \qquad W_2^{0,l/2}(Q_T) = W_2^{l/2}\big((0,T), L_2(\Omega)\big);$$

$Q_T = \Omega \times (0,T)$, the squares of norms in these spaces coincide, respectively, with

$$\|u\|^2_{W_2^{l,0}(Q_T)} = \int_0^T \|u(\,\cdot\,,t)\|^2_{W_2^l(\Omega)}\,dt, \quad \|u\|^2_{W_2^{0,l/2}(Q_T)} = \int_\Omega \|u(x,\,\cdot\,)\|^2_{W_2^{l/2}(0,T)}\,dx.$$

The space $W_2^{l,l/2}(Q_T) \equiv W_2^{l,0}(Q_T) \cap W_2^{0,l/2}(Q_T)$ can be supplied with the norm

$$\|u\|_{W_2^{l,l/2}(Q_T)} \equiv \|u\|_{W_2^{l,0}(Q_T)} + \|u\|_{W_2^{0,l/2}(Q_T)}.$$

There exist many other equivalent norms in $W_2^{l,l/2}(Q_T)$; some of them will be used below.

The Sobolev–Slobodetskiĭ spaces of functions given on smooth surfaces, in particular, on $S_{R_0}$ and on $G_T = S_{R_0} \times (0,T)$, $T \le \infty$, are introduced in the standard way, with the help of local maps and partition of unity.

Moreover, we introduce also the norm

$$|u|^{(s+l,l/2)}_{G_T} = \|u\|_{W_2^{s+l,0}(G_T)} + \|u\|_{W_2^{l/2}(0,T;W_2^s(S_{R_0}))}.$$

Finally, we set

$$\|u\|^2_{W_2^l(\cup B^\pm)} \equiv \|u\|^2_{W_2^l(B^+)} + \|u\|^2_{W_2^l(B^-)}, \quad \|u\|_\Omega \equiv \|u\|_{L_2(\Omega)}.$$

Now, the main result of the paper is stated.

Theorem 1.2 (Global Existence). *Let* $\Sigma \in W_2^{3/2+l}$, $\boldsymbol{u}_0 \in \boldsymbol{W}_2^{1+l}(\cup B^\pm)$, $r_0 \in W_2^{2+l}(S_{R_0})$ *with* $l \in (1/2,1)$, *and the compatibility and smallness conditions*

$$(1.9)\qquad \begin{aligned} &\nabla\cdot\boldsymbol{u}_0 = l_2(\boldsymbol{u}_0,r_0), \qquad && [\mu^\pm\Pi_0\mathbb{S}(\boldsymbol{u}_0)\boldsymbol{N}]|_{S_{R_0}} = \boldsymbol{l}_3(\boldsymbol{u}_0,r_0),\\ &[\boldsymbol{u}_0]|_{S_{R_0}} = 0, && \boldsymbol{u}_0|_\Sigma = 0,\end{aligned}$$

$$(1.10)\qquad \|\boldsymbol{u}_0\|_{\boldsymbol{W}_2^{1+l}(\cup B^\pm)} + \|r_0\|_{W_2^{2+l}(S_{R_0})} \le \varepsilon$$

*are satisfied. Moreover, assume that* $\boldsymbol{f}$ *has finite norms*

$$\sup_{\tau>0}\|\mathcal{D}_x^{\boldsymbol{i}}\boldsymbol{f}\|_{Q_{\tau,\tau+T_0}}, \qquad \|\mathrm{e}^{bt}\boldsymbol{f}\|_{W_2^{l,l/2}(Q_\infty)},$$

*where* $Q_\infty = \Omega\times(0,\infty)$, $T_0 > 2$ *is an appropriate fixed number, and*

$$(1.11)\qquad \|\mathrm{e}^{bt}\boldsymbol{f}\|_{W_2^{l,l/2}(Q_\infty)} \le \varepsilon, \quad b>0, \quad \sup_{\tau>0}\|\mathcal{D}_x^{\boldsymbol{i}}\boldsymbol{f}\|_{Q_{\tau,\tau+T_0}} \le \varepsilon, \quad |\boldsymbol{i}| = 1,2.$$

*Then problem* (1.5) *has a unique solution* $(\boldsymbol{u},q,r)$, *and it satisfies the inequality*

$$(1.12)\qquad \begin{aligned} &\|\mathrm{e}^{at}\boldsymbol{u}\|_{W_2^{2+l,1+l/2}(\cup D_\infty^\pm)} + \|\mathrm{e}^{at}\nabla q\|_{W_2^{l,l/2}(\cup D_\infty^\pm)} + \|\mathrm{e}^{at}q\|_{W_2^{0,l/2}(\cup D_\infty^\pm)}\\ &\quad + \|\mathrm{e}^{at}r\|_{W_2^{5/2+l,5/4+l/2}(G_\infty)} + \|\mathrm{e}^{at}\mathcal{D}_t r\|_{W_2^{3/2+l,3/4+l/2}(G_\infty)}\\ &\quad\le c_1(\varepsilon)\Big\{\|\mathrm{e}^{at}\boldsymbol{f}\|_{W_2^{l,l/2}(Q_\infty)} + \|\boldsymbol{u}_0\|_{\boldsymbol{W}_2^{1+l}(\cup B^\pm)} + \|r_0\|_{W_2^{2+l}(S_{R_0})}\Big\}\end{aligned}$$

*with a certain* $a < b$, $D^\pm_\infty = B^\pm \times (0,\infty)$. $c(\varepsilon)$ *is a bounded function of* $\varepsilon$.

We note that similar results in the Hölder spaces were obtained without and with mass forces in [9] and [7], respectively.

Theorem 1.2 guarantees solution stability understood in the sense that velocity vector field differs a little from zero as well as pressure function does a little from a step function for small initial data and mass forces. In addition, limit interface is a sphere $S_{R_0}(h_\infty)$ of the radius $R_0$; however, the center $h_\infty$ of limit sphere may be displaced slightly with respect to the origin, the barycenter of $\Omega_0^+$. This displacement will be evaluated by inequality (3.23) at the end of Section 3. There will be also given an estimate of the initial distance between the outer boundary and fluid interface sufficient for preventing the intersection of the surfaces in the future.

The proof of Theorem 1.2 consists of several steps. It is based on an exponential energy inequality for a solution of a linear problem, which implies an exponential decay of a global solution to the problem.

## 2. Linear problem

Along with (1.5), we consider the linear problem

$$
\begin{cases}
\mathcal{D}_t \boldsymbol{v} - \nu^\pm \nabla^2 \boldsymbol{v} + \dfrac{1}{\rho^\pm}\nabla p = \boldsymbol{f}, \quad \nabla\cdot\boldsymbol{v} = f & \text{in } B^\pm,\ t>0,\\
\boldsymbol{v}(y,0) = \boldsymbol{v}_0(y) & \text{in } B^\pm,\\
r(y,0) = r_0(y) & \text{on } S_{R_0},\\
[\boldsymbol{v}]|_{S_{R_0}} = 0, \quad [\mu^\pm \Pi_0 \mathbb{S}(\boldsymbol{v})\boldsymbol{N}]|_{S_{R_0}} = \boldsymbol{b}, \quad \boldsymbol{v}|_\Sigma = 0, & \\
[\boldsymbol{N}\cdot\mathbb{T}(\boldsymbol{v},p)\boldsymbol{N}]|_{S_{R_0}} - \sigma\mathcal{B}_0 r|_{S_{R_0}} = b, & \\
\mathcal{D}_t r - \left(\boldsymbol{v}\cdot\boldsymbol{N} - \dfrac{\boldsymbol{N}}{|B^+|}\cdot\displaystyle\int_{B^+}\boldsymbol{v}(y',t)\,dy'\right)\Big|_{S_{R_0}} = g, & \\
\displaystyle\int_\Omega p(y,t)\,dy = 0. &
\end{cases}
\tag{2.1}
$$

THEOREM 2.1 (Local Solvability of the Linear Problem). *Let* $\Sigma \in W_2^{3/2+l}$, $r_0 \in W_2^{2+l}(S_{R_0})$ *with* $l \in (1/2,1)$. *For arbitrary* $\boldsymbol{f} \in \boldsymbol{W}_2^{l,l/2}(\cup D_T^\pm)$, $f \in W_2^{1+l,0}(\cup D_T^\pm)$, $f = \nabla\cdot\boldsymbol{F}$, $\boldsymbol{F} \in \boldsymbol{W}_2^{0,1+l/2}(\cup D_T^\pm)$, $[F_{\boldsymbol{N}}]|_{S_{R_0}} = 0$, $\boldsymbol{v}_0 \in W_2^{1+l}(\cup B^\pm)$, $\boldsymbol{b} \in \boldsymbol{W}_2^{l+1/2,l/2+1/4}(G_T)$, $b \in W_2^{l+1/2,0}(G_T) \cap W_2^{l/2}\big(0,T;W_2^{1/2}(S_{R_0})\big)$, $g \in W_2^{3/2+l,3/4+l/2}(G_T)$, *where* $D_T^\pm = B^\pm \times (0,T)$, $G_T = S_{R_0}\times(0,T)$, $T<\infty$, *satisfying the compatibility conditions*

$$
\begin{gathered}
\nabla\cdot\boldsymbol{v}_0 = f(y,0), \qquad [\mu^\pm\Pi_0\mathbb{S}(\boldsymbol{v}_0)\boldsymbol{N}]|_{S_{R_0}} = \boldsymbol{b}(y,0),\\
\boldsymbol{N}\cdot\boldsymbol{b}(y,t) = 0, \quad t\ge 0,\\
[\boldsymbol{v}_0]|_{S_{R_0}} = 0, \quad \boldsymbol{v}_0|_\Sigma = 0,
\end{gathered}
\tag{2.2}
$$

*problem* (2.1) *has a unique solution* $(\boldsymbol{v}, p, r)$: $\boldsymbol{v} \in \boldsymbol{W}_2^{2+l,1+l/2}(\cup D_T^\pm)$, $\nabla p \in \boldsymbol{W}_2^{l,l/2}(\cup D_T^\pm)$, $r(\,\cdot\,, t) \in W_2^{2+l}(S_{R_0})$ *for all* $t \in (0, T)$, *and*

$$\begin{aligned}(2.3)\qquad \|\boldsymbol{v}\|_{\boldsymbol{W}_2^{2+l,1+l/2}(\cup D_T^\pm)} &+ \|\nabla p\|_{\boldsymbol{W}_2^{l,l/2}(\cup D_T^\pm)} \\ &+ \|p\|_{W_2^{0,l/2}(\cup D_T^\pm)} + \|r\|_{W_2^{5/2+l,5/4+l/2}(G_T)} + \|\mathcal{D}_t r\|_{W_2^{3/2+l,3/4+l/2}(G_T)} \\ &\le c(T)\big\{\|\boldsymbol{f}\|_{\boldsymbol{W}_2^{l,l/2}(\cup D_T^\pm)} + \|f\|_{W_2^{1+l,0}(\cup D_T^\pm)} \\ &+ \|\boldsymbol{F}\|_{W_2^{0,1+l/2}(\cup D_T^\pm)} + \|\boldsymbol{b}\|_{\boldsymbol{W}_2^{l+1/2,l/2+1/4}(G_T)} + |b|_{G_T}^{(1/2+l,l/2)}\big) \\ &+ \|g\|_{W_2^{3/2+l,3/4+l/2}(G_T)} + \|\boldsymbol{v}_0\|_{\boldsymbol{W}_2^{1+l}(\cup B^\pm)} + \|r_0\|_{W_2^{2+l}(S_{R_0})}\big\}.\end{aligned}$$

Remark 2.2. From the trace theorem for $\rho \in W_2^{1,1}(G_T)$, it follows that

$$\|\rho(\,\cdot, t)\|_{W_2^{1/2}(S_{R_0})} \le c\big\{\|\rho\|_{W_2^{1,0}(G_T)} + \|\mathcal{D}_t\rho\|_{G_T}\big\}, \quad t \in [0, T],$$

which implies

$$\|r(\,\cdot\,, t)\|_{W_2^{2+l}(S_{R_0})} \le c\big\{\|r\|_{W_2^{5/2+l,0}(G_T)} + \|\mathcal{D}_t r\|_{W_2^{3/2+l,0}(G_T)}\big\}.$$

This means that $\Gamma_t \in W_2^{2+l}$ for all $t \in [0, T]$.

Proof. Let $r_1$ be a function satisfying the conditions

$$\begin{aligned} r_1(y, 0) &= r_0(y), \\ \mathcal{D}_t r_1(y, 0) &= g(y, 0) + \bigg(\boldsymbol{v}_0(y) \cdot \boldsymbol{N}(y) - \frac{\boldsymbol{N}(y)}{|B^+|} \cdot \int_{B^+} \boldsymbol{v}_0(y')\, dy'\bigg) \equiv r_0'(y)\end{aligned}$$

and the estimates

$$\begin{aligned}(2.4)\qquad |r_1|_{G_T}^{(5/2+l,l/2)} &+ \|\mathcal{D}_t r_1\|_{W_2^{3/2+l,3/4+l/2}(G_T)} \\ &\le c\Big\{\|r_1\|_{W_2^{5/2+l,5/4+l/2}(G_T)} + \|\mathcal{D}_t r_1\|_{W_2^{3/2+l,3/4+l2}(G_T)}\Big\} \\ &\le c\Big\{\|r_0\|_{W_2^{2+l}(S_{R_0})} + \|r_0'\|_{W_2^{l+1/2}(S_{R_0})}\Big\}.\end{aligned}$$

Such $r_1$ exits due to Proposition 4.1 in [22] and equivalent normalizations of the Sobolev–Slobodetskiĭ spaces. Then we can write

$$\begin{aligned}\mathcal{B}_0 r(y, t) &= \mathcal{B}_0 r_1(y, t) + \int_0^t \mathcal{B}_0\big(\mathcal{D}_t(r(y, \tau) - r_1(y, \tau))\big)\, d\tau \\ &= \mathcal{B}_0 r_1(y, t) + \int_0^t \mathcal{B}_0\big(g(y, \tau) + \boldsymbol{v}(y, \tau) \cdot \boldsymbol{N}(y) - \mathcal{D}_t r_1(y, \tau)\big)\, d\tau,\end{aligned}$$

because $\mathcal{B}_0\boldsymbol{N} = 0$ in view of the fact that $\boldsymbol{N}$ is an eigenvector of $\Delta_{S_{R_0}}$ with the eigenvalue $-2R_0^{-2}$. Hence, (2.1) can be written in the form

$$\left\{\begin{array}{ll} \mathcal{D}_t\boldsymbol{v} - \nu^\pm\nabla^2\boldsymbol{v} + \frac{1}{\rho^\pm}\nabla p = \boldsymbol{f}, \quad \nabla\cdot\boldsymbol{v} = f & \text{in } B^\pm,\ t>0, \\ \boldsymbol{v}(y,0) = \boldsymbol{v}_0(y) & \text{in } B^\pm, \\ [\boldsymbol{v}]|_{S_{R_0}} = 0, \quad [\mu^\pm\Pi_0\mathbb{S}(\boldsymbol{v})\boldsymbol{N}]\big|_{S_{R_0}} = \boldsymbol{b}, \quad \boldsymbol{v}|_\Sigma = 0, & \\ [\boldsymbol{N}\cdot\mathbb{T}(\boldsymbol{v},p)\boldsymbol{N}]\big|_{S_{R_0}} - \sigma\boldsymbol{N}\cdot\mathcal{B}_0\displaystyle\int_0^t \boldsymbol{v}|_{S_{R_0}}\,d\tau & \\ \qquad = b' + \sigma\displaystyle\int_0^t B'\,d\tau + 2\sigma\int_0^t \nabla_S\boldsymbol{v}:\nabla_S\boldsymbol{N}\,d\tau & \text{on } S_{R_0}, \\ \displaystyle\int_\Omega p(y,t)\,dy = 0 & \text{for } t>0, \end{array}\right. \tag{2.5}$$

where $b' = b + \sigma\mathcal{B}_0 r_1$, $B' = \mathcal{B}_0(g - \mathcal{D}_t r_1)$, $\nabla_S$ is the surface gradient on $S_{R_0}$; $\mathbb{S}:\mathbb{T} \equiv S_{ij}T_{ij}$. Problems of this type were studied in [3], where the solvability of (2.5) without the term $2\sigma\int_0^t \nabla_S\boldsymbol{v}:\nabla_S\boldsymbol{N}\,d\tau$ and the estimate

$$\begin{aligned} &\|\boldsymbol{v}\|_{W_2^{2+l,l+1/2}(\cup D_T^\pm)} + \|\nabla p\|_{\boldsymbol{W}_2^{l,l/2}(\cup D_T^\pm)} + \|p\|_{W_2^{0,l/2}(\cup D_T^\pm)} \\ &\qquad \le c(T)\Big\{\|\boldsymbol{f}\|_{\boldsymbol{W}_2^{l,l/2}(\cup D_T^\pm)} + \|f\|_{W_2^{1+l,0}(\cup D_T^\pm)} \\ &\qquad\quad + \|\boldsymbol{F}\|_{\boldsymbol{W}_2^{0,1+l/2}(\cup D_T^\pm)} + \|\boldsymbol{b}\|_{\boldsymbol{W}_2^{l+1/2,l/2+1/4}(G_T)} \\ &\qquad\quad + |b'|_{G_T}^{(1/2+l,l/2)} + \|B'\|_{W_2^{l-1/2,l/2-1/4}(G_T)} + \|\boldsymbol{v}_0\|_{\boldsymbol{W}_2^{1+l}(\cup B^\pm)}\Big\} \end{aligned}$$

of a solution were established. Together with (2.4), this inequality implies (2.3) because the additional term is weak and has no essential influence on the final result. □

We also consider problem (2.1) with $\boldsymbol{f} = 0$, $f = 0$, $\boldsymbol{b} = 0$, $b = 0$, $g = 0$ and with $r_0(y)$ satisfying the orthogonality conditions

$$\int_{S_{R_0}} r_0(y)\,dS = 0, \qquad \int_{S_{R_0}} r_0(y)y_j\,dS = 0, \quad j = 1,2,3, \tag{2.6}$$

obtained by linearization of (1.8). Since

$$\int_{S_{R_0}} \mathcal{D}_t r(y,t)\,dS = \int_{S_{R_0}} \boldsymbol{v}\cdot\boldsymbol{N}\,dS = \int_{B^+} \nabla\cdot\boldsymbol{v}(y,t)\,dy = 0,$$
$$\int_{S_{R_0}} \mathcal{D}_t r(y,t)y_j\,dS = \int_{S_{R_0}} y_j\boldsymbol{v}\cdot\boldsymbol{N}\,dS - \int_{B^+} v_j(y,t)\,dy = 0,$$

conditions (2.6) are satisfied also for $r(y,t)$, $t > 0$ :

$$\int_{S_{R_0}} r(y,t)\,dS = 0, \qquad \int_{S_{R_0}} r(y,t)y_j\,dS = 0, \quad j = 1,2,3. \tag{2.7}$$

Theorem 2.3 (Global Solvability of the Linear Homogeneous Problem). *Problem* (2.1) *with* $\boldsymbol{f}=0$, $f=0$, $\boldsymbol{b}=0$, $b=0$, $g=0$ *and with* $\boldsymbol{v}_0\in W_2^{1+l}(\cup B^{\pm})$, $r_0\in W_2^{2+l}(S_{R_0})$, $l\in(1/2,1)$, *satisfying compatibility conditions* (2.2), *i.e.*

$$\nabla\cdot\boldsymbol{v}_0=0,\qquad [\mu^{\pm}\Pi_0\mathbb{S}(\boldsymbol{v}_0)\boldsymbol{N}]|_{S_{R_0}}=0,\qquad [\boldsymbol{v}_0]|_{S_{R_0}}=0,\qquad \boldsymbol{v}_0|_{\Sigma}=0, \tag{2.8}$$

*and orthogonality conditions* (2.6), *has a unique solution* $(\boldsymbol{v},p,r)$, *such that* $\boldsymbol{v}\in\boldsymbol{W}_2^{2+l,1+l/2}(\cup D_\infty^{\pm})$, $\nabla p\in\boldsymbol{W}_2^{l,l/2}(\cup D_\infty^{\pm})$, $r(\,\cdot\,,t)\in W_2^{2+l}(S_{R_0})$ *for all* $t\in(0,\infty)$, *it is subject to the inequality*

$$\begin{aligned}\|\mathrm{e}^{\beta t}\boldsymbol{v}\|^2_{\boldsymbol{W}_2^{2+l,1+l/2}(\cup D_\infty^{\pm})}&+\|\mathrm{e}^{\beta t}\nabla p\|^2_{\boldsymbol{W}_2^{l,l/2}(\cup D_\infty^{\pm})}+\|\mathrm{e}^{\beta t}p\|^2_{W_2^{0,l/2}(\cup D_\infty^{\pm})}\\&+\|\mathrm{e}^{\beta t}r\|^2_{W_2^{5/2+l,5/4+l/2}(G_\infty)}+\|\mathrm{e}^{\beta t}\mathcal{D}_t r\|^2_{W_2^{3/2+l,3/4+l/2}(G_\infty)}\\&\le c\Big\{\|\boldsymbol{v}_0\|^2_{\boldsymbol{W}_2^{1+l}(\cup B^{\pm})}+\|r_0\|^2_{W_2^{2+l}(S_{R_0})}\Big\}\end{aligned} \tag{2.9}$$

*with a certain* $\beta>0$.

We outline the proof of (2.9). At first, weighted $L_2$-estimates of $\boldsymbol{v},r$ are obtained.

Proposition 2.4. *A solution of* (2.1), (2.6) *with* $\boldsymbol{f}=0$, $f=0$, $\boldsymbol{b}=0$, $b=0$, $g=0$ *satisfies the inequality*

$$\big\|e^{\beta_1 t}\boldsymbol{v}(\,\cdot\,,t)\big\|^2_{\Omega}+\big\|e^{\beta_1 t}r(\,\cdot\,,t)\big\|^2_{W_2^1(S_{R_0})}\le c\Big\{\|\boldsymbol{v}_0\|^2_{\Omega_0}+\|r_0\|^2_{W_2^1(S_{R_0})}\Big\}, \tag{2.10}$$

*where* $\beta_1>0$, $c$ *is independent of* $t$.

Proof. Inequality (2.10) is obtained in the same way as inequality (2.8) in [9] and even easier because the triple $(\boldsymbol{v},p,r)$ solves a linear problem. The proof is based on the energy relation

$$\frac12\frac{d}{dt}\big\|\sqrt{\rho^{\pm}}\boldsymbol{v}\big\|^2_{\Omega}-\sigma\int_{S_{R_0}}\boldsymbol{v}\cdot\boldsymbol{N}\mathcal{B}_0 r\,dS+\frac12\big\|\sqrt{\mu^{\pm}}\mathbb{S}(\boldsymbol{v})\big\|^2_{\Omega}=0 \tag{2.11}$$

which, in view of the last boundary condition in (2.1) and the self-adjointness of the operator $\mathcal{B}_0$, implies

$$\frac12\frac{d}{dt}\bigg(\big\|\sqrt{\rho^{\pm}}\boldsymbol{v}\big\|^2_{\Omega}-\sigma\int_{S_{R_0}}r\mathcal{B}_0 r\,dS\bigg)+\frac12\big\|\sqrt{\mu^{\pm}}\mathbb{S}(\boldsymbol{v})\big\|^2_{\Omega}=0.$$

Similarly to (2.11), one can deduce the equality

$$\begin{aligned}\frac{d}{dt}\int_{\Omega}\rho^{\pm}\boldsymbol{v}\cdot\boldsymbol{W}\,dx&-\int_{\Omega}\rho^{\pm}\boldsymbol{v}\cdot\mathcal{D}_t\boldsymbol{W}\,dx\\&+\int_{\Omega}\frac{\mu^{\pm}}{2}\mathbb{S}(\boldsymbol{v}):\mathbb{S}(\boldsymbol{W})\,dx-\sigma\int_{S_{R_0}}r\mathcal{B}_0 r\,dS=0,\end{aligned} \tag{2.12}$$

where $\boldsymbol{W}$ is an auxiliary vector field satisfying the relations (see [9])

$$\nabla\cdot\boldsymbol{W}(x,t)=0 \quad \text{in } \Omega, \qquad \boldsymbol{W}\cdot\boldsymbol{N}|_{S_{R_0}}=r, \qquad [\boldsymbol{W}]|_{S_{R_0}}=0, \qquad \boldsymbol{W}|_{\Sigma}=0,$$
$$\|\boldsymbol{W}\|_{\boldsymbol{W}_2^1(\Omega)}\le c\|r\|_{W_2^{1/2}(S_{R_0})},$$
$$\|\mathcal{D}_t\boldsymbol{W}\|_{\Omega}\le c\|\mathcal{D}_t r\|_{S_{R_0}}\le c\big\{\|\boldsymbol{v}\cdot\boldsymbol{N}\|_{S_{R_0}}+\|\boldsymbol{v}\|_{\Omega}\big\}.$$

We multiply (2.12) by a small $\gamma>0$ and add it to (2.11). Taking account of the fact that the form $-\int_{S_{R_0}} r\mathcal{B}_0 r\,dS=\int_{S_{R_0}}(|\nabla_S r|^2-2R_0^{-2}r^2)\,dS$ is positive definite if $r$ satisfies (2.7) (see [18]) and making use of (2.4) and of the Korn inequality for $\boldsymbol{v}$, we show that for the so-called generalized energy [13]

$$\mathcal{E}(t)=\frac{1}{2}\big\|\sqrt{\rho^{\pm}}\boldsymbol{v}\big\|_{\Omega}^2-\sigma\int_{S_{R_0}} r\mathcal{B}_0 r\,dS+\gamma\int_{\Omega}\rho^{\pm}\boldsymbol{v}\cdot\boldsymbol{W}\,dx,$$

the estimate

$$\frac{d}{dt}\,\mathcal{E}(t)+2\beta_1\mathcal{E}(t)\le 0$$

is valid, where $\beta_1=\mathit{const}>0$. Since $\mathcal{E}$ is controlled by $c\big(\|\boldsymbol{v}\|_{\Omega}^2+\|r\|^2_{W_2^1(S_{R_0})}\big)$ from above and from below if $\gamma$ is small enough, by the Gronwall lemma, we have (2.10). $\square$

For obtaining bounds for higher order norms of the solution similar to (2.10), we invoke a local in time estimate of the solution. Keeping in mind forthcoming applications, we assume that $T>2$.

PROPOSITION 2.5. *Let $T>2$. The solution of problem* (2.1), (2.6) *with $\boldsymbol{f}=0$, $f=0$, $\boldsymbol{b}=0$, $b=0$, $g=0$ is subject to the inequality*

$$\begin{aligned}(2.13)\quad &\|\boldsymbol{v}\|_{\boldsymbol{W}_2^{2+l,1+l/2}(\cup D^{\pm}_{t_0-1,t_0})}+\|\nabla p\|_{\boldsymbol{W}_2^{l,l/2}(\cup D^{\pm}_{t_0-1,t_0})}\\ &\quad+\|p\|_{W_2^{0,l/2}(\cup D^{\pm}_{t_0-1,t_0})}+\|r\|_{W_2^{5/2+l,5/4+l/2}(G_{t_0-1,t_0})}\\ &\quad+\|\mathcal{D}_t r\|_{W_2^{3/2+l,3/4+l/2}(G_{t_0-1,t_0})}\le c\big(\|\boldsymbol{v}\|_{Q_{t_0-2,t_0}}+\|r\|_{G_{t_0-2,t_0}}\big),\end{aligned}$$

*where $2<t_0\le T$, $D^{\pm}_{t_1,t_2}=B^{\pm}\times(t_1,t_2)$, $Q_{t_1,t_2}=\Omega\times(t_1,t_2)$, $G_{t_1,t_2}=S_{R_0}\times(t_1,t_2)$.*

PROOF. We fix $t_0\in(2,T)$ and multiply (2.13) by a cutoff function $\zeta_\lambda(t)$, smooth, monotone, equal to zero for $t\le t_0-2+\lambda/2$ and to one for $t\ge t_0-2+\lambda$, where $\lambda\in(0,1]$, and such that for $\dot\zeta_\lambda(t)\equiv d\zeta_\lambda(t)/dt$ and $\ddot\zeta_\lambda(t)$, the inequalities

$$\sup_{t\in\mathbb{R}}\big|\dot\zeta_\lambda(t)\big|\le c\lambda^{-1},\qquad \sup_{t\in\mathbb{R}}\big|\ddot\zeta_\lambda(t)\big|\le c\lambda^{-2}$$

hold. Then, for $\boldsymbol{v}_\lambda = \boldsymbol{v}\zeta_\lambda$, $p_\lambda = p\zeta_\lambda$, $r_\lambda = r\zeta_\lambda$, we obtain

$$\tag{2.14} \begin{cases} \mathcal{D}_t\boldsymbol{v}_\lambda - \nu^\pm\nabla^2\boldsymbol{v}_\lambda + \dfrac{1}{\rho^\pm}\nabla p_\lambda = \boldsymbol{v}\dot{\zeta}_\lambda, \quad \nabla\cdot\boldsymbol{v}_\lambda = 0 & \text{in } B^\pm,\ t>0,\\ \boldsymbol{v}_\lambda(y,0)=0 & \text{in } B^\pm,\\ r_\lambda(y,0)=0 & \text{on } S_{R_0},\\ [\boldsymbol{v}_\lambda]|_{S_{R_0}}=0, \quad [\mu^\pm\Pi_0\mathbb{S}(\boldsymbol{v}_\lambda)\boldsymbol{N}]\big|_{S_{R_0}}=0, \quad \boldsymbol{v}_\lambda|_\Sigma = 0, \\ [\boldsymbol{N}\cdot\mathbb{T}(\boldsymbol{v}_\lambda,p_\lambda)\boldsymbol{N}]\big|_{S_{R_0}} - \sigma\mathcal{B}_0 r_\lambda|_{S_{R_0}} = 0, \\ \displaystyle\int_\Omega p_\lambda(y,t)\,dy = 0, \\ \mathcal{D}_t r_\lambda - \bigg(\boldsymbol{v}_\lambda\cdot\boldsymbol{N} - \dfrac{\boldsymbol{N}}{|B^+|}\cdot\displaystyle\int_{B^+}\boldsymbol{v}_\lambda(y',t)\,dy'\bigg)\bigg|_{S_{R_0}} = r\dot{\zeta}_\lambda(t). \end{cases}$$

By Theorem 2.1 applied to system (2.14), estimate (2.3) for $\boldsymbol{v}_\lambda$, $p_\lambda$, $r_\lambda$ is valid whence it follows that

$$\tag{2.15} \begin{aligned} &\|\boldsymbol{v}\|_{\boldsymbol{W}_2^{2+l,1+l/2}(\cup D^\pm_{t_1+\lambda,t_0})} + \|\nabla p\|_{\boldsymbol{W}_2^{l,l/2}(\cup D^\pm_{t_1+\lambda,t_0})} + \|p\|_{W_2^{0,l/2}(\cup D^\pm_{t_1+\lambda,t_0})} \\ &\quad + \|r\|_{W_2^{5/2+l,5/4+l/2}(G_{t_1+\lambda,t_0})} + \|\mathcal{D}_t r\|_{W_2^{3/2+l,3/4+l/2}(G_{t_1+\lambda,t_0})} \\ &\qquad \le c\lambda^{-2}\Big\{\|\boldsymbol{v}\|_{\boldsymbol{W}_2^{l,l/2}(\cup D^\pm_{t_1+\lambda/2,t_0})} + \|r\|_{W_2^{3/2+l,3/4+l/2}(G_{t_1+\lambda/2,t_0})}\Big\}, \end{aligned}$$

where $t_1 = t_0 - 2$.

Now, we apply the interpolation inequalities

$$\begin{aligned} \|\boldsymbol{v}\|_{\boldsymbol{W}_2^{l,l/2}(\cup D^\pm_{t_1+\lambda/2,t_0})} &\le \varkappa^2\|\boldsymbol{v}\|_{\boldsymbol{W}_2^{2+l,1+l/2}(\cup D^\pm_{t_1+\lambda/2,t_0})} + c\varkappa^{-l}\|\boldsymbol{v}\|_{Q_{t_1+\lambda/2,t_0}}, \\ \|r\|_{W_2^{3/2+l,0}(G_{t_1+\lambda/2})} &\le \varkappa^2\|r\|_{W_2^{5/2+l,0}(G_{t_1+\lambda/2,t_0})} + c\varkappa^{-3-2l}\|r\|_{G_{t_1+\lambda/2,t_0}}, \\ \|r\|_{W_2^{0,3/4+l/2}(G_{t_1+\lambda/2})} &\le \varkappa^2\|\mathcal{D}_t r\|_{W_2^{0,3/4+l/2}(G_{t_1+\lambda/2,t_0})} + c\varkappa^{-3/2-l}\|r\|_{G_{t_1+\lambda/2,t_0}}, \end{aligned}$$

which leads to

$$\Psi(\lambda) \le c_1\varkappa^2\lambda^{-2}\Psi(\lambda/2) + c_2\varkappa^{-m}\lambda^{-2}K,$$

here $\Psi(\lambda)$ denotes the left-hand side of (2.15), $K = \|\boldsymbol{v}\|_{Q_{t_1,t_0}} + \|r\|_{G_{t_1,t_0}}$, $m = 3+2l$. Setting $\varkappa = \delta\lambda \le 1$, we obtain

$$\lambda^{m+2}\Psi(\lambda) \le c_1\delta^2 2^{m+2}(\lambda/2)^{m+2}\Psi(\lambda/2) + c_2\delta^{-m}K.$$

This implies

$$\Psi(\lambda) \le c_3(\delta)\lambda^{-m-2}(K + 2^{-1}K + 2^{-2}K + \ldots) \le \frac{c_3\lambda^{-m-2}}{1-1/2}K \le 2c_3\lambda^{-m-2}K$$

if $c_1\delta^2 2^{m+2} < 1/2$. For $\lambda = 1$ this inequality is equivalent to (2.13). $\square$

Proof of Theorem 2.3. By Theorem 2.1 and Proposition 2.5, one has

$$
\begin{aligned}
(2.16)\qquad & e^{2\beta(T-j)}\Big\{\|\boldsymbol{v}\|^2_{\boldsymbol{W}_2^{2+l,1+l/2}(\cup D^{\pm}_{T-j-1,T-j})} \\
&+ \|\nabla p\|^2_{\boldsymbol{W}_2^{l,l/2}(\cup D^{\pm}_{T-j-1,T-j})} + \|p\|^2_{W_2^{0,l/2}(\cup D^{\pm}_{T-j-1,T-j})} \\
&+ \|r\|^2_{W_2^{5/2+l,5/4+l/2}(G_{T-j-1,T-j})} + \|\mathcal{D}_t r\|^2_{W_2^{3/2+l,3/4+l/2}(G_{T-j-1,T-j})}\Big\} \\
&\le c e^{2\beta(T-j)}\Big\{\|\boldsymbol{v}\|^2_{Q_{T-j-2,T-j}} + \|r\|^2_{G_{T-j-2,T-j}}\Big\},
\end{aligned}
$$

for $j = 0, \dots, [T]-2$. Taking the sum of (2.16) from $j = 0$ to $j = [T]-2$, we obtain the inequality that implies

$$
(2.17)\quad Y^2_{T-[T]+1,T}\big(e^{\beta t}\boldsymbol{v}, e^{\beta t}p, e^{\beta t}r\big) \le c\int_{T-[T]}^{T} e^{2\beta t}\big(\|\boldsymbol{v}(\,\cdot\,,t)\|^2_{\Omega} + \|r(\,\cdot\,,t)\|^2_{S_{R_0}}\big)\,dt,
$$

where

$$
\begin{aligned}
(2.18)\quad Y_{t_1,t_2}(\boldsymbol{u},q,r) = {}& \|\boldsymbol{u}\|_{\boldsymbol{W}_2^{2+l,1+l/2}(\cup D^{\pm}_{t_1,t_2})} \\
&+ \|\nabla q\|_{\boldsymbol{W}_2^{l,l/2}(\cup D^{\pm}_{t_1,t_2})} + \|q\|_{W_2^{0,l/2}(\cup D^{\pm}_{t_1,t_2})} \\
&+ \|r\|_{W_2^{5/2+l,5/4+l/2}(G_{t_1,t_2})} + \|\mathcal{D}_t r\|_{W_2^{3/2+l,3/4+l/2}(G_{t_1,t_2})}.
\end{aligned}
$$

By adding the estimate

$$
Y^2_{0,2}(\boldsymbol{v},p,r) \le c\Big\{\|\boldsymbol{v}_0\|^2_{\boldsymbol{W}_2^{1+l}(\cup B^{\pm})} + \|r_0\|^2_{W_2^{2+l}(S_{R_0})}\Big\}
$$

to (2.17), choosing $\beta < \beta_1$ and making use of (2.10), we arrive at an inequality equivalent to (2.9). □

## 3. Nonlinear problem

We start with the construction of a solution to problem (1.5) in a finite time interval $(0, T_0)$ with $T_0$ to be fixed later on.

Theorem 3.1 (Local Solvability of the Nonlinear Problem). *Let* $T_0 < \infty$ *and let compatibility conditions* (1.9) *of Theorem* 1.2 *be satisfied. Then there exists a value* $\varepsilon(T_0) \ll 1$ *such that problem* (1.5) *with small data*:

$$
(3.1)\quad
\begin{gathered}
\|\boldsymbol{u}_0\|_{\boldsymbol{W}_2^{1+l}(\cup B^{\pm})} + \|r_0\|_{W_2^{2+l}(S_{R_0})} + \|\boldsymbol{f}\|_{\boldsymbol{W}_2^{l,l/2}(Q_{T_0})} + \|\nabla \boldsymbol{f}\|_{Q_{T_0}} \le \varepsilon, \\
\|\nabla \boldsymbol{f}\|_{\boldsymbol{W}_2^{l,l/2}(Q_{T_0})} \le \varepsilon,
\end{gathered}
$$

*has a unique solution* $(\boldsymbol{u}, q, r)$ *on the interval* $(0, T_0)$ *and the inequalities*

$$
(3.2)\qquad Y_{0,T_0}(\boldsymbol{u},q,r) \le c\Big\{N(\boldsymbol{u}_0, r_0) + \|\boldsymbol{f}\|_{\boldsymbol{W}_2^{l,l/2}(Q_{T_0})}\Big\},
$$

$$
(3.3)\qquad N(\boldsymbol{u}(\,\cdot\,,T_0), r(\,\cdot\,,T_0)) \le \theta N(\boldsymbol{u}_0, r_0) + c\|\boldsymbol{f}\|_{\boldsymbol{W}_2^{l,l/2}(Q_{T_0})},
$$

*hold, where* $\theta < 1$, $Y_{0,T_0}$ *is calculated by* (2.18) *and*

$$
N(\boldsymbol{w},\rho) = \|\boldsymbol{w}\|_{\boldsymbol{W}_2^{1+l}(\cup B^{\pm})} + \|\rho\|_{W_2^{2+l}(S_{R_0})}.
$$

The proof of Theorem 3.1 relies on Theorem 2.1 and on the following estimates of the nonlinear terms.

Proposition 3.2. *If*

$$\|r(\,\cdot\,,t)\|_{W_2^{3/2+l}(S_{R_0})} + \|\mathcal{D}_t r\,(\,\cdot\,,t)\|_{W_2^{1/2+l}(S_{R_0})} + \|\boldsymbol{u}(\,\cdot\,,t)\|_\Omega \le \delta, \quad t \le T, \tag{3.4}$$

*where $\delta$ is a certain small positive number, then nonlinear terms* (1.6) *and* $\widehat{\boldsymbol{f}}(y,t) \equiv \boldsymbol{f}(e_{r,h}(y,t),t)$ *are subject to the inequalities*

$$\begin{aligned} Z(\boldsymbol{u},q,r) \equiv{}& \|\boldsymbol{l}_1(\boldsymbol{u},r)\|_{\boldsymbol{W}_2^{l,l/2}(\cup D_T^\pm)} + \|l_2(\boldsymbol{u},r)\|_{W_2^{1+l,0}(\cup D_T^\pm)} \\ &+ \|\boldsymbol{L}(\boldsymbol{u},r)\|_{W_2^{0,1+l/2}(Q_T)} + \|\boldsymbol{l}_3(\boldsymbol{u},r)\|_{W_2^{1/2+l,1/4+l/2}(G_T)} \\ &+ |l_4(\boldsymbol{u},r)|_{G_T}^{(1/2+l,l2)} + |l_5(r)|_{G_T}^{(1/2+l,l/2)} \\ &+ \|l_6(\boldsymbol{u},r)\|_{W_2^{3/2+l,3/4+l/2}(G_T)} \le cY^2(\boldsymbol{u},q,r), \end{aligned} \tag{3.5}$$

$$\begin{aligned} \|\widehat{\boldsymbol{f}}\|_{\boldsymbol{W}_2^{l,l/2}(Q_T)} \le c\Big\{&\|\boldsymbol{f}\|_{\boldsymbol{W}_2^{l,l/2}(Q_T)} \\ &+ \|\nabla\boldsymbol{f}\|_{Q_T} \sup_{t<T}\big(\|\mathcal{D}_t r(\,\cdot\,,t)\|_{W_2^{l+1/2}(S_{R_0})} + \|\boldsymbol{u}(\,\cdot\,,t)\|_\Omega\big)\Big\}. \end{aligned} \tag{3.6}$$

*If* $(\boldsymbol{u},r)$ *and* $(\boldsymbol{u}',r')$ *satisfy* (3.4), *then*

$$\begin{aligned} Z(\boldsymbol{u}-\boldsymbol{u}',q-q',r-r') &\le c\delta Y(\boldsymbol{u}-\boldsymbol{u}',q-q',r-r'), \\ \|\widehat{\boldsymbol{f}}-\widehat{\boldsymbol{f}}'\|_{\boldsymbol{W}_2^{l,l/2}(Q_T)} &\le c\delta Y(\boldsymbol{u}-\boldsymbol{u}',q-q',r-r'), \end{aligned} \tag{3.7}$$

*where*

$$\widehat{\boldsymbol{f}}' = \boldsymbol{f}(e_{r',\boldsymbol{h}'}(y,t),t), \qquad \boldsymbol{h}' = |B^+|^{-1}\int_0^t\int_{B^+} \boldsymbol{u}'(y,\tau)L'(y,\tau)\,dy\,d\tau$$

*and $L'$ is the Jacobian of the transformation $e_{r',\boldsymbol{h}'}$.*

Proof. Inequality (3.5) is established as in [14]. The inequality

$$\|\widehat{\boldsymbol{f}}\|_{Q_T} = \|\boldsymbol{f}(e_{r,\boldsymbol{h}}(y,t),t)\|_{Q_T} \le c\|\boldsymbol{f}\|_{Q_T}$$

is obtained by the passage to the Eulerian coordinates $x = e_{r,\boldsymbol{h}}(y,t)$ under the integral sign and taking account of the boundedness of the Jacobian $L$ which follows from (3.4). The estimates

$$\begin{aligned} \int_0^T\int_\Omega\int_\Omega & \frac{|\boldsymbol{f}(e_{r,\boldsymbol{h}}(y,t),t) - \boldsymbol{f}(e_{r,\boldsymbol{h}}(z,t),t)|^2}{|y-z|^{3+2l}}\,dz\,dy\,dt \\ &\le c\int_0^T\int_\Omega\int_\Omega \frac{|\boldsymbol{f}(x,t)-\boldsymbol{f}(x',t)|^2}{|x-x'|^{3+2l}}\,dx'\,dx\,dt, \end{aligned}$$

$$\begin{aligned} \int_0^T dt\int_0^t d\tau\int_\Omega & \frac{|\boldsymbol{f}(e_{r,\boldsymbol{h}}(y,t),t) - \boldsymbol{f}(e_{r,\boldsymbol{h}}(y,t),t-\tau)|^2}{\tau^{1+l}}\,dy \\ &\le c\int_0^T\int_0^t\int_\Omega \frac{|\boldsymbol{f}(x,t)-\boldsymbol{f}(x,t-\tau)|^2}{\tau^{1+l}}\,dx\,d\tau\,dt \end{aligned}$$

are proved in the same manner (for small $\delta$). Finally, assuming that the function $\boldsymbol{f}$ is extended outside $\Omega$ with preservation of class and making use of the relation

$$
\begin{aligned}
&\boldsymbol{f}(e_{r,\boldsymbol{h}}(y,t),t) - \boldsymbol{f}\big(e_{r,\boldsymbol{h}}(y,t-\tau),t\big)\\
&= \int_0^1 \nabla\boldsymbol{f}\bigg(e_{r,\boldsymbol{h}}(y,t) - \lambda\int_0^\tau \big(\boldsymbol{N}^*(y)\mathcal{D}_t r^*(y,t-\tau') + \dot{\boldsymbol{h}}(t-\tau')\chi(y)\big)\,d\tau', t\bigg)\,d\lambda\\
&\qquad\qquad \times \int_0^\tau \big(\boldsymbol{N}^*(y)\mathcal{D}_t r^*(y,t-\tau') + \dot{\boldsymbol{h}}(t-\tau')\chi(y)\big)\,d\tau',
\end{aligned}
$$

we obtain, in view of (1.3),

$$
\begin{aligned}
&\int_0^T dt\int_0^t \frac{d\tau}{\tau^{1+l}}\int_\Omega \big|\boldsymbol{f}(e_{r,\boldsymbol{h}}(y,t),t) - \boldsymbol{f}(e_{r,\boldsymbol{h}}(y,t-\tau),t)\big|^2\,dy\\
&\qquad \le c\,\|\nabla\boldsymbol{f}\|^2_{Q_T}\Big(\sup_{Q_T}|\mathcal{D}_t r^*(y,t)| + \sup_{t<T}\|\boldsymbol{u}(\,\cdot\,,t)\|_\Omega\Big)^2\\
&\qquad \le c\|\nabla\boldsymbol{f}\|^2_{Q_T}\Big\{\sup_{t<T}\|\mathcal{D}_t r(\,\cdot\,,t)\|^2_{W_2^{l+1/2}(S_{R_0})} + \sup_{t<T}\|\boldsymbol{u}(\,\cdot\,,t)\|^2_\Omega\Big\}.
\end{aligned}
$$

Inequality (3.7) is proved by applying the above estimates to

$$
\begin{aligned}
\boldsymbol{f}(e_{r,\boldsymbol{h}},t) - \boldsymbol{f}(e_{r',\boldsymbol{h}'},t) = \int_0^1 &\nabla\boldsymbol{f}(e_{r',\boldsymbol{h}'} + \lambda(\boldsymbol{N}^*(y,t)(r-r')\\
&+(\boldsymbol{h}-\boldsymbol{h}')\chi(y)),t)\,d\lambda(\boldsymbol{N}^*(y,t)(r-r') + (\boldsymbol{h}-\boldsymbol{h}')\chi(y)). \ \square
\end{aligned}
$$

Proof of Theorem 3.1. We go back to problem (1.5). The solution is sought in the form

$$
\boldsymbol{u} = \boldsymbol{u}' + \boldsymbol{u}'', \qquad q = q' + q'', \qquad r = r' + r'',
$$

where $(\boldsymbol{u}', q', r')$ and $(\boldsymbol{u}'', q'', r'')$ are solutions to the problems

$$
\begin{cases}
\mathcal{D}_t\boldsymbol{u}' - \nu^\pm\nabla^2\boldsymbol{u}' + \dfrac{1}{\rho^\pm}\nabla q' = 0, \quad \nabla\cdot\boldsymbol{u}' = 0 & \text{in } B^\pm,\ t>0,\\
\boldsymbol{u}'(y,0) = \boldsymbol{u}'_0(y) & \text{in } B^\pm,\\
r'(y,0) = r'_0(y) & \text{on } S_{R_0},\\
[\boldsymbol{u}']|_{S_{R_0}} = 0, \quad [\mu^\pm\Pi_0\mathbb{S}(\boldsymbol{u}')\boldsymbol{N}]\big|_{S_{R_0}} = 0, \quad \boldsymbol{u}'|_\Sigma = 0, & \\
[-q' + \mu^\pm\boldsymbol{N}\cdot\mathbb{S}(\boldsymbol{u}')\boldsymbol{N}]\big|_{S_{R_0}} - \sigma\mathcal{B}_0 r' = 0, & \\
\mathcal{D}_t r' - \bigg(\boldsymbol{u}' - \dfrac{1}{|B^+|}\displaystyle\int_{B^+}\boldsymbol{u}'\,dy\bigg)\cdot\boldsymbol{N} = 0 & \text{on } S_{R_0},\\
\displaystyle\int_\Omega q'(y,t)\,dy = 0, &
\end{cases} \tag{3.8}
$$

$$
(3.9)\qquad \begin{cases}
\mathcal{D}_t\boldsymbol{u}'' - \nu^{\pm}\nabla^2\boldsymbol{u}'' + \dfrac{1}{\rho^{\pm}}\nabla q'' = \boldsymbol{l}_1(\boldsymbol{u},q,r) + \widehat{\boldsymbol{f}}(y,t), & \\
\nabla\cdot\boldsymbol{u}'' = l_2(\boldsymbol{u},r) & \text{in } B^{\pm},\ t>0,\\
\displaystyle\int_\Omega q''(y,t)\,dy = 0, & \\
\boldsymbol{u}''(y,0) = \boldsymbol{u}_0''(y) & \text{in } B^{\pm},\\
r''(y,0) = r_0''(y) & \text{on } S_{R_0},\\
[\boldsymbol{u}'']\big|_{S_{R_0}} = 0,\quad [\mu^{\pm}\Pi_0\mathbb{S}(\boldsymbol{u}'')\boldsymbol{N}]\big|_{S_{R_0}} = \boldsymbol{l}_3(\boldsymbol{u},r),\quad \boldsymbol{u}''|_{\Sigma} = 0, & \\
[-q'' + \mu^{\pm}\boldsymbol{N}\cdot\mathbb{S}(\boldsymbol{u}'')\boldsymbol{N}]\big|_{S_{R_0}} - \sigma\mathcal{B}_0 r'' = l_4(\boldsymbol{u},r) + \sigma l_5(r), & \\
\mathcal{D}_t r'' - \left(\boldsymbol{u}'' - \dfrac{1}{|B^+|}\displaystyle\int_{B^+}\boldsymbol{u}''\,dy\right)\cdot\boldsymbol{N} = l_6(\boldsymbol{u},r) & \text{on } S_{R_0},
\end{cases}
$$

here the nonlinear terms $l_i$ are given by (1.6).

The couples of initial data $(\boldsymbol{u}_0', r_0')$ and $(\boldsymbol{u}_0'', r_0'')$ are defined as follows: $(\boldsymbol{u}_0'', r_0'')$ should satisfy the relations

$$
\begin{gathered}
\int_{S_{R_0}} r_0''(y)\,dS = \int_{S_{R_0}}\left(r_0 - \frac{\varphi(y,r_0)}{3R_0^2}\right)dS,\\
\int_{S_{R_0}} r_0''(y)y_j\,dS = \int_{S_{R_0}}\left(r_0 y_j - \frac{\psi_j(y,r_0)}{4R_0^3}\right)dS,\quad j=1,2,3,\\
[\boldsymbol{u}_0'']|_{S_{R_0}} = 0,\quad [\mu^{\pm}\Pi_0\mathbb{S}(\boldsymbol{u}_0'')\boldsymbol{N}]\big|_{S_{R_0}} = \boldsymbol{l}_3(\boldsymbol{u}_0,r_0),\\
\nabla\cdot\boldsymbol{u}_0'' = l_2(\boldsymbol{u}_0,r_0)\quad \text{in } B^+\cup B^-,\qquad \boldsymbol{u}_0''=0\quad\text{on }\Sigma,
\end{gathered}
$$

where $\varphi(y,r) = (R_0+r)^3 - R_0^3$, $\psi_j(y,r) = y_j((R_0+r)^4 - R_0^4)$ (see (1.8)), and the inequality

$$
(3.10)\quad \|\boldsymbol{u}_0''\|_{\boldsymbol{W}_2^{1+l}(\cup B^{\pm})} + \|r_0''\|_{W_2^{2+l}(S_{R_0})} \le c\varepsilon\big\{\|\boldsymbol{u}_0\|_{\boldsymbol{W}_2^{1+l}(\cup B^{\pm})} + \|r_0\|_{W_2^{2+l}(S_{R_0})}\big\}.
$$

The functions $(\boldsymbol{u}_0'', r_0'')$ can be defined in the following way (cf. [21]):

$$
r_0''(y) = \frac{I\boldsymbol{N}(y)\cdot\boldsymbol{y}}{3|B^+|} + \frac{\boldsymbol{I}\cdot\boldsymbol{N}(y)}{|B^+|},
$$

where

$$
\begin{aligned}
I &= -\int_{S_{R_0}}\frac{3r_0^2R_0 + r_0^3}{3R_0^2}\,dS,\\
I_j &= -\int_{S_{R_0}}\frac{y_j(6r_0^2R_0^2 + 4r_0^3R_0 + r_0^4)}{4R_0^3}\,dS,\quad j=1,2,3.
\end{aligned}
$$

The vector field $\boldsymbol{u}_0''$ can be taken in the form $\boldsymbol{u}_0'' = \boldsymbol{u}_1 + \boldsymbol{u}_2$, where

$$
\nabla\cdot\boldsymbol{u}_1(y) = l_2(\boldsymbol{u}_0,r_0) = \nabla\cdot(\mathbb{I}-\widehat{\mathbb{L}})\boldsymbol{u}_0 = (\mathbb{I}-\widehat{\mathbb{L}}^T)\nabla\cdot\boldsymbol{u}_0,\quad [\boldsymbol{u}_1]|_{S_{R_0}} = 0,\quad \boldsymbol{u}_1|_{\Sigma} = 0,
$$

and $\boldsymbol{u}_2^- = 0$, $\boldsymbol{u}_2^+ = \operatorname{rot}\boldsymbol{\Phi}(y)$, where

$$\boldsymbol{\Phi}|_{S_{R_0}} = \frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{N}}\Big|_{S_{R_0}} = 0,$$

$$\mu^+\frac{\partial^2\boldsymbol{\Phi}}{\partial\boldsymbol{N}^2}\Big|_{S_{R_0}} = \big(\boldsymbol{l}_3(\boldsymbol{u}_0, r_0) - [\mu^\pm\Pi_0\mathbb{S}(\boldsymbol{u}_1)\boldsymbol{N}]\big|_{S_{R_0}}\big)\times\boldsymbol{N}\big|_{S_{R_0}}.$$

Since $[\boldsymbol{u}_0] = 0$ and $[\mathbb{L}] = 0$ on $S_{R_0}$, the sufficient compatibility condition $[\boldsymbol{N}\cdot(\mathbb{I}-\widehat{\mathbb{L}})\boldsymbol{u}_0]|_{S_{R_0}} = 0$ is fulfilled, so it can be shown that $\boldsymbol{u}_0''$ satisfies (3.10).

The functions $\boldsymbol{u}_0' = \boldsymbol{u}_0 - \boldsymbol{u}_0''$, $r_0' = r_0 - r_0''$ satisfy orthogonality conditions (2.6) and compatibility ones (2.8). Consequently, by Theorem 2.3, problem (3.8) is solvable on an infinite time interval and

$$\begin{aligned}(3.11)\quad &\|e^{\beta t}\boldsymbol{u}'\|_{\boldsymbol{W}_2^{2+l,1+l/2}(\cup D_T^\pm)} + \|e^{\beta t}\nabla q'\|_{\boldsymbol{W}_2^{l,l/2}(\cup D_T^\pm)} + \|e^{\beta t}q'\|_{W_2^{0,l/2}(\cup D_T^\pm)}\\ &\quad + \|e^{\beta t}r'\|_{W_2^{5/2+l,5/4+l/2}(G_T)} + \|e^{\beta t}\mathcal{D}_t r'\|_{W_2^{3/2+l,3/4+l/2}(G_T)}\\ &\qquad\le c_0\Big\{\|\boldsymbol{u}_0'\|_{\boldsymbol{W}_2^{1+l}(\cup B^\pm)} + \|r_0'\|_{W_2^{2+l}(S_{R_0})}\Big\}\end{aligned}$$

for all $T\le\infty$ which implies (see Remark 2.2)

$$\begin{aligned}(3.12)\quad &\|\boldsymbol{u}'(\,\cdot\,,T)\|^2_{\boldsymbol{W}_2^{1+l}(\cup B^\pm)} + \|r'(\,\cdot\,,T)\|^2_{W_2^{2+l}(S_{R_0})}\\ &\qquad\le c_1 e^{-2\beta T}\Big\{\|\boldsymbol{u}_0'\|^2_{\boldsymbol{W}_2^{1+l}(\cup B^\pm)} + \|r_0'\|^2_{W_2^{2+l}(S_{R_0})}\Big\}.\end{aligned}$$

In order to prove (3.3), we fix $T = T_0$ such that $c_1 e^{-\beta T_0}\le\theta/2 < 1/2$ (one can put $\theta = e^{-bT_0}$ with $b<\beta$ and require $T_0$ to be such that $2c_1 < e^{(\beta-b)T_0}$).

As for problem (3.9), it is solvable in the case of sufficiently small $\varepsilon(T_0)$ in (3.1). Indeed, a solution can be constructed by successive approximations according to the scheme

$$(3.13)\quad\begin{cases}\mathcal{D}_t\boldsymbol{u}''_{m+1} - \nu^\pm\nabla^2\boldsymbol{u}''_{m+1} + \dfrac{1}{\rho^\pm}\nabla q''_{m+1} = \boldsymbol{l}_1(\boldsymbol{u}_m, q_m, r_m) + \widehat{\boldsymbol{f}}_m(y,t),\\ \nabla\cdot\boldsymbol{u}''_{m+1} = l_2(\boldsymbol{u}_m, r_m) \qquad \text{in } B^\pm,\ t>0,\\ \displaystyle\int_\Omega q''_{m+1}(y,t)\,dy = 0,\\ \boldsymbol{u}''_{m+1}(y,0) = \boldsymbol{u}_0''(y) \qquad \text{in } B^\pm,\\ r''_{m+1}(y,0) = r_0''(y) \qquad \text{on } S_{R_0},\\ [\boldsymbol{u}''_{m+1}]\big|_{S_{R_0}} = 0,\\ [\mu^\pm\Pi_0\mathbb{S}(\boldsymbol{u}''_{m+1})\boldsymbol{N}]\big|_{S_{R_0}} = \boldsymbol{l}_3(\boldsymbol{u}_m, r_m),\ \boldsymbol{u}''_{m+1}\big|_\Sigma = 0,\\ [-q''_{m+1} + \mu^\pm\boldsymbol{N}\cdot\mathbb{S}(\boldsymbol{u}''_{m+1})\boldsymbol{N}]\big|_{S_{R_0}} - \sigma\mathcal{B}_0 r''_{m+1}\\ \qquad = l_4(\boldsymbol{u}_m, r_m) + \sigma l_5(r_m),\\ \mathcal{D}_t r''_{m+1} - \Big(\boldsymbol{u}''_{m+1} - \dfrac{1}{|B^+|}\displaystyle\int_{B^+}\boldsymbol{u}''_{m+1}\,dy\Big)\cdot\boldsymbol{N} = l_6(\boldsymbol{u}_m, r_m) \quad \text{on } S_{R_0},\end{cases}$$

where $m = 1,2,\ldots$, $\boldsymbol{u}_m = \boldsymbol{u}' + \boldsymbol{u}''_m$, $q_m = q' + q''_m$, $r_m = r' + r''_m$, $\widehat{\boldsymbol{f}}_m = \boldsymbol{f}\big(e_{r_m,\boldsymbol{h}_m}(y,t),t\big)$, $\boldsymbol{h}_m = |B^+|^{-1}\int_0^t\int_{B^+}\boldsymbol{u}_m(y,\tau)L_m(y,\tau)\,dy\,d\tau$, $L_m = L|_{\boldsymbol{u}=\boldsymbol{u}_m}$. For $m=0$, we set $q''_0 = 0$, while let $\boldsymbol{u}''_0(y,t)$ and $r''_0(y,t)$ be functions satisfying the initial conditions $\boldsymbol{u}''_0(y,0) = \boldsymbol{u}''_0(y)$, $r''_0(y,0) = r''_0(y)$, $\mathcal{D}_t r''_0(y,0) = 0$ $\big(\boldsymbol{u}''_0(y), r''_0(y)$ are constructed above$\big)$ and the inequalities

$$
\begin{aligned}
(3.14)\qquad \|\boldsymbol{u}''_0\|_{\boldsymbol{W}_2^{2+l,1+l/2}(\cup D^\pm_{T_0})} &+ \|r''_0\|_{W_2^{5/2+l,5/4+l/2}(G_{T_0})} \\
&+ \|\mathcal{D}_t r''_0\|_{W_2^{3/2+l,3/4+l/2}(G_{T_0})} \\
&\le c\Big\{\|\boldsymbol{u}''_0\|_{\boldsymbol{W}_2^{1+l}(\cup B^\pm)} + \|r''_0\|_{W_2^{2+l}(S_{R_0})}\Big\} \\
&\le c_1\varepsilon\Big\{\|\boldsymbol{u}_0\|_{\boldsymbol{W}_2^{1+l}(\cup B^\pm)} + \|r_0\|_{W_2^{2+l}(S_{R_0})}\Big\}.
\end{aligned}
$$

Such $\boldsymbol{u}''_0$, $r''_0$ exist in view of inverse trace theorem and (3.10).

If $\boldsymbol{u}''_m$, $q''_m$, $r''_m$ are known, then $\boldsymbol{u}''_{m+1}$, $q''_{m+1}$, $r''_{m+1}$ can be found by Theorem 2.1 as a solution of (3.13). In view of (2.3), (3.5) and (3.14),

$$
(3.15)\quad Y''_{m+1} \equiv Y(\boldsymbol{u}''_{m+1}, q''_{m+1}, r''_{m+1}) \le c(T_0)\big\{\|\widehat{\boldsymbol{f}}_m\|_{\boldsymbol{W}_2^{l,l/2}(Q_{T_0})} + \varepsilon N_0 + Y_m''^2\big\},
$$

where

$$
\begin{aligned}
N_0 &\equiv \|\boldsymbol{u}_0\|_{\boldsymbol{W}_2^{1+l}(\cup B^\pm)} + \|r_0\|_{W_2^{2+l}(S_{R_0})}, \\
Y(\boldsymbol{u},q,r) &\equiv \|\boldsymbol{u}\|_{\boldsymbol{W}_2^{2+l,1+l/2}(\cup D^\pm_{T_0})} + \|\nabla q\|_{\boldsymbol{W}_2^{l,l/2}(\cup D^\pm_{T_0})} + \|q\|_{W_2^{0,l/2}(\cup D^\pm_{T_0})} \\
&\quad + \|r\|_{W_2^{5/2+l,5/4+l/2}(G_{T_0})} + \|\mathcal{D}_t r\|_{W_2^{3/2+l,3/4+l/2}(G_{T_0})}, \\
\widehat{\boldsymbol{f}}_m &\equiv \boldsymbol{f}\big(e_{r_m,\boldsymbol{h}_m}(y,t),t\big), \\
e_{r_m,\boldsymbol{h}_m}(y,t) &\equiv y + r^*_m(y,t)\boldsymbol{N}^*(y) + \chi(y)\boldsymbol{h}_m(t).
\end{aligned}
$$

We also set

$$
\begin{gathered}
Y_m \equiv Y(\boldsymbol{u}_m, q_m, r_m), \qquad Y' \equiv Y(\boldsymbol{u}', q', r'), \\
N^{(m)}(T_0) \equiv \|\boldsymbol{u}_m(\,\cdot\,,T_0)\|_{\boldsymbol{W}_2^{1+l}(\cup B^\pm)} + \|r_m(\,\cdot\,,T_0)\|_{W_2^{2+l}(S_{R_0})}, \quad m \ge 1.
\end{gathered}
$$

We show by induction that (3.15) yields an estimate for the norm $Y''_m$ (and $Y_m$) uniform on $m$.

Thus, assume that $\boldsymbol{u}_m, r_m$ satisfy (3.4) with $\delta$ so small that by (3.6), (3.11)

$$
\begin{aligned}
\|\widehat{\boldsymbol{f}}_m\|_{\boldsymbol{W}_2^{l,l/2}(Q_{T_0})} &\le c_f\|\boldsymbol{f}\|_{\boldsymbol{W}_2^{l,l/2}(Q_{T_0})} + c'_f\|\nabla\boldsymbol{f}\|_{Q_{T_0}} Y(\boldsymbol{u}_m, 0, r_m) \\
&\le c_f\|\boldsymbol{f}\|_{\boldsymbol{W}_2^{l,l/2}(Q_{T_0})} + c'_f\|\nabla\boldsymbol{f}\|_{Q_{T_0}}\big(c_0 N_0 + Y''_m\big) \\
&\le c_f\|\boldsymbol{f}\|_{\boldsymbol{W}_2^{l,l/2}(Q_{T_0})} + c_2\varepsilon N_0 + c'_f\varepsilon Y''_m.
\end{aligned}
$$

Moreover, let

$$
(3.16)\qquad Y''_m \le 2c(T_0)\Big(c_f\|\boldsymbol{f}\|_{\boldsymbol{W}_2^{l,l/2}(Q_{T_0})} + (c_2+1)\varepsilon N_0\Big).
$$

Then, on one hand, by (3.11),

$$Y_m \leq Y' + Y''_m \leq cN_0 + 2c(T_0)\Big(c_f\|\boldsymbol{f}\|_{\boldsymbol{W}_2^{l,l/2}(Q_{T_0})} + (c_2+1)\varepsilon N_0\Big) \leq c\varepsilon \tag{3.17}$$

which can guarantee the smallness of $\delta$, and on the other hand, by (3.15), (3.16),

$$\begin{aligned} Y''_{m+1} &\leq c(T_0)\Big\{c_f\|\boldsymbol{f}\|_{\boldsymbol{W}_2^{l,l/2}(Q_{T_0})} + (c_2+1)\varepsilon N_0 + (c'_f\varepsilon + Y''_m)Y''_m\Big\} \\ &\leq 2c(T_0)\Big\{c_f\|\boldsymbol{f}\|_{\boldsymbol{W}_2^{l,l/2}(Q_{T_0})} + (c_2+1)\varepsilon N_0\Big\}, \end{aligned}$$

provided that

$$2c(T_0)c'_f\varepsilon + 4c^2(T_0)\Big\{c_f\|\boldsymbol{f}\|_{\boldsymbol{W}_2^{l,l/2}(Q_{T_0})} + (c_2+1)\varepsilon N_0\Big\} \leq 1.$$

By virtue of (3.14), inequality (3.16) holds for $m = 0$, hence, it is satisfied for all $m$. In addition, from (3.12) and (3.16) it follows that

$$\begin{aligned} N^{(m)}(T_0) &\leq c\big(\mathrm{e}^{-\beta T_0} + c'(T_0)\varepsilon\big)N_0 + c''(T_0)\|\boldsymbol{f}\|_{\boldsymbol{W}_2^{l,l/2}(Q_{T_0})} \\ &\leq \theta N_0 + c''\|\boldsymbol{f}\|_{\boldsymbol{W}_2^{l,l/2}(Q_{T_0})} \end{aligned} \tag{3.18}$$

if $c\big(\mathrm{e}^{-\beta T_0} + c'(T_0)\varepsilon\big)N_0 \leq \theta$. The convergence of $(\boldsymbol{u}''_m, q''_m, r''_m)$ to a solution of (3.9) follows from inequalities (2.3) and (3.7). Letting $m \to \infty$ in (3.17), (3.18), we arrive at (3.2) and (3.3). □

Now we can complete the proof of Theorem 1.2.

Proof of Theorem 1.2. We extend a solution of (1.5) guaranteed by Theorem 3.1 into the interval $t > 0$ step by step: first to the interval $(T_0, 2T_0)$, then to $(2T_0, 3T_0)$ and so forth. Let us suppose the solution is already found for $t < kT_0$. Then it can be defined for $t \in (kT_0, (k+1)T_0)$ as a solution to the problem with the initial conditions

$$\boldsymbol{u}(y, kT_0) = \boldsymbol{u}(y, kT_0 - 0) \equiv \boldsymbol{u}_k(y), \qquad r(y, kT_0) = r(y, kT_0 - 0) \equiv r_k(y).$$

We write transformation (1.4) for $t > kT_0$ as

$$x = y + \boldsymbol{h}(kT_0)\chi(y) + \boldsymbol{k}(t,k)\chi(y) + \boldsymbol{N}^*(y)r^*(y,t), \tag{3.19}$$

where $\boldsymbol{h}(kT_0)$ is already found and $\boldsymbol{k}(t,k) = \boldsymbol{h}(t) - \boldsymbol{h}(kT_0)$. The elements of the Jacobi matrix of this transformation are given by

$$\mathbb{L}_{ij} = \left\{\delta_{ij} + (h_i(kT_0) + k_i(t,k))\frac{\partial\chi(y)}{\partial y_j} + \frac{\partial N_i^* r^*(y)}{\partial y_j}\right\}_{i,j=1}^3.$$

Proposition 3.2 can be reformulated as follows.

Proposition 3.3. *Let $k \in \mathbb{N}$. If inequality* (3.4) *holds for $t > kT_0$ and $|\boldsymbol{h}(kT_0)| \leq \delta$, then*

$$Z_k(\boldsymbol{u}, q, r) \leq c\{\delta Y_k(\boldsymbol{u}, q, r) + Y_k^2(\boldsymbol{u}, q, r)\},$$

*where $Z_k$ and $Y_k$ are norms* (3.5) *and* (2.18)*, respectively, computed for* $t \in (kT_0, (k+1)T_0)$*. Moreover,* $\widehat{\boldsymbol{f}}$ *satisfies inequalities* (3.6) *and* (3.7) *on this time interval as well.*

Let us consider the case $k = 1$. From (3.2) and (3.3) it follows that

$$N_1 \equiv N(\boldsymbol{u}_1, r_1) \le C\varepsilon,$$

hence by replacing $\varepsilon$ with $C^{-1}\varepsilon$ we see that this problem is solvable in the time interval $(T_0, 2T_0)$ and the estimates

$$\begin{aligned} Y_1(\boldsymbol{u}, q, r) &\le c\Big\{N_1 + \|\boldsymbol{f}\|_{\boldsymbol{W}_2^{l,l/2}(Q_{T_0,2T_0})}\Big\}, \\ N_2 &\le \theta N_1 + c\|\boldsymbol{f}\|_{\boldsymbol{W}_2^{l,l/2}(Q_{T_0,2T_0})} \le C\varepsilon, \end{aligned}$$

are satisfied, where $N_k = N(\boldsymbol{u}_k, r_k)$. The constants in these estimates need not coincide with the constants in (3.2), (3.3), because of the presence of the extra term with $\boldsymbol{h}(T_0)$ in (3.19), but, as will be shown below, the differences between these constants are of order $\delta$ for all $k > 0$. If the solution is found for $t < kT_0$ and the inequalities

$$\begin{aligned} &N_j^2 \le \theta^2 N_{j-1}^2 + c\|\boldsymbol{f}\|^2_{W_2^{l,l/2}(Q_{(j-1)T_0,jT_0})}, \quad \theta < 1, \\ &Y_j^2 \le c\Big\{N_{j-1}^2 + \|\boldsymbol{f}\|^2_{W_2^{l,l/2}(Q_{(j-1)T_0,jT_0})}\Big\}, \quad j = 1, \dots, k-1, \end{aligned} \tag{3.20}$$

are proved, then

$$N_j^2 \le \dots \le \theta^{2j} N_0^2 + c\sum_{i=0}^{j-1} \theta^{2(j-1-i)} \|\boldsymbol{f}\|^2_{W_2^{l,l/2}(Q_{iT_0,(i+1)T_0})} \le c\theta^{2(j-1)}\varepsilon^2 \tag{3.21}$$

with the constants $c$ independent of $j$ (we have used inequalities (1.11) for $\boldsymbol{f}$). Since $\theta^j \to 0$ as $j \to \infty$, the right-hand side of (3.21) is less than $\varepsilon^2$ for $j \ge j_0$, and the replacement of $\varepsilon$ with $C^{-1}\varepsilon$ can be made only a finite number of times.

The estimate of $\boldsymbol{h}(jT_0)$ can be obtained at every step. Let $\theta_1 > \theta$ ($\theta_1 = \mathrm{e}^{-aT_0}$, $0 < a < b$). We take the sum of (3.20) multiplied by $\theta_1^{-2j}$. This leads to

$$\begin{aligned} \sum_{j=0}^{k} \theta_1^{-2j} N_j^2 &\le N_0^2 + \frac{\theta^2}{\theta_1^2}\sum_{j=1}^{k} \theta_1^{-2j+2} N_{j-1}^2 + c\sum_{j=1}^{k} \theta_1^{-2j}\|\boldsymbol{f}\|^2_{W_2^{l,l/2}(Q_{(j-1)T_0,jT_0})} \\ &\le \frac{\theta_1^2}{\theta_1^2 - \theta^2} N_0^2 + \frac{c\theta_1^2}{\theta_1^2 - \theta^2}\sum_{j=1}^{k} \theta_1^{-2j}\|\boldsymbol{f}\|^2_{W_2^{l,l/2}(Q_{(j-1)T_0,jT_0})}. \end{aligned}$$

Hence, by embedding theorem,

$$\begin{aligned} |\boldsymbol{h}(kT_0)| &= \frac{3}{4\pi R_0^3}\bigg|\int_0^{kT_0}\int_{\Omega_t^+} \boldsymbol{v}(\,\cdot\,,t)\,dx\,dt\bigg| \\ &\le c\sqrt{T_0}\bigg(\sum_{j=0}^{k-1} \theta_1^{-2j}\int_{jT_0}^{(j+1)T_0} \|\boldsymbol{u}(\,\cdot\,,t)\|^2_{W_2^{l+1}(B^+)}\,dt\bigg)^{1/2} \end{aligned} \tag{3.22}$$

$$\leq c\bigg(N_0^2 + \sum_{j=0}^{k-1} \theta_1^{-2j} \|\boldsymbol{f}\|^2_{W_2^{l,l/2}(Q_{jT_0,(j+1)T_0})}\bigg)^{1/2} \leq c\varepsilon$$

with the constants $c$ independent of $k$.

Finally, by passing to the limit as $k \to \infty$ in

$$\sum_{j=0}^{k} \theta_1^{-2j} Y_j^2(\boldsymbol{u}, q, r) \leq c\bigg\{N_0^2 + \sum_{j=0}^{k} \theta_1^{-2j} \|\boldsymbol{f}\|^2_{W_2^{l,l/2}(Q_{jT_0,(j+1)T_0})}\bigg\},$$

we arrive at an inequality equivalent to (1.12). In addition, the passage to the limit in (3.22) allows us to estimate the limit position of inner drop barycenter $h(\infty)$:

$$|h(\infty)| \leq c_2\Big\{\|\mathrm{e}^{at}\boldsymbol{f}\|_{W_2^{l,l/2}(Q_\infty)} + \|\boldsymbol{u}_0\|_{\boldsymbol{W}_2^{1+l}(\cup B^\pm)} + \|r_0\|_{W_2^{2+l}(S_{R_0})}\Big\} \leq 2c_2\varepsilon. \tag{3.23}$$

From (1.12) and embedding theorems, it follows that

$$\max_{G_\infty} |r| \leq c_1\Big\{\|\mathrm{e}^{at}\boldsymbol{f}\|_{W_2^{l,l/2}(Q_\infty)} + \|\boldsymbol{u}_0\|_{\boldsymbol{W}_2^{1+l}(\cup B^\pm)} + \|r_0\|_{W_2^{2+l}(S_{R_0})}\Big\} \leq 2c_1\varepsilon.$$

It is clear that if $2(c_1 + c_2)\varepsilon$ is less than the initial distance between the surfaces $\Gamma_t$ and $\Sigma$, the intersection of these surfaces will be never possible. □

Let us show that one can construct a solution to problem (1.5) under less restrictive assumptions on $\boldsymbol{f}$. We introduce the norms

$$\begin{aligned} |||\boldsymbol{u}, q, r||| &= \sum_{j=0}^{\infty} Y_j(\boldsymbol{u}, q, r), \\ |||\boldsymbol{f}||| &= \sum_{j=0}^{\infty} \|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_{jT_0,(j+1)T_0})}, \\ |||\boldsymbol{f}|||_\eta &= \sum_{j=0}^{\infty} \eta_j^{-1} \|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_{jT_0,(j+1)T_0})}, \end{aligned} \tag{3.24}$$

where $\{\eta_j\}_0^\infty \in (0,1)$, $\eta_{j+1} \leq \eta_j$ and $\eta_j \to 0$ as $j \to \infty$.

Theorem 3.4. *Let $\boldsymbol{u}_0 \in W_2^{l+1}(\cup B^\pm)$, $r_0 \in W_2^{l+2}(S_{R_0})$ and let $\boldsymbol{f}$ have finite norms* (3.24)*. Assume that compatibility conditions* (1.9)*, as well as smallness conditions* (1.10) *and the inequalities*

$$\sup_{\tau > 0} \|\mathcal{D}_x^{\boldsymbol{i}} \boldsymbol{f}\|_{Q_{\tau,\tau+T_0}} \leq \varepsilon, \quad |\boldsymbol{i}| = 1, 2, \qquad |||\boldsymbol{f}||| + |||\boldsymbol{f}|||_\eta \leq \varepsilon$$

*are satisfied. Then there exists a solution of* (1.5) *defined for $t > 0$ and the estimate*

$$|||\boldsymbol{u}, q, r||| \leq c\Big\{\|\boldsymbol{u}_0\|_{W_2^{l+1}(\cup B^\pm)} + \|r_0\|_{W_2^{l+2}(S_{R_0})} + |||\boldsymbol{f}|||\Big\} \tag{3.25}$$

*holds.*

Proof. We follow the arguments in the proof of Theorem 1.1 presented above. From the inequalities

$$\begin{aligned} &N_j \le \theta N_{j-1} + c\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_{(j-1)T_0,jT_0})},\\ &Y_j \le c\Big(N_{j-1} + c\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_{jT_0,(j+1)T_0})}\Big), \quad j \in \mathbb{N}, \end{aligned}$$

equivalent to (3.20), it follows that

$$N_j \le \theta^j N_0 + c\sum_{i=0}^{j-1}\theta^{j-1-i}\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_{iT_0,(i+1)T_0})} \le \theta^j N_0 + c\varkappa_j|||\boldsymbol{f}|||_\eta,$$

where

$$\varkappa_j = \max_{i\le(j-1)}\theta^{j-1-i}\eta_i \le \max\big(\theta^{[(j-1)/2]},\eta_{[(j-1)/2]}\big) \to 0 \quad \text{as } j\to\infty,$$

$[k]$ means the integral part of $k$. Hence the solution of (1.5) is extendable to the whole half-axis $t>0$ (if $h(kT_0)$ is small); moreover, we have

$$\begin{aligned} \sum_{j=1}^k N_j &\le \frac{1}{1-\theta}\bigg\{N_0 + c\sum_{i=0}^k \|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_{iT_0,(i+1)T_0})}\bigg\},\\ \sum_{j=0}^k Y_j(\boldsymbol{u},q,r) &\le c\bigg\{N_0 + \sum_{i=0}^{k-1}\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_{iT_0,(i+1)T_0})}\bigg\}. \end{aligned}$$

Using this inequality we estimate $h(kT_0)$: $|h(kT_0)| \le c\,\varepsilon$ and, letting $k\to\infty$, we arrive at (3.25). □

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