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ON THE FAEDO–GALERKIN METHOD FOR A FREE BOUNDARY PROBLEM FOR INCOMPRESSIBLE VISCOUS MAGNETOHYDRODYNAMICS

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I devote the paper to Professor Marek Burnat my boss and friend
(Wojciech Zajączkowski)

Abstract. The motion of incompressible magnetohydrodynamics (mhd) in a domain bounded by a free surface and coupled through it with external electromagnetic field is considered. Transmission conditions for electric currents and magnetic fields are prescribed on the free surface. Although we show the idea of a proof of local existence by the method of successive approximations, we are not going to prove neither local nor global existence of solutions. The existence of solutions of the linearized problems (the Stokes system for velocity and pressure and the linear transmission problem for the electromagnetic fields) is the main step in the proof of existence to the considered problem. This can be done either by the Faedo–Galerkin method or by the technique of regularizer. We concentrate our considerations to the Faedo–Galerkin method. For this we need an existence of a fundamental basis. We have to find the basis for the Stokes system and mhd system. We concentrate our considerations on the mhd system because this for the Stokes system is well known. We have to emphasize that the considered mhd system is obtained after linearization and transformation to the initial domains by applying the Lagrangian coordinates. This is the main aim of this paper.

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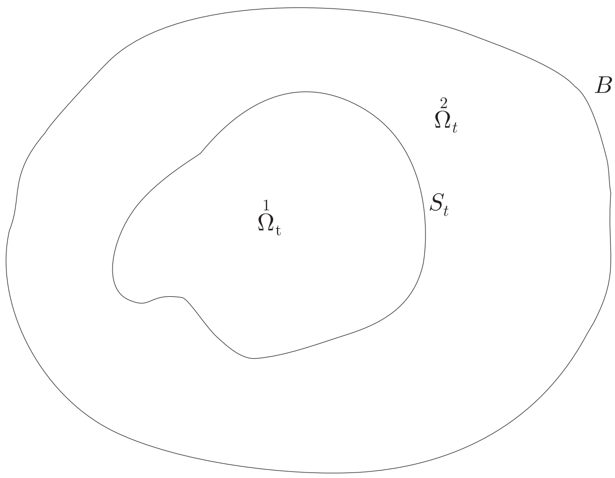
1. Introduction

We consider a free boundary problem for magnetohydrodynamic motions in a domain $\overset{1}{\Omega}_t$ interacted through free surface S_t with an electromagnetic field located in $\overset{2}{\Omega}_t$. In $\overset{1}{\Omega}_t$ the magnetohydrodynamic motion is described by the system of equations



Picture 1

$$
\begin{gathered}
v_{,t} + v \cdot \nabla v - \operatorname{div} \mathbb{T}(v,p) - \mu_1 \overset{1}{H} \cdot \nabla \overset{1}{H} + \frac{1}{2}\mu_1 \nabla \overset{1}{H}{}^2 = f, \\
\operatorname{div} v = 0, \qquad \mu_1 \overset{1}{H}_{,t} = -\operatorname{rot} \overset{1}{E}, \\
\operatorname{rot} \overset{1}{H} = \sigma_1(\overset{1}{E} + \mu_1 v \times \overset{1}{H}), \qquad \operatorname{div} \overset{1}{H} = 0,
\end{gathered} \tag{1.1}
$$

where $v = v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t)) \in \mathbb{R}^3$ is the velocity of the fluid, $p = p(x,t) \in \mathbb{R}$ is the pressure, $\overset{1}{H}(x,t) = \big(\overset{1}{H}_1(x,t), \overset{1}{H}_2(x,t), \overset{1}{H}_3(x,t)\big) \in \mathbb{R}^3$ is the magnetic field, $\overset{1}{E} = \overset{1}{E}(x,t) = \big(\overset{1}{E}_1(x,t), \overset{1}{E}_1(x,t), \overset{1}{E}_3(x,t)\big) \in \mathbb{R}^3$ is the electric field, $f = f(x,t) = (f_1(x,t), f_2(x,t), f_3(x,t)) \in \mathbb{R}$ is the external force field per unit mass, $x = (x_1, x_2, x_3)$ are the Cartesian coordinates. Moreover, μ_1 is the constant magnetic permeability and σ_1 the constant electric conductivity. By $\mathbb{T}(v,p)$, we denote the stress tensor of the form

$$
\mathbb{T}(v,p) = \nu \mathbb{D}(v) - pI, \tag{1.2}
$$

where ν is positive viscosity coefficient, I is the unit matrix and $\mathbb{D}(v)$ is the dilatation tensor of the form

$$
\mathbb{D}(v) = \{v_{i,x_j} + v_{j,x_i}\}_{i,j=1,2,3}. \tag{1.3}
$$

For system (1.1) the following initial and boundary conditions are prescribed

$$
\begin{gathered}
\overline{n}\cdot\mathbb{T}(v,p)+\mu_1\overline{n}\cdot\mathbb{T}(\overset{1}{H})=p_0\overline{n}\quad\text{on } S_t,\\
v|_{t=0}=v(0),\qquad \overset{1}{H}\big|_{t=0}=\overset{1}{H}(0),\qquad \overset{1}{\Omega}_t\big|_{t=0}=\overset{1}{\Omega}_0,\qquad S_t|_{t=0}=S_0,
\end{gathered}\tag{1.4}
$$

where $\overline{n}$ is the unit outward to $\overset{1}{\Omega}_t$ and normal to S_t vector,

$$
\mathbb{T}(\overset{1}{H})=\Big\{\overset{1}{H}_i\overset{1}{H}_j-\frac{1}{2}\overset{1}{H}{}^2\delta_{ij}\Big\}_{i,j=1,2,3}.\tag{1.5}
$$

Finally, $(1.4)_1$ implies the following compatibility conditions

$$
\begin{gathered}
\overline{n}_0\cdot\mathbb{D}(v(0))\cdot\overline{\tau}_{\alpha0}+\mu_1\overline{n}_0\cdot\overset{1}{H}(0)\overline{\tau}_{\alpha0}\cdot\overset{1}{H}(0)=0,\\
\alpha=1,2,\qquad \overline{n}_0=\overline{n}|_{t=0},\qquad \overline{\tau}_{\alpha0}=\overline{\tau}_\alpha|_{t=0}.
\end{gathered}
$$

In $\overset{2}{\Omega}_t$ we have a motionless dielectric gas under the constant pressure p_0. Therefore, we have only an electromagnetic field described by the following system of equations

$$
\mu_2\overset{2}{H}_{,t}=-\operatorname{rot}\overset{2}{E},\qquad \sigma_2\overset{2}{E}=\operatorname{rot}\overset{2}{H},\qquad \operatorname{div}\overset{2}{H}=0.\tag{1.6}
$$

For system (1.6) the following system of initial and boundary conditions is prescribed

$$
\overset{2}{H}\big|_{t=0}=\overset{2}{H}(0),\qquad \overset{2}{\Omega}_t\big|_{t=0}=\overset{2}{\Omega}_0,\qquad \overset{2}{H}\big|_B=0.\tag{1.7}
$$

The homogeneous boundary condition on B is assumed for simplicity only. We can prescribe here either a magnetic field or an electric current. Electromagnetic fields in domains $\overset{1}{\Omega}_t$ and $\overset{2}{\Omega}_t$, respectively, are coupled through S_t by the following transmission conditions

$$
\begin{gathered}
\overset{1}{E}\cdot\overline{\tau}_\alpha=\overset{2}{E}\cdot\overline{\tau}_\alpha|_{S_t},\\
\overline{n}\times\overline{\tau}_\alpha\cdot\overset{1}{H}=\overline{n}\times\overline{\tau}_\alpha\cdot\overset{2}{H}\big|_{S_t},\quad \alpha=1,2,\\
\mu_1\overset{1}{H}\cdot\overline{n}=\mu_2\overset{2}{H}\cdot\overline{n}|_{S_t},
\end{gathered}\tag{1.8}
$$

where $\overline{\tau}_1$, $\overline{\tau}_2$, $\overline{n}$ is an orthonormal system of vectors in a neighbourhood of S_t such that $\overline{n}|_{S_t}$ is normal to S_t and $\overline{\tau}_1$, $\overline{\tau}_2|_{S_t}$ are tangent to S_t.

Now we explain the physical meaning of the transmission conditions (1.8). The currents are defined by $j_i=\sigma_i\overset{i}{E}$, $i=1,2$. Therefore $(1.8)_1$ means that the jump of the tangent components of currents is described, by the equality

$$
\frac{1}{\sigma_1}j_1\cdot\overline{\tau}_\alpha=\frac{1}{\sigma_1}j_2\cdot\overline{\tau}_\alpha,\quad \alpha=1,2,\ \text{on } S_t,\ \text{for } \sigma_1\neq\sigma_2.
$$

Conditions $(1.8)_2$ means that tangent components of magnetic field are continuous passing through S_t. To describe $(1.8)_3$ we recall the magnetic induction

$\overset{i}{B} = \mu_i \overset{i}{H}$, $i = 1, 2$. Since $\operatorname{div} \overset{i}{B} = 0$ in $\overset{i}{\Omega}_t$ we derive $(1.8)_3$. It means that the normal component of the magnetic induction is continuous on S_t. Hence there is no jump of the magnetic induction flux.

To prove existence of solutions to problem (1.1)–(1.8) we transform it into two problems: problem for the fluid motion and problem for the electromagnetic field. Therefore, for given $\overset{1}{H}$ we have the problem for (v, p):

$$
\begin{aligned}
&v_t + v \cdot \nabla v - \operatorname{div} \mathbb{T}(v, p) = f + \mu_1 \operatorname{div} \mathbb{T}\big(\overset{1}{H}\big) && \text{in } \overset{1}{\Omega}_t,\\
&\operatorname{div} v = 0 && \text{in } \overset{1}{\Omega}_t,\\
&\overline{n} \cdot \mathbb{T}(v, p) = p_0 \overline{n} - \mu_1 \cdot \mathbb{T}\big(\overset{1}{H}\big) && \text{on } S_t,\\
&v|_{t=0} = v(0) && \text{in } \Omega_0.
\end{aligned} \tag{1.9}
$$

Next, for given v, the electromagnetic field is determined by the problem

$$
\begin{aligned}
&\mu_1 \overset{1}{H}_{,t} = -\operatorname{rot} \overset{1}{E}, \qquad \operatorname{rot} \overset{1}{H} = \sigma_1\big(\overset{1}{E} + \mu_1 v \times \overset{1}{H}\big), && \text{in } \overset{1}{\Omega}_t,\\
&\mu_2 \overset{2}{H}_{,t} = -\operatorname{rot} \overset{2}{E}, \qquad \sigma_2 \overset{2}{E} = \operatorname{rot} \overset{2}{H}, && \text{in } \overset{2}{\Omega}_t,\\
&\overset{1}{H}|_{t=0} = \overset{1}{H}(0), \qquad \operatorname{div} \overset{1}{H}(0) = 0, && \text{in } \overset{1}{\Omega}_0,\\
&\overset{2}{H}|_{t=0} = \overset{2}{H}(0), \qquad \operatorname{div} \overset{2}{H}(0) = 0, && \text{in } \overset{2}{\Omega}_0,\\
&\overset{2}{H}|_B = 0,\\
&\overset{1}{E} \cdot \overline{\tau}_\alpha = \overset{2}{E} \cdot \overline{\tau}_\alpha, \quad \overline{n} \times \overline{\tau}_\alpha \cdot \overset{1}{H} = \overline{n} \times \overline{\tau}_\alpha \cdot \overset{2}{H}, && \alpha = 1, 2,\\
&\mu_1 \overset{1}{H} \cdot \overline{n} = \mu_2 \overset{2}{H} \cdot \overline{n}, && \text{on } S_t.
\end{aligned} \tag{1.10}
$$

Since (1.9), (1.10) are free boundary problems; the natural way to treat them is passing to the Lagrangian coordinates (see [24]). Therefore domains $\overset{1}{\Omega}_t$, $\overset{2}{\Omega}_t$ and S_t are determined by the velocity v of the fluid. However, equations (1.10) are not in the form appropriate for the use of the Lagrangian coordinates. Moreover, in $\overset{2}{\Omega}_t$ there is no motion, so there is no velocity guaranteeing existence of Lagrangian coordinates. Therefore, we construct an artificial velocity $\overset{2}{v}$ in $\overset{2}{\Omega}_t$ as a solution to problem (3.2). Moreover, to the field equations $(1.10)_{1,2}$,

$$
\mu_i \overset{i}{H}_t = -\operatorname{rot} \overset{i}{E}, \quad i = 1, 2, \tag{1.11}
$$

we add the term $\mu_i \overset{i}{v} \cdot \nabla \overset{i}{H}$, $i = 1, 2$, to both sides of (1.11), so we have

$$
\mu_i\big(\overset{i}{H}_t + \overset{i}{v} \cdot \nabla \overset{i}{H}\big) = -\operatorname{rot} \overset{i}{E} + \mu_i \cdot \overset{i}{v} \cdot \nabla \overset{i}{H}, \quad i = 1, 2. \tag{1.12}
$$

These equations can be formulated in the Lagrangian coordinates and the term on the r.h.s. of (1.12) is of the lower order. Therefore the natural way to show the existence of solutions to (1.9), (1.10) is the method of successive approximations described in Section 3. In this paper we are not going to prove existence of solutions to problem (1.9), (1.10) (the result is shown in [8], [18], [19]). Our aim is a justification of the energy method used in [8], [9] to prove the existence of solutions to problem (1.9), (1.10). The existence is proved by using the Faedo–Galerkin method applied to linearized problem (1.9), (1.10) formulated in the fixed initial domains. Problems (3.21) and (3.27) are exactly such problems. In Sections 4 and 5 the existence of the fundamental bases necessary for applying the Faedo–Galerkin method are shown. Hence to formulate the main results we recall problem (3.21) and (3.27). To solve the eigenvalue problem for (3.21) we examine the elliptic problem (see (5.13))

$$
\begin{aligned}
&\mu_i\overset{i}{\psi} + \frac{1}{\sigma_i}\operatorname{rot}^2\overset{i}{\psi} = \lambda\overset{i}{\psi}, && \text{in } \overset{i}{\Omega}_0,\ i=1,2,\\
&\frac{1}{\sigma_1}\operatorname{rot}\overset{i}{\psi}\cdot\overline{\tau}_\alpha = \frac{1}{\sigma_2}\operatorname{rot}\overset{2}{\psi}\cdot\overline{\tau}_\alpha + g\cdot\overline{\tau}_\alpha, && \alpha=1,2,\ \text{on } S_0,\\
&\overset{1}{\psi}\cdot\overline{n}\times\overline{\tau}_\alpha = \overset{2}{\psi}\cdot\overline{n}\times\overline{\tau}_\alpha, && \alpha=1,2,\\
&\overset{2}{\psi}\big|_B = 0,
\end{aligned}
\tag{1.13}
$$

and, for problem (3.27), the problem

$$
\begin{aligned}
\nu\varphi - \operatorname{div}\mathbb{T}(\varphi,q) &= \lambda\varphi && \text{in } \Omega_0,\\
\operatorname{div}\varphi &= g && \text{in } \Omega_0,\\
\overline{n}\cdot\mathbb{T}(\varphi,q) &= h && \text{on } S_0,
\end{aligned}
\tag{1.14}
$$

where λ is an eigenvalue and μ_1, μ_2, ν are positive numbers.

Theorem 1.1. *There exists eigenvalues and eigenfunctions for problems* (1.13) *and* (1.14).

Proof. In [23] Temam proved existence of eigenvalues and eigenfuctions to problem (1.14). The same properties for problem (1.13) are proved in Section 5.□

Hence the paper is organized in the following way. In Section 2 a notation and some auxiliary results are introduced. In Section 3 the method of successive approximations is formulated. In Sections 4 and 5 the existence of fundamental basis to problems (3.27) and (3.21) is proved, respectively.

The existence of the fundamental basis for the Faedo–Galerkin for problem (3.27) is presented in Definition 4.1 and for problem (3.21) in Lemmas 5.2, 5.3.

Having the existence of weak solutions to the linearized problems by the Faedo–Galerkin method, the existence of solutions to nonlinear problems (1.9), (1.10) is proved by the method of successive approximations described in Section 3 in [8]. Global existence is proved in [9] for sufficiently small initial data. The existence of local solutions to (1.9), (1.10) is proved in [8] by the energy method. To examine problem (1.10) it is necessary to use the transmission conditions (1.8). In Lemma 2.5 transmission conditions (1.8) are generalized to the form (2.16) and then the fundamental energy identity takes the form (2.17). Transmission condition (2.16) implies relations between tangent components of the magnetic fields. To show Lemma 2.3, which gives estimate for solutions to problem (2.6), we also need a transmission condition for the normal component of the magnetic fields (see $(2.6)_3$). To relax condition $(2.6)_3$ we see that equations (2.18), (2.19) are invariant with respect to the homotetia transformation

$$\overset{i}{H}{}' = b_i \overset{i}{H}, \quad \overset{i}{E}{}' = b_i \overset{i}{E}, \quad i = 1,2, \tag{1.15}$$

where b_i, $i = 1,2$, are constants. Then the transmission condition (2.16) for tangent components of magnetic fields and the homotetia transformation (1.15) imply that the following transmission condition for magnetic fields is admissible

$$\alpha_1 \overset{1}{H} = \alpha_2 \overset{2}{H} \quad \text{on } S_t, \tag{1.16}$$

where α_1, α_2 are arbitrary constants.

The transmission condition (1.16) in the case $\alpha_1 = \alpha_2 = 1$ simplifies very strongly the proof of local and global estimates in [8, 9], respectively. However, condition (1.16) implies a jump of tangent components of the electric field which is described by relations $(1.15)_2$ and $(2.25)_1$. But, (1.16) realizes only in the energy equality (2.17). We have to emphasize that this paper plays a fundamental role in the proof of local existence of solutions to problem (1.1)–(1.8) in [8]. The idea of the proof in [8] is based on the following steps. In the first, by applying linearization described in Section 3, the existence of weak solutions is proved by the Faedo–Galerkin method, so the existence of the fundamental basis presented in this paper is crucial. Next by the standard technique the regularity of weak solutions is increased. Finally, by the method of successive approximations the existence of local solutions is proved.

However the Maxwell equations form a symmetric hyperbolic system the analitical treatment is difficult because they are expressed in the form of the rotation operators. Bykhovsky in [1] derived many analitical results for solutions to elliptic rot-div systems. Magnetohydrodynamics (mhd) is a coupling of the Navier–Stokes equations with the Maxwell equations under neglecting the displacement currents and taking into account the electrical conduction. The mhd equations can be found in [15, Chapter 8].

The first result on solvability of a transmission problem for mhd system was proved by Ladyzhenskaya and Solonnikov in [14]. In this paper fixed domain were considered. The first results on existence of solutions to problem (1.1)–(1.10) were shown by Kacprzyk [5]–[7]. The free boundary problems to mhd system were also considered by Padula–Solonnikov [16], Frolova–Solonnikov [4] and Frolova [3].

In these papers the external magnetic field satisfies the elliptic system

$$\operatorname{rot} \overset{2}{H} = 0, \qquad \operatorname{div} \overset{2}{H} = 0. \tag{1.17}$$

However, transmission condition (1.4) is different because it contains additionally the surface tension.

Hence, on the one hand the problem considered in [16], [4], [3] is simpler than problem (1.1)–(1.10) but, on the other hand, is more complicated because the surface tension is taken into account. Moreover, in papers [16], [4], [3] the passing from the free boundary to a problem with the fixed boundary is made by using the Hanzawa transformation. This is an essential difference with respect to [5]–[9], where the Lagrangian coordinates were used. Finally, in [18], [19] problem (1.1)–(1.10) was considered by Shibata–Zajączkowski in the L_p-approach using also the Lagrangian coordinates.

The paper is organized in the following form. In Section 2 we introduce some notation and the Lagrangian coordinates. Next we prove Lemma 2.3 necessary for showing existence of eigenvalues and eigenfunctions to problem (5.7). In Lemma 2.5 the existence of very general transmission conditions is shown. In Section 3 the idea of the proof of local existence by the method of successive approximations is presented. In Section 4 the existence of the fundamental basis to the Stokes system is presented. In this case the results of [23] are used. Finally, in Section 5, the existence of eigenfunctions and eigenvalues to problem (5.7) is shown. The problem follows from (1.10) by transformation to the Lagrangian coordinates, linearization and the elliptic problem by an appropriate Laplace transformation. Therefore, we do not need that function $\overset{i}{\psi}$, $i = 1, 2$, appearing in (5.7) are divergence free. Next it is shown that natural space $\overline{H}(\Omega)$ for problem (5.7) defined by (5.10) is equivalent to space $H(\Omega)$ (defined by (5.12)) on solutions to problem (5.7) (see Lemma 5.1). Next the existence of full system of eigenvectors is proved (see Lemma 5.2).

Remark 1.2. In this paper the following transmission condition for the magnetic field is considered

$$\overset{1}{H} = \overset{2}{H} \quad \text{on } S_t. \tag{1.18}$$

This condition is motivated by one of the following two physical restrictions

(a) $\mu_1 = \mu_2$.
(b) $\mu_1 \overset{1}{H}_\tau = \mu_2 \overset{2}{H}_\tau$ on S_t.

Since in this paper the main technique to derive estimates for the magnetic fields is energy type estimate (2.17) we need transmission condition $(2.16)_2$ for the tangent components of magnetic fields.

Assuming (b) and using that $\mu_1 \overset{1}{H}_n = \mu_2 \overset{2}{H}_n$ we get by homotetia transformation (1.17) transmission condition (1.18).

2. Notation and auxiliary results

First we introduce notation employed in this paper. We do not distinguish between norms of scalar and vector-valued functions. Let ω be a vector, $\omega = (\omega_1, \dots, \omega_n)$. Then

$$|\omega| = \bigg(\sum_{i=1}^n |\omega_i|^2\bigg)^{1/2}.$$

Let

$$L_p(\Omega) = \bigg\{u : \int_\Omega |u|^p\,dx < \infty\bigg\}, \quad p \in [1, \infty].$$

By $V_2^0(\Omega^T)$ we denote a space of functions with the finite norm

$$\|u\|_{V_2^0(\Omega^T)} = \|u\|_{L_\infty(0,T;L_2(\Omega))} + \|u\|_{L_2(0,T;H^1(\Omega))}.$$

We shall use the notation

$$H^l(\Omega) = \bigg\{u : \sum_{|\alpha| \le l} \|D_x^\alpha u\|_{L_2(\Omega)} < \infty\bigg\},$$

where $D_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$, $|\alpha| = \alpha_1+\alpha_2+\alpha_3$, $\alpha_i \in \mathbb{N}_0$, $i = 1,2,3$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\Omega \subset \mathbb{R}^3$.

By c we denote a generic constant which changes its value from formula to formula. Similarly we denote by φ a generic function which is always positive and increasing.

To examine free boundary problems in hydrodynamics we use the Lagrangian coordinates which are the initial data to the following Cauchy problem

$$\frac{dx}{dt} = v(x,t), \quad x|_{t=0} = \xi \in \overset{1}{\Omega}_0. \tag{2.1}$$

Therefore,

$$x = x_v(\xi,t) \equiv \xi + \int_0^t \overline{v}(\xi,s)\,ds, \tag{2.2}$$

where $\overline{v}(\xi,t) = v(x_v(\xi,t),t)$. To define the Lagrangian coordinates in $\overset{2}{\Omega}_t$ we need

Lemma 2.1 (see [20]). *Let $X(\overset{1}{\Omega}_t)$ be some Sobolev space. Let $v \in X(\overset{1}{\Omega}_t)$ be a divegence free. Then there exists an extension v' of v on $\overset{1}{\Omega}_t \cup \overset{2}{\Omega}_t$ such that v' is divergence free, $v'|_{\overset{1}{\Omega}_t} = v$ and there exists a constant c such that*

$$\|v'\|_{X\left(\overset{1}{\Omega}_t \cup \overset{2}{\Omega}_t\right)} \le c\|v\|_{X\left(\overset{1}{\Omega}_t\right)}. \tag{2.3}$$

We make the extension more precisely using Lemma 2.4 to problem (3.2). In view of the definition of Lagrangian coordinates we have

$$\overset{1}{\Omega}_t = \Big\{x \in \mathbb{R}^3 : x = x_v(\xi,t),\ \xi \in \overset{1}{\Omega}_0\Big\}, \quad S_t = \{x \in \mathbb{R}^3 : x = x_v(\xi,t),\ \xi \in S_0\},$$
$$\overset{1}{\Omega}_t \cup \overset{2}{\Omega}_t = \Big\{x \in \mathbb{R}^3 : x = x_{v'}(\xi,t),\ \xi \in \overset{1}{\Omega}_0 \cup \overset{2}{\Omega}_0\Big\}.$$

To formulate our problem in the Lagrangian coordinates we need the notation

$$\begin{aligned} &\nabla_{\overline{v}} = \frac{\partial \xi_k}{\partial x}\frac{\partial}{\partial \xi_k}, && \mathbb{D}_{\overline{v}}\overline{u} = \nabla_{\overline{v}}\overline{u} + (\nabla_{\overline{v}}\overline{u})^T, \\ &\mathbb{T}_{\overline{v}}(\overline{u},\overline{p}) = \mathbb{D}_{\overline{v}}(\overline{u}) - \overline{p}\mathbb{I}, && \operatorname{div}_{\overline{v}}\overline{v} = \partial_{x_i}\xi_k\partial_{\xi_k}\overline{v}_i = \nabla_{\overline{v}}\cdot\overline{v}, \end{aligned} \tag{2.4}$$

where the summation over the repeated indices is assumed, $\xi = \xi(x,t)$ is the inverse transformation to $x = x_{\overline{v}}(\xi,t)$. From [24], [22] we have

LEMMA 2.2. *Let $\Omega \subset \mathbb{R}^3$ be a given bounded domain. Let $v \in L_2(\Omega)$ be such that*

$$E_\Omega(v) = \int_\Omega (v_{j,x_i} + v_{i,x_j})^2\,dx. \tag{2.5}$$

Then there exists a constant c such that

$$\|v\|^2_{H^1(\Omega)} \le \big(E_\Omega(v) + \|v\|^2_{L_2(\Omega)}\big).$$

Let us consider the set of functions

$$\operatorname{div}\overset{i}{H} = 0 \quad \text{in } \overset{i}{\Omega}_0,\ i = 1,2, \qquad \overset{1}{H} = \overset{2}{H} \quad \text{on } S_0, \qquad \overset{2}{H}|_B = 0. \tag{2.6}$$

LEMMA 2.3. *Assume that* $\operatorname{rot}\overset{i}{H} \in L_2\big(\overset{i}{\Omega}_0\big)$, $i = 1,2$, *and* $S_0 \in H^{3-1/2}$. *Then, for functions from the set* (2.6) *such that* $\overset{i}{H} \in H^1\big(\overset{i}{\Omega}_0\big)$, $i = 1,2$, *the estimate holds*

$$\sum_{i=1}^2 \big\|\overset{i}{H}\big\|_{H^1(\overset{i}{\Omega}_0)} \le c\sum_{i=1}^2 \big\|\operatorname{rot}\overset{i}{H}\big\|_{L_2(\overset{i}{\Omega}_0)}. \tag{2.7}$$

PROOF. Note that without loss of generality we may assume that functions satisfying (2.6) are regular. This can be made by using mollifiers. Indeed if Ω_0 is a Lipschitz domain then it can be decomposed in a finite set of star-shape sets. Therefore we can use standard approximation method to approximate arbitrary H^1-function by a sequence of smooth one. Then to prove the lemma we introduce the identity (see [1])

$$\int_{\overset{i}{\Omega}} \big(\operatorname{rot}\overset{i}{H}\big)^2\,dx = \int_{\overset{i}{\Omega}} \big|\nabla\overset{i}{H}\big|^2\,dx - \int_S \bigg[\big(\overset{i}{H}\times\operatorname{rot}\overset{i}{H}\big)\cdot\overset{i}{\overline{n}} + \sum_{k=1}^3 \overset{i}{H}_k\nabla\overset{i}{H}_k\cdot\overset{i}{\overline{n}}\bigg]\,dS, \tag{2.8}$$

where $i = 1,2$, $\overset{i}{\overline{n}}$ is the unit outward vector to $\overset{i}{\Omega}$ normal to S and index 0 in $\overset{i}{\Omega}$ and S is dropped.

To prove (2.7) we have to examine the boundary term in (2.8). According to homotetia transformation (1.15) and Lemma 2.5 we can make such transmission conditions that H is continuous passing the free boundary S. Moreover, the tangent components of electric field E are also continuous passing S. Therefore, to examine the boundary term in (2.8) we introduce a partition of unity $\zeta^{(k)}$ such that $S \subset \bigcup_k \operatorname{supp} \zeta^{(k)}$. In each $\operatorname{supp} \zeta^{(k)}$ we introduce local coordinates and perform transformation making $S^{(k)} = S \cap \operatorname{supp} \zeta^{(k)}$ flat. We denote it by $\hat{S}^{(k)}$. Assume that coordinates are such that $\hat{S}^{(k)}$ is the plane $x_3 = 0$. Then $\overline{n} = (0,0,1)$ and the transformed vector H is $H = (H_1, H_2, H_3)$. Then

$$\begin{aligned} &\overline{n} \times H = (-H_2, H_1, 0), \\ &\operatorname{rot} H = (H_{2,x_3} - H_{3,x_2}, -H_{1,x_3} + H_{3,x_1}, H_{1,x_2} - H_{2,x_1}). \end{aligned}$$

Hence

$$\begin{aligned} I &\equiv \overline{n} \times H \cdot \operatorname{rot} H + H_i H_{i,x_3} \\ &= -H_2(H_{2,x_3} - H_{3,x_2}) + H_1(-H_{1,x_3} + H_{3,x_1}) \\ &\quad + H_1 H_{1,x_3} + H_2 H_{2,x_3} + H_3 H_{3,x_3} \\ &= H_1 H_{3,x_1} + H_2 H_{3,x_2} + H_3 H_{3,x_3} \end{aligned}$$

where $H_{3,x_3} = -H_{1,x_1} - H_{2,x_2}$. It follows that I is continuous passing S.

In virtue of the performed transformations we get the additional term

$$I_1 = \sum_k a_{ij}^k H_i H_j$$

expressed in local coordinates which is also continuous on S. Then, from (2.8), we have

$$\sum_{i=1}^{2} \big\|\nabla \overset{i}{H}\big\|^2_{L_2(\overset{i}{\Omega})} \leq c \sum_{i=1}^{2} \big\|\operatorname{rot} \overset{i}{H}\big\|^2_{L_2(\overset{i}{\Omega})}. \tag{2.9}$$

Hence (2.9) holds for regular functions. Then, by the density argument, (2.9) holds also for $\overset{i}{H} \in H^1(\overset{i}{\Omega})$, $i = 1, 2$.

Finally, we obtain an estimate for $\sum_{i=1}^{2} \|\overset{i}{H}\|^2_{L_2(\overset{i}{\Omega})}$. For simplicity we introduce the notation: $\overset{i}{g} = \operatorname{rot} \overset{i}{H}$, $i = 1, 2$.

Applying the Poincaré inequality in $\overset{2}{\Omega}$ and using that $\overset{2}{H}\big|_B = 0$ we have

$$\big\|\overset{2}{H}\big\|_{L_2(\overset{2}{\Omega})} \leq c \big\|\nabla \overset{2}{H}\big\|_{L_2(\overset{2}{\Omega})} \leq c \sum_{i=1}^{2} \big\|\overset{i}{g}\big\|_{L_2(\overset{i}{\Omega})}. \tag{2.10}$$

Hence, we have the estimate

$$\big\|\overset{2}{H}\big\|_{H^1(\overset{2}{\Omega})} \leq c \sum_{i=1}^{2} \big\|\overset{i}{g}\big\|_{L_2(\overset{i}{\Omega})}. \tag{2.11}$$

Since $\overset{2}{H} \in H^1(\overset{2}{\Omega})$ the trace of $\overset{2}{H}$ on S belongs to $H^{1/2}(S)$. By the transmission conditions $(2.6)_2$ we have

$$\|\overset{1}{H}\|_{H^{1/2}(S)} \le c\|\overset{2}{H}\|_{H^{1/2}(S)} \le c\sum_{i=1}^{2}\|\overset{i}{g}\|_{L_2(\overset{i}{\Omega})}. \tag{2.12}$$

Applying again the Poincaré inequality in $\overset{1}{\Omega}$ we derive

$$\|\overset{1}{H}\|_{L_2(\overset{1}{\Omega})} \le c\Big(\|\nabla\overset{1}{H}\|_{L_2(\overset{1}{\Omega})} + \|\overset{1}{H}\|_{L_2(S)}\Big) \le c\sum_{i=1}^{2}\|\overset{i}{g}\|_{L_2(\overset{i}{\Omega})}. \tag{2.13}$$

From (2.9), (2.10) and (2.13) we obtain estimate (2.7). $\square$

Let us consider the Stokes problem in a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary S

$$\begin{aligned} \omega_{,t} - \operatorname{div}\mathbb{T}(\omega,q) &= f && \text{in } \Omega^T = \Omega\times(0,T),\\ \operatorname{div}\omega &= 0 && \text{in } \Omega^T,\\ \omega &= b && \text{on } S^T = S\times(0,T),\\ \omega|_{t=0} &= \omega_0 && \text{in } \Omega. \end{aligned} \tag{2.14}$$

Lemma 2.4 (see [20]). *Assume that* $f \in L_p(\Omega^T)$, $b \in W_p^{2-1/p,1-1/2p}(S^T)$, $\omega_0 \in W_p^{2-2/p}(\Omega)$, $p \in (1,\infty)$, $S \in C^2$. *Then there exists a solution to problem* (2.14) *such that* $\omega \in W_p^{2,1}(\Omega^T)$, $\nabla q \in L_p(\Omega^T)$ *and the estimate holds*

$$\begin{aligned} &\|\omega\|_{W_p^{2,1}(\Omega^T)} + \|\nabla q\|_{L_p(\Omega^T)}\\ &\qquad \le c(\|f\|_{L_p(\Omega^T)} + \|b\|_{W_p^{2-1/p,1-1/2p}(S^T)} + \|\omega_0\|_{W_p^{2-2/p}(\Omega)}). \end{aligned} \tag{2.15}$$

Now we justify the transmission condition from (1.10) To obtain the energy type estimate for solutions to problem (1.10) we need

Lemma 2.5. *Assume the following transmission conditions on* S_t

$$a_1^{\nu_1}\overset{1}{E}\cdot\overline{\tau}_\alpha = a_2^{\nu_1}\overset{2}{E}\cdot\overline{\tau}_\alpha, \qquad a_1^{\nu_2}\overline{n}\times\overline{\tau}_\alpha\cdot\overset{1}{H} = a_2^{\nu_2}\overline{n}\times\overline{\tau}_\alpha\cdot\overset{2}{H}, \tag{2.16}$$

where $\alpha = 1,2$, $\nu_1+\nu_2 = 1$, $0 \le \nu_i \le 1$, $i = 1,2$, a_1, a_2 *are positive constants. Then the following equality holds*

$$\sum_{i=1}^{2}\bigg[a_i\mu_i\int_{\overset{i}{\Omega}_t}\overset{i}{H}_{,t}\cdot\overset{i}{H}\,dx + a_i\int_{\overset{i}{\Omega}_t}\overset{i}{E}\cdot\operatorname{rot}\overset{i}{H}\,dx\bigg] = 0. \tag{2.17}$$

Proof. We write equations $(1.10)_{1,2}$ in the form

$$\mu_1\overset{1}{H}_{,t} = -\operatorname{rot}\overset{1}{E}, \quad \overset{1}{E} = \frac{1}{\sigma_1}\operatorname{rot}\overset{1}{H} - \mu_1 v\times\overset{1}{H}, \quad \operatorname{div}\overset{1}{H} = 0 \qquad \text{in } \overset{1}{\Omega}_t, \tag{2.18}$$

$$\mu_2\overset{2}{H}_{,t} = -\operatorname{rot}\overset{2}{E}, \quad \overset{2}{E} = \frac{1}{\sigma_2}\operatorname{rot}\overset{2}{H}, \quad \operatorname{div}\overset{2}{H} = 0 \quad \text{in } \overset{2}{\Omega}_t. \tag{2.19}$$

From (2.18) and (2.19) we have

$$\sum_{i=1}^{2}\int_{\overset{i}{\Omega}_t} a_i\mu_i\overset{i}{H}_{,t}\cdot\overset{i}{H}\,dx+\sum_{i=1}^{2}\int_{\overset{i}{\Omega}_t} a_i\mathrm{rot}\,\overset{i}{E}\cdot\overset{i}{H}\,dx=0. \tag{2.20}$$

To obtain any energy type estimate we have to integrate by parts in the second term. Moreover, we need that any boundary term on S_t does not appear. For this we shall need the transmission condition (2.16).

Let us recall the identity

$$\int_{\Omega_t}\mathrm{rot}\,H\cdot\psi\,dx=\int_{\Omega_t}H\cdot\mathrm{rot}\,\psi\,dx-\int_{S_t}\overline{n}\times H\cdot\psi\,dS_t, \tag{2.21}$$

where $\overline{n}$ is the unit exterior vector to Ω_t and normal to S_t. From (2.21) we have

$$\int_{\overset{1}{\Omega}_t}\mathrm{rot}\,\overset{1}{E}\cdot\overset{1}{H}\,dx=\int_{\overset{1}{\Omega}_t}\overset{1}{E}\cdot\mathrm{rot}\,\overset{1}{H}\,dx-\int_{S_t}\overset{1}{\overline{n}}\times\overset{1}{E}\cdot\overset{1}{H}\,dS_t, \tag{2.22}$$

$$\int_{\overset{2}{\Omega}_t}\mathrm{rot}\,\overset{2}{E}\cdot\overset{2}{H}\,dx=\int_{\overset{2}{\Omega}_t}\overset{2}{E}\cdot\mathrm{rot}\,\overset{2}{H}\,dx-\int_{S_t}\overset{2}{\overline{n}}\times\overset{2}{E}\cdot\overset{2}{H}\,dS_t, \tag{2.23}$$

where $\overset{i}{\overline{n}}$ is exterior to $\overset{i}{\Omega}_t$ and $\overset{1}{\overline{n}}=-\overset{2}{\overline{n}}$. Using (2.22) and (2.23) in (2.20) we derive

$$\begin{aligned}\sum_{i=1}^{2}\int_{\overset{i}{\Omega}_t} a_i\mu_i\overset{i}{H}_{,t}\cdot Hidx+\sum_{i=1}^{2}\int_{\overset{i}{\Omega}_t} a_i\overset{i}{E}\cdot\mathrm{rot}\,\overset{i}{H}\,dx&\\ -\int_{S_t}\big(a_1\overset{1}{\overline{n}}\times\overset{1}{E}\cdot\overset{1}{H}-a_2\overset{1}{\overline{n}}\times\overset{2}{E}\cdot\overset{2}{H}\big)\,dS_t&=0.\end{aligned} \tag{2.24}$$

The boundary term must vanish because otherwise (2.24) does not imply any estimate. The boundary term only contains tangent components of $\overset{i}{E}$ and $\overset{i}{H}$, $i=1,2$. Let $\overline{\tau}_1$, $\overline{\tau}_2$, $\overline{n}$ be an orthonormal system of vectors (see [11, Chapter 2, Section 18]). Then we have the expansion

$$\overset{i}{E}=\sum_{\alpha=1}^{2}\overset{i}{E}\cdot\overline{\tau}_\alpha\overline{\tau}_\alpha+\overset{i}{E}\cdot\overline{nn},\quad i=1,2, \tag{2.25}$$

where $\overline{n}=\overset{1}{\overline{n}}$. Then the boundary term in (2.24) equals

$$I=-\sum_{\alpha=1}^{2}\int_{S_t}\big[a_1\overset{1}{E}\cdot\overline{\tau}_\alpha\overline{n}\times\overline{\tau}_\alpha\cdot\overset{1}{\overline{H}}-a_2\overset{2}{E}\cdot\overline{\tau}_\alpha\overline{n}\times\overline{\tau}_\alpha\cdot\overset{2}{H}\big]\,dS_t.$$

Hence, the transmission condition (2.16) implies that I vanishes.

In the case $a_1=a_2=1$, transmission conditions (2.16) assume the form $(1.10)_6$. □

Let us consider the problem

$$
(2.26)\qquad
\begin{aligned}
v_t - \operatorname{div}\mathbb{T}(v,q) &= f && \text{in } \Omega^T,\\
\operatorname{div} v &= 0 && \text{in } \Omega^T,\\
\overline{n}\cdot\mathbb{T}(v,q) &= g && \text{on } S^T,\\
v|_{t=0} &= v_0 && \text{in } \Omega.
\end{aligned}
$$

Lemma 2.6 (see [21]). *Assume that $f\in L_p(\Omega^T)$, $g\in W_p^{1-1/p,1/2-1/2p}(S^T)$, $v_0\in W_p^{2-2/p}(\Omega)$, $p\in(1,\infty)$, $S\in C^2$. Then there exists a solution to problem (2.26) such that $v\in W_p^{2,1}(\Omega^T)$, $\nabla q\in L_p(\Omega^T)$ and the estimate holds*

$$
(2.27)\qquad \|v\|_{W_p^{2,1}(\Omega^T)}+\|\nabla q\|_{L_p(\Omega^T)} \le c(\|f\|_{L_p(\Omega^T)}+\|g\|_{W_p^{1-1/p,1/2-1/2p}(S^T)}+\|v_0\|_{W_p^{2-2/p}(\Omega)}).
$$

3. Method of successive approximations

Let $v_n=v_n(x,t)$ be given, $x\in\overset{1}{\Omega}_t$.

Definition 3.1. The Lagrangian coordinates in $\overset{1}{\Omega}_0$ are the initial data to the Cauchy problem

$$
(3.1)\qquad \frac{dx}{dt}=v_n(x,t),\qquad x|_{t=0}=\xi\in\overset{1}{\Omega}_0.
$$

Hence domain $\overset{1}{\Omega}_{nt}$ is defined by

$$
\overset{1}{\Omega}_{nt}=\left\{x\in\mathbb{R}^3: x=x^{(n)}(\xi,t)=\xi+\int_0^t\overline{v}_n(\xi,t')dt',\ \xi\in\overset{1}{\Omega}_0\right\},
$$

where $\overline{v}_n(\xi,t)=v_n(x^{(n)}(\xi,t),t)$.

In free boundary problems in hydrodynamics the free boundary is built up from the same fluid particles because $v_n|_{S_{nt}}$ is tangent to S_{nt} and

$$
S_{nt}=\{x\in\mathbb{R}^3: x=x^{(n)}(\xi,t),\ \xi\in S_0\}.
$$

To formulate problem (1.10) in the Lagrangian coordinates we have to introduce them in the domain $\overset{2}{\Omega}_0$. Since there is no velocity in $\overset{2}{\Omega}_t$ we have to introduce it artificially (see Lemma 2.1).

Definition 3.2. Let us denote $\overset{1}{v}_n = v_n$ in $\overset{1}{\Omega}_t$ and define $\overset{2}{v}_n$ in $\overset{2}{\Omega}_t$ as a solution to the nonstationary Stokes system

$$
\begin{aligned}
\overset{2}{v}_{n,t} - \operatorname{div} \mathbb{T}(\overset{2}{v}_n, q_n) &= 0 \qquad \text{in } \overset{2}{\Omega}_t, \\
\operatorname{div} \overset{2}{v}_n &= 0 \qquad \text{in } \overset{2}{\Omega}_t, \\
\overset{2}{v}_n\big|_{S_t} = \overset{1}{v}_n\big|_{S_t}, \quad \overset{2}{v}_n\big|_B &= 0, \\
\overset{2}{v}_n\big|_{t=0} &= \overset{2}{v}(0) \quad \text{in } \overset{2}{\Omega}_0,
\end{aligned}
\tag{3.2}
$$

where q_n plays a role of a pressure but it is not important for any estimate for $\overset{2}{v}_n$. The initial data $\overset{2}{v}(0)$ is an extension of $\overset{1}{v}(0)$ through the fixed given boundary S_0, because $\overset{2}{v}(0)\big|_{S_0} = \overset{1}{v}(0)\big|_{S_0}$. The extension can be made by applying Lemma 2.1. The existence of solutions to (3.2) follows from Lemma 2.4.

Now, we introduce the Lagrangian coordinates $\overset{1}{\xi}$, $\overset{2}{\xi}$ as the initial data to the problems

$$
\frac{d\overset{i}{x}}{dt} = \overset{i}{v}_n(x,t), \quad x^i|_{t=0} = \overset{i}{\xi} \in \overset{i}{\Omega}_0, \quad i = 1,2.
\tag{3.3}
$$

Then

$$
\begin{aligned}
\overset{i}{\Omega}_{nt} = \bigg\{ \overset{i}{x} \in \mathbb{R}^3 : \overset{i}{x} = \overset{i}{x}{}^{(n)}(\overset{i}{\xi}, t) &= \overset{i}{\xi} + \int_0^t \overset{i}{v}_n(\overset{i}{x}, t')\, dt' \\
&= \overset{i}{\xi} + \int_0^t \overset{i}{\overline{v}}_n(\overset{i}{\xi}, t') dt', \overset{i}{\xi} \in \overset{i}{\Omega}_0 \bigg\},
\end{aligned}
\tag{3.4}
$$

where $\overset{i}{\overline{v}}_n(\overset{i}{\xi}, t) = \overset{i}{v}_n(\overset{i}{x}{}^{(n)}(\overset{i}{\xi}, t), t)$, $\overset{i}{\xi} \in \overset{i}{\Omega}_0$, $i = 1,2$.

Formulation of the method of successive approximations. Let v in problem (1.10) be given. We set $v = v_n$. To emphasize that v_n describes the motion in $\overset{1}{\Omega}_t$ we write $v_n = \overset{1}{v}_n$. Then, by Definition 3.2, we have $\overset{2}{v}_n$ in $\overset{2}{\Omega}_t$. Passing to the Lagrangian coordinates expressed by (3.3), (3.4) we can write problem (1.10) as a problem for $\overset{i}{\overline{H}}_n$, $i = 1,2$, in the form

$$
\begin{aligned}
\mu_1 \overset{1}{\overline{H}}_{nt} + \operatorname{rot}_{\overset{1}{\overline{v}}_n} \bigg[\frac{1}{\sigma_1} \Big(\operatorname{rot}_{\overset{1}{\overline{v}}_n} \overset{1}{\overline{H}}_n - \mu_1 \overset{1}{\overline{v}}_n \times \overset{1}{\overline{H}}_n \Big) \bigg] &= \mu_1 \overset{1}{\overline{v}}_n \cdot \nabla_{\overset{1}{\overline{v}}_n} \overset{1}{\overline{H}}_n, \\
\operatorname{div}_{\overset{1}{\overline{v}}_n} \overset{1}{\overline{H}}_n &= 0
\end{aligned}
\tag{3.5}
$$

in $\overset{1}{\Omega}_0 \times (0,t)$,

$$
\begin{aligned}
&\mu_2 \overset{2}{H}_{nt} + \operatorname{rot}_{\overset{2}{\overline{v}}_n} \left[\frac{1}{\sigma_2} \operatorname{rot}_{\overset{2}{\overline{v}}_n} \overset{2}{H} \right] = \mu_2 \overset{2}{\overline{v}}_n \cdot \nabla_{\overset{2}{\overline{v}}_n} \overset{2}{H}_n, \\
&\operatorname{div}_{\overset{2}{\overline{v}}_n} \overset{2}{H}_n = 0,
\end{aligned} \tag{3.6}
$$

in $\overset{2}{\Omega}_0 \times (0,t)$,

$$
\frac{1}{\sigma_1} \left(\operatorname{rot}_{\overset{1}{\overline{v}}_n} \overset{1}{H}_n - \mu_1 \overset{1}{\overline{v}}_n \times \overset{1}{H}_n \right) \cdot \overline{\tau}_{\overline{v}_n\alpha} = \frac{1}{\sigma_2} \operatorname{rot}_{\overset{2}{\overline{v}}_n} \overset{2}{H}_n \cdot \overline{\tau}_{\overline{v}_n\alpha}, \quad \alpha = 1,2, \quad \text{on } S_0 \times (0,t), \tag{3.7}
$$

$$
\overline{n}_{\overline{v}_n} \times \overline{\tau}_{\overline{v}_n\alpha} \cdot \left(\overset{1}{H} - \overset{2}{H} \right) = 0, \quad \alpha = 1,2, \quad \text{on } S_0 \times (0,t), \tag{3.8}
$$

$$
\mu_1 \overline{n}_{\overline{v}_n} \cdot \overset{1}{H}_n = \mu_2 \overline{n}_{\overline{v}_n} \cdot \overset{2}{H}_n \qquad \text{on } S_0 \times (0,t), \tag{3.9}
$$

$$
\overset{2}{H}_n \big|_B = 0 \qquad \text{on } B \times (0,t), \tag{3.10}
$$

$$
\overset{i}{H} \big|_{t=0} = \overset{i}{H}(0), \qquad \text{in } \overset{i}{\Omega}_0, \ i = 1,2, \tag{3.11}
$$

where

$$
\nabla_{\overline{v}_n} = \frac{\partial \xi}{\partial x^{(n)}} \bigg|_{x^{(n)} = x^{(n)}(\xi,t)} \cdot \nabla_\xi
$$

and any operator with index $\overline{v}_n$ means that it contains the transformed gradient $\nabla_{\overline{v}_n}$. Moreover, $\overset{1}{\overline{v}}_n = \overset{2}{\overline{v}}_n = \overline{v}_n$ on S_0.

For $\overline{v}_n$ given we have $\overset{i}{\overline{v}}_n$, $i = 1,2$, and also domains $\overset{i}{\Omega}_{nt}$, $i = 1,2$, by formulations (3.4). Hence, by the Lagrangian coordinates, problem (3.5)–(3.11) for quantities $\overset{i}{H}_n$, $i = 1,2$, is formulated in the fixed (independent of time) domains $\overset{i}{\Omega}_0$, $i = 1,2$.

We have to emphasize that terms on the r.h.s. of $(3.5)_1$ and $(3.6)_1$ follow from expressing problem (1.10) in the Lagrangian coordinates. This means that the Lagrangian coordinates for problem (1.10) are introduced artificially. But, they are introduced because the formulation of a free boundary problem for the fluid mechanics equations in these coordinates is very natural and simple. The main point is that the derived problem is formulated in a fixed initial domain.

For given $\overset{i}{\overline{v}}_n$, $i = 1,2$, problem (3.5)–(3.11) for variable $\overline{H}^i_n$, $i = 1,2$, is linear. However, to prove the existence of solutions we have to use the method of successive approximations. For this purpose we write problem (3.5)–(3.11) in the form

$$\text{(3.12)}\quad \mu_1\overset{1}{\overline{H}}_{nt}+\frac{1}{\sigma_1}\operatorname{rot}{}^2_\xi\overset{1}{\overline{H}}_n=\frac{1}{\sigma_1}\Big(\operatorname{rot}{}^2_\xi\overset{1}{\overline{H}}_n-\operatorname{rot}{}^2_{\overset{1}{\overline{v}}_n}\overset{1}{\overline{H}}_n\Big)+\mu_1\operatorname{rot}_{\overset{1}{\overline{v}}_n}\Big(\overset{1}{\overline{v}}_n\times\overset{1}{\overline{H}}_n\Big)+\mu_1\overset{1}{\overline{v}}_n\cdot\nabla_{\overset{1}{\overline{v}}_n}\overset{1}{\overline{H}}_n\equiv\overset{1}{f},$$

in $\overset{1}{\Omega}_0\times(0,t)$,

$$\text{(3.13)}\qquad \operatorname{div}_\xi\overset{1}{\overline{H}}_n=\operatorname{div}_\xi\overset{1}{\overline{H}}-\operatorname{div}_{\overset{1}{\overline{v}}_n}\overset{1}{\overline{H}}\equiv\overset{1}{g},$$

$$\text{(3.14)}\quad \mu_2\overset{2}{\overline{H}}_{nt}+\frac{1}{\sigma_2}\operatorname{rot}{}^2_\xi\overset{2}{\overline{H}}_n=\frac{1}{\sigma_2}\bigg(\operatorname{rot}{}^2_\xi\overset{2}{\overline{H}}_n-\operatorname{rot}{}^2_{\overset{2}{\overline{v}}_n}\overset{2}{\overline{H}}_n\bigg)+\mu_2\overset{2}{\overline{v}}_n\cdot\nabla_{\overset{2}{\overline{v}}_n}\overset{2}{\overline{H}}_n\equiv\overset{2}{f},$$

in $\overset{2}{\Omega}_0\times(0,t)$,

$$\text{(3.15)}\qquad \operatorname{div}_\xi\overset{2}{\overline{H}}_n=\operatorname{div}_\xi\overset{2}{\overline{H}}_n-\operatorname{div}_{\overset{2}{\overline{v}}_n}\overset{2}{\overline{H}}_n\equiv\overset{2}{g},$$

$$\text{(3.16)}\quad \bigg(\frac{1}{\sigma_1}\operatorname{rot}_\xi\overset{1}{\overline{H}}_n-\frac{1}{\sigma_2}\operatorname{rot}_\xi\overset{2}{\overline{H}}_n\bigg)\cdot\overline{\tau}_\alpha=\frac{1}{\sigma_1}\bigg(\operatorname{rot}_\xi\overset{1}{\overline{H}}_n\cdot\overline{\tau}_\alpha-\operatorname{rot}_{\overset{1}{\overline{v}}_n}\overset{1}{\overline{H}}_n\cdot\overline{\tau}_{\overline{v}_n\alpha}\bigg)-\frac{1}{\sigma_2}(\operatorname{rot}_\xi\overset{2}{\overline{H}}_n\cdot\overline{\tau}_\alpha-\operatorname{rot}_{\overset{2}{\overline{v}}_n}\overset{2}{\overline{H}}_n\cdot\overline{\tau}_{\overline{v}_n\alpha})+\frac{\mu_1}{\sigma_1}\overset{1}{\overline{v}}_n\times\overset{1}{\overline{H}}_n\cdot\overline{\tau}_{\overline{v}_n\alpha}\equiv g_\alpha,$$

for $\alpha=1,2$, on $S_0\times(0,t)$,

$$\text{(3.17)}\quad \overline{n}\times\overline{\tau}_\alpha\cdot\bigg(\overset{1}{\overline{H}}_n-\overset{2}{\overline{H}}_n\bigg)=(\overline{n}\times\overline{\tau}_\alpha-\overline{n}_{\overline{v}_n}\times\overline{\tau}_{\overline{v}_n\alpha})\cdot\bigg(\overset{1}{\overline{H}}_n-\overset{2}{\overline{H}}_n\bigg)\equiv k_\alpha,$$

for $\alpha=1,2$, on $S_0\times(0,t)$,

$$\text{(3.18)}\qquad \mu_1\overline{n}\cdot\overset{1}{\overline{H}}_n-\mu_2\overline{n}\cdot\overset{2}{\overline{H}}_n=\mu_1(\overline{n}-\overline{n}_{\overline{v}_n})\cdot\overset{1}{\overline{H}}_n-\mu_2(\overline{n}-\overline{n}_{\overline{v}_n})\cdot\overset{2}{\overline{H}}\equiv l$$

on $S_0\times(0,t)$,

$$\text{(3.19)}\qquad \overset{2}{\overline{H}}_n\Big|_B=0\qquad\text{on } B\times(0,t),$$

$$\text{(3.20)}\qquad \overset{i}{\overline{H}}_n\Big|_{t=0}=\overset{i}{H}(0)\quad\text{in } \overset{i}{\Omega}_0,\ i=1,2.$$

For given $\overset{1}{\overline{v}}_n$, $\overset{2}{\overline{v}}_n$ we prove the existence of solutions to problem (3.12)–(3.20) in two steps.

First we consider the problem with constant coefficients

$$
\begin{aligned}
\mu_i \overset{i}{\overline{H}}_{nt} + \frac{1}{\sigma_i}\operatorname{rot}{}^2_\xi \overset{i}{\overline{H}}_n = \overset{i}{f}, \quad \operatorname{div}{}_\xi \overset{i}{\overline{H}}_n &= \overset{i}{g} && \text{in } \overset{i}{\Omega}_0 \times (0,t),\ i=1,2,\\
\left(\frac{1}{\sigma_1}\operatorname{rot}{}_\xi \overset{1}{\overline{H}}_n - \frac{1}{\sigma_2}\operatorname{rot}{}_\xi \overset{2}{\overline{H}}_n\right)\cdot \overline{\tau}_\alpha &= g_\alpha, && \alpha = 1,2, \text{ on } S_0 \times (0,t),\\
(\overline{n}\times\overline{\tau}_\alpha)\cdot(\overset{1}{\overline{H}}_n - \overset{2}{\overline{H}}_n) &= k_\alpha, && \alpha = 1,2, \text{ on } S_0 \times (0,t),\\
\mu_1 \overline{n}\cdot \overset{1}{\overline{H}}_n &= \mu_2 \overline{n}\cdot \overset{2}{\overline{H}}_n && \text{on } S_0\times(0,t),\\
\overset{2}{\overline{H}}_n &= 0 && \text{on } B\times(0,t),\\
\overset{i}{\overline{H}}_n\Big|_{t=0} &= \overset{i}{H}(0) && \text{in } \overset{i}{\Omega}_0,\ i=1,2.
\end{aligned}
\tag{3.21}
$$

The existence of solutions to (3.21) can be shown either by applying the Faedo-Galerkin method (see [8]) or by the technique of regularizer (see [18], [19]). To use the Faedo–Galerkin method we need the fundamental basis for problem (3.21). Existence of such fundamental basis will be shown in Section 5.

Having the existence of solutions to problem (3.21) with an appropriate regularity of the r.h.s. functions we show the existence of solutions to problem (3.12)–(3.20) by the method of successive approximations for sufficiently small time. For this we replace in the r.h.s. functions of (3.12)–(3.20) $\overset{i}{\overline{H}}_n$ by $\overset{i}{\overline{H}}{}^{(m)}_n$, $m \in \mathbb{N}$, and in the l.h.s. functions by $\overset{i}{\overline{H}}{}^{(m+1)}_n$. Then problem (3.12)–(3.20) implies the mapping

$$
\Big(\overset{1}{\overline{H}}{}^{(m+1)}_n, \overset{2}{\overline{H}}{}^{(m+1)}_n\Big) = \Phi\Big(\overset{1}{\overline{H}}{}^{(m)}_n, \overset{2}{\overline{H}}{}^{(m)}_n\Big),
\tag{3.22}
$$

which for a sufficiently small time and regular $\overset{1}{\overline{v}}_n$, $\overset{2}{\overline{v}}_n$ has a fixed point implied by the method of successive approximations. In this way the existence of solutions to problem (3.12)–(3.20) is proved and the following functional dependence holds

$$
\Big(\overset{1}{\overline{H}}_n, \overset{2}{\overline{H}}_n\Big) = F(\overline{v}_n),
\tag{3.23}
$$

where we used that $\overset{1}{\overline{v}}_n = \overline{v}_n$ and $\overset{2}{\overline{v}}_n$ is described by problem (3.2).

The functional dependence (3.23) is expressed by the following estimate

$$
\sum_{i=1}^{2} \Big\|\overset{i}{\overline{H}}_n\Big\|_{1,\overset{i}{\Omega}{}^t_0} \le \varphi\Big(\|\overline{v}_n\|_{2,\overset{1}{\Omega}{}^t_0}\Big),
\tag{3.24}
$$

where the norms $\|\cdot\|_{1,\overset{i}{\Omega}{}^t_0}$, $\|\cdot\|_{2,\overset{1}{\Omega}{}^t_0}$, $\overset{i}{\Omega}{}^t_0 = \overset{i}{\Omega}_0 \times (0,t)$, $i=1,2$, are found in [8], [18], [19] and φ is an increasing positive function.

To calculate the next step v_{n+1} in the method of successive approximations applied to problem (1.1)–(1.8) we use problem (1.9). Expressing (1.9) in the Lagrangian coordinates we have

$$
\begin{aligned}
\overline{v}_{n+1,t} - \operatorname{div}_{\overline{v}_n}\mathbb{T}_{\overline{v}_n}(\overline{v}_{n+1},\overline{p}_{n+1}) &= \overline{f} + \mu_1 \operatorname{div}_{\overline{v}_n}\mathbb{T}(\overline{\overset{1}{H}}_n) && \text{in } \overset{1}{\Omega}{}^t_0,\\
\operatorname{div}_{\overline{v}_n}\overline{v}_{n+1} &= 0 && \text{in } \overset{1}{\Omega}{}^t_0,\\
\overline{n}_{\overline{v}_n}\cdot\mathbb{T}_{\overline{v}_n}(\overline{v}_{n+1},\overline{p}_{n+1}) &= p_0\overline{n}_{\overline{v}_n} - \mu_1\overline{n}_{\overline{v}_n}\mathbb{T}(\overline{\overset{1}{H}}_n) && \text{on } S^t_0,\\
\overline{v}_{n+1}|_{t=0} &= v(0) && \text{in } \overset{1}{\Omega}_0,
\end{aligned}
\tag{3.25}
$$

where we used the simplified notation $\overline{v}_n = \overline{\overset{1}{v}}_n$, $\overline{v}_{n+1} = \overline{\overset{1}{v}}_{n+1}$. Moreover, $\overline{v}_n$ and $\overline{\overset{1}{H}}_n$ are given, where $\overline{v}_n$ describes prescribed n-th step in the method of successive approximations and $\overline{\overset{1}{H}}_n$ depends on $\overline{v}_n$ by formula (3.23).

To prove the existence of solutions to problem (3.25) we formulate it in the form

$$
\begin{aligned}
&\overline{v}_{n+1,t} - \operatorname{div}_{\xi}\mathbb{T}_{\xi}\big(\overline{v}_{n+1},\overline{p}_{n+1}\big) = -\big(\operatorname{div}_{\xi}\mathbb{T}_{\xi}(\overline{v}_{n+1},\overline{p}_{n+1})\\
&\qquad - \operatorname{div}_{\overline{v}_n}\mathbb{T}_{\overline{v}_n}(\overline{v}_{n+1},\overline{p}_{n+1})\big) + \mu_1\operatorname{div}_{\overline{v}_n}\mathbb{T}\Big(\overline{\overset{1}{H}}_n\Big) + \overline{f} \equiv f_0 && \text{in } \overset{1}{\Omega}{}^t_0,\\
&\operatorname{div}_{\xi}\overline{v}_{n+1} = \operatorname{div}_{\xi}\overline{v}_{n+1} - \operatorname{div}_{\overline{v}_n}\overline{v}_{n+1} \equiv g_0 && \text{in } \overset{1}{\Omega}{}^t_0,\\
&\overline{n}_{\xi}\mathbb{T}_{\xi}(\overline{v}_{n+1},\overline{p}_{n+1}) = \overline{n}_{\xi}\mathbb{T}_{\xi}(\overline{v}_{n+1},\overline{p}_{n+1})\\
&\qquad - \overline{n}_{\overline{v}_n}\mathbb{T}_{\overline{v}_n}(\overline{v}_{n+1},\overline{p}_{n+1}) + p_0\overline{n}_{\overline{v}_n} - \mu_1\overline{n}_{\overline{v}_n}\cdot\mathbb{T}(\overline{\overset{1}{H}}_n) \equiv k_0 && \text{on } S^t_0,\\
&\overline{v}_{n+1}|_{t=0} = v(0) && \text{in } \overset{1}{\Omega}_0.
\end{aligned}
\tag{3.26}
$$

To describe the way of proving the existence of solutions to problem (3.26) we repeat the approach applied to problem (3.12)–(3.20). Therefore, we first consider the problem

$$
\begin{aligned}
\overline{v}_{n+1,t} - \operatorname{div}_{\xi}\mathbb{T}_{\xi}(\overline{v}_{n+1},\overline{p}_{n+1}) &= f_0,\\
\operatorname{div}_{\xi}\overline{v}_{n+1} &= g_0,\\
\overline{n}_{\xi}\cdot\mathbb{T}_{\xi}(\overline{v}_{n+1},\overline{p}_{n+1}) &= k_0,\\
\overline{v}_{n+1}|_{t=0} &= v(0).
\end{aligned}
\tag{3.27}
$$

The existence of solutions to problem (3.27) can be proved either by the Faedo–Galerkin method (see Section 4) or by the technique of regularizer (see Lemma 2.6). Having the existence of solutions to problem (3.27) we prove existence of solutions to (3.26) by the method of successive approximations such that the r.h.s. functions depend on $\overline{v}^{(m)}_{n+1}$, $\overline{p}^{(m)}_{n+1}$ and the l.h.s. on $\overline{v}^{(m+1)}_{n+1}$, $\overline{p}^{(m+1)}_{n+1}$. Hence

(3.26) implies the mapping

$$\left(\overline{v}_{n+1}^{(m+1)},\overline{p}_{n+1}^{(m+1)}\right)=\Phi\left(\overline{v}_{n+1}^{(m)},\overline{p}_{n+1}^{(m)}\right), \tag{3.28}$$

where $m\in\mathbb{N}$. Hence for sufficiently small time and given $\overline{v}_n$ the mapping (3.28) has a fixed point which is a solution to problem (3.26) for given $\overline{v}_n$. Then we get the functional dependence

$$\overline{v}_{n+1}=F(\overline{v}_n). \tag{3.29}$$

Hence, again, by the method of successive approximations and assumption that $\overline{v}_0$ is some extension of the initial data we show for sufficiently small time an existence of the fixed point to mapping (3.29). In this way we show the existence of solutions to problem (1.1)–(1.8).

4. The existence of solutions to problem (3.27)

For simplicity we write problem (3.27) in the form

$$\begin{aligned} v_t-\operatorname{div}_\xi\mathbb{T}_\xi(v,p)&=f_0,\\ \operatorname{div}_\xi v&=g_0,\\ \overline{n}_\xi\cdot\mathbb{T}_\xi(v,p)&=k_0,\\ v|_{t=0}&=v(0). \end{aligned} \tag{4.1}$$

We construct a function G satisfying the problem

$$\operatorname{div}_\xi G=g_0,\qquad G|_{S_0}=0. \tag{4.2}$$

Applying the Bogovskiĭ operator B solutions to (4.2) can be written in the form

$$G=B*g_0. \tag{4.3}$$

Introducing the new function

$$u=v-G \tag{4.4}$$

we see that (u,p) is a solution to the problem

$$\begin{aligned} u_{,t}-\operatorname{div}_\xi\mathbb{T}_\xi(u,p)&=-G_{,t}+\operatorname{div}_\xi\mathbb{D}(G)+f_0\equiv F,\\ \operatorname{div}u&=0,\\ \overline{n}_\xi\cdot\mathbb{T}_\xi(u,p)&=-\overline{n}_\xi\mathbb{D}_\xi(G)+k_0\equiv H,\\ u|_{t=0}&=v(0)-G|_{t=0}\equiv u(0). \end{aligned} \tag{4.5}$$

Simplifying (4.5) yields

$$
\begin{aligned}
u_{,t} - \operatorname{div}_{\xi}\mathbb{T}_{\xi}(u,p) &= F && \text{in } \overset{1}{\Omega}{}_0^T,\\
\operatorname{div} u &= 0 && \text{in } \overset{1}{\Omega}{}_0^T,\\
\overline{n}_{\xi}\cdot\mathbb{T}_{\xi}(u,p) &= H && \text{on } S_0^T\\
u|_{t=0} &= u(0) && \text{in } \overset{1}{\Omega}{}_0.
\end{aligned}
\tag{4.6}
$$

For needs of [8] we solve the problem using the Faedo–Galerkin method. The Faedo-Galerkin method implies the existence of solutions to problem (4.6) in the L_2-approach.

Definition 4.1. By a weak solution to problem (4.6) we mean any solution to the integral identity

$$
\int_{\overset{1}{\Omega}{}_0^T} u_{,t}\cdot\eta\,dx\,dt + \int_{\overset{1}{\Omega}{}_0^T}\mathbb{D}(u)\cdot\mathbb{D}(\eta)\,dx\,dt = \int_{S_0^T} H\cdot\eta\,dS_0\,dt + \int_{\overset{1}{\Omega}{}_0^T} F\cdot\eta\,dx\,dt,
\tag{4.7}
$$

where $B^T = B\times[0,T]$, which holds for any $\eta\in L_2\big(0,T;H^1\big(\overset{1}{\Omega}{}_0\big)\cap V\big(\overset{1}{\Omega}{}_0\big)\big)$, where the time derivative is understandable in the weak sense and the following spaces are introduced

$$
\vartheta = \Big\{u\in C^{\infty}\Big(\overset{1}{\Omega}{}_0\Big) : \operatorname{div} u = 0\Big\},\qquad V = \text{closure of } \vartheta \text{ in } W_2^1\Big(\overset{1}{\Omega}{}_0\Big).
$$

Since V is separable there exists a sequence of linearly-independent elements $\varphi_1,\ldots,\varphi_m,\ldots$, which is a base in V. The existence of the fundamental base for Stokes system (4.6) is proved in [23].

Therefore, we are looking for approximate solutions to (4.7) in the form

$$
\begin{aligned}
u_m &= \sum_{i=1}^m c_{im}(t)\varphi_i(x),\\
\sum_{i=1}^m \dot c_{im}\int_{\overset{1}{\Omega}{}_0}\varphi_i\cdot\varphi_l\,dx + \sum_{i=1}^m c_{im}&\int_{\overset{1}{\Omega}{}_0}\mathbb{D}(\varphi_i)\cdot\mathbb{D}(\varphi_l)\,dx\\
&= \int_{S_0} H\cdot\varphi_l\,dS_0 + \int_{\overset{1}{\Omega}{}_0} F\cdot\varphi_l\,dx,
\end{aligned}
\tag{4.8}
$$

$l = 1,\ldots,m$, $u_m|_{t=0} = u_m(0)$ and $\dot c_{im} = c_{im,t}$.

Since $\varphi_1,\ldots,\varphi_m$ are linearly independent then $\det\int_{\overset{1}{\Omega}}\varphi_i\cdot\varphi_j\,dx \ne 0$. Therefore, (4.8) implies the following linear system with constant coefficients and time dependent r.h.s. We express it in the form

$$
\dot c_{im} + \sum_{j=1}^m \alpha_{ij}c_{jm} = \sum_{j=1}^m \beta_{ij}K_j,\quad i = 1,\ldots,m,\qquad c_{im}|_{t=0} = c_{im}(0),
\tag{4.9}
$$

where

$$K_j = \int_{S_0} H \cdot \varphi_j \, dS + \int_{\overset{1}{\Omega}_0} F \cdot \varphi_j \, dx, \qquad u_m(0) = \sum_{i=1}^{m} c_{im}(0)\varphi_i.$$

Lemma 4.2. *Assume that* $H \in L_2(0,T; L_2(S_0))$, $F \in L_2\big(0,T; L_2\big(\overset{1}{\Omega}_0\big)\big)$, $u(0) \in L_2(\overset{1}{\Omega}_0)$. *Then there exists a weak solution to problem* (4.6) *such that* $u \in V_2^0(\overset{1}{\Omega}{}_0^T)$ *and the estimate holds*

$$\|u\|_{V_2^0(\overset{1}{\Omega}{}_0^T)} \le c\Big(\|F\|_{L_2\left(\overset{1}{\Omega}{}_0^T\right)} + \|H\|_{L_2(S_0^T)} + \|u(0)\|_{L_2\left(\overset{1}{\Omega}_0\right)}\Big). \tag{4.10}$$

Proof. Since (4.9) is a system of linear ordinary differential equations the existence of solutions is well known. To prove the existence of weak solutions in $V_2^0(\overset{1}{\Omega}{}_0^T)$ we multiply (4.8) by c_{lm} and sum over l from 1 to m. Integrating the result with respect to time we get

$$\|u_m\|_{V_2^0\left(\overset{1}{\Omega}{}_0^T\right)} \le c\Big(\|F\|_{L_2\left(\overset{1}{\Omega}{}_0^T\right)} + \|H\|_{L_2(S_0^T)} + \|u(0)\|_{L_2\left(\overset{1}{\Omega}_0\right)}\Big), \tag{4.11}$$

where we used that $\|u_m(0)\|_{L_2\left(\overset{1}{\Omega}_0\right)} \le \|u(0)\|_{L_2\left(\overset{1}{\Omega}_0\right)}$. Using the weak convergence in $L_2\big(0,T; H^1\big(\overset{1}{\Omega}_0\big)\big)$ and weak star convergence in $L_\infty\big(0,T; L_2\big(\overset{1}{\Omega}_0\big)\big)$ we show that the limit function belongs to $V_2^0\big(\overset{1}{\Omega}{}_0^T\big)$ and estimate (4.10) holds. □

5. The existence of solutions to problem (3.21)

Dropping the index n in (3.21) we write it in the simple form

$$\begin{aligned}
&\mu_i \overset{i}{H}_t + \frac{1}{\sigma_i}\operatorname{rot}_\xi^2 \overset{i}{H} = \overset{i}{f}, \quad \operatorname{div}_\xi \overset{i}{H} = \overset{i}{g} && \text{in } \overset{i}{\Omega}{}_0^t,\ i = 1,2,\\
&\left(\frac{1}{\sigma_1}\operatorname{rot}_\xi \overset{1}{H} - \frac{1}{\sigma_2}\operatorname{rot}_\xi \overset{2}{H}\right)\cdot \overline{\tau}_\alpha = g_\alpha, && \alpha = 1,2, \text{ on } S_0^t,\\
&\overline{n} \times \overline{\tau}_\alpha \cdot \left(\overset{1}{H} - \overset{2}{H}\right) = k_\alpha, && \alpha = 1,2, \text{ on } S_0^t,\\
&\mu_1 \overline{n}\cdot \overset{1}{H} - \mu_2 \overline{n}\cdot \overset{1}{H} = l && \text{on } S_0^t,\\
&\overset{2}{H}\big|_B = 0 && \text{on } B^t,\\
&\overset{i}{H}\big|_{t=0} = \overset{i}{H}(0) && \text{in } \overset{i}{\Omega}_0,\ i = 1,2,
\end{aligned} \tag{5.1}$$

where $\overset{i}{\overline{H}}$ was replaced by $\overset{i}{H}$, $i = 1,2$.

To prove the existence of solutions to problem (5.1) by Faedo–Galerkin method we need a weak formulation to problem (5.1). For this purpose we multiply (5.1) by $\overset{i}{\psi}$, a sufficiently regular function, and integrate over $\overset{i}{\Omega}_0$, $i = 1,2$.

Then we have

$$\sum_{i=1}^{2}\int_{\overset{i}{\Omega}_0}\bigg(\mu_i\overset{i}{H}_t+\frac{1}{\sigma_i}\operatorname{rot}_\xi^2\overset{i}{H}\bigg)\cdot\overset{i}{\psi}\,d\xi=\sum_{i=1}^{2}\int_{\overset{i}{\Omega}_0}\overset{i}{f}\cdot\overset{i}{\psi}\,d\xi. \tag{5.2}$$

Integration by parts yields

$$\begin{aligned}\sum_{i=1}^{2}\int_{\overset{i}{\Omega}_0}\bigg(\mu_i\overset{i}{H}_t\cdot\overset{i}{\psi}\,d\xi+\frac{1}{\sigma_i}\operatorname{rot}_\xi\overset{i}{H}\cdot\operatorname{rot}\overset{i}{\psi}\bigg)\,d\xi&\\-\sum_{i=1}^{2}\int_{S_0}\frac{1}{\sigma_i}\overset{i}{\overline{n}}\times\operatorname{rot}\overset{i}{H}\cdot\overset{i}{\psi}\,dS_0&=\sum_{i=1}^{2}\int_{\overset{i}{\Omega}_0}\overset{i}{f}\cdot\overset{i}{\psi}\,d\xi,\end{aligned} \tag{5.3}$$

where $\overset{i}{\overline{n}}$ is the unit vector normal to S_0 which is exterior to $\overset{i}{\Omega}_0$. Therefore choosing $\overline{n}=\overset{1}{\overline{n}}$ we get that $\overset{2}{\overline{n}}=-\overline{n}$. Then the boundary term in (5.3) takes the form

$$I=-\int_{S_0}\frac{1}{\sigma_1}\overline{n}\times\operatorname{rot}\overset{1}{H}\cdot\overset{1}{\psi}\,d\xi+\int_{S_0}\frac{1}{\sigma_2}\overline{n}\times\operatorname{rot}\overset{2}{H}\cdot\overset{2}{\psi}\,d\xi.$$

In the integrals of I only the tangent coordinates of $\overset{1}{\psi}$ and $\overset{2}{\psi}$ appear. Therefore, using the decomposition

$$\overset{i}{\psi}=\sum_{\alpha=1}^{2}\overset{i}{\psi}\cdot\overline{\tau}_\alpha\overline{\tau}_\alpha+\psi\cdot\overline{n}\,\overline{n}$$

the expression I takes the form

$$I=\sum_{\alpha=1}^{2}\bigg[-\int_{S_0}\frac{1}{\sigma_1}\overline{n}\times\operatorname{rot}\overset{1}{H}\cdot\overline{\tau}_\alpha\overset{1}{\psi}\cdot\overline{\tau}_\alpha\,dS_0+\int_{S_0}\frac{1}{\sigma_2}\overline{n}\times\operatorname{rot}\overset{2}{H}\cdot\overline{\tau}_\alpha\overset{2}{\psi}\cdot\overline{\tau}_\alpha\,dS_0\bigg].$$

In view of the transmission conditions (1.8) we have

$$\overset{1}{E}_{\tau_\alpha}=\overset{2}{E}_{\tau_\alpha},\quad \overset{1}{H}_{\tau_\alpha}=\overset{2}{H}_{\tau_\alpha},\quad \alpha=1,2,\quad\text{on}\quad S_0. \tag{5.4}$$

Recalling that

$$\overset{1}{E}_{\tau_\alpha}=\bigg(\frac{1}{\sigma_1}\operatorname{rot}\overset{1}{H}-\mu_1v\times\overset{1}{H}\bigg)\cdot\overline{\tau}_\alpha,\qquad \overset{2}{E}_{\tau_\alpha}=\frac{1}{\sigma_2}\operatorname{rot}\overset{2}{H}\cdot\overline{\tau}_\alpha$$

we obtain

$$I=-\sum_{\alpha=1}^{2}\int_{S_0}\mu_1\overline{\tau}_\alpha\times\overline{n}\cdot v\times\overset{1}{H}\psi\cdot\overline{\tau}_\alpha\,dS_0 \tag{5.5}$$

where we used that $\psi\cdot\overline{\tau}_\alpha=\overset{1}{\psi}\cdot\overline{\tau}_\alpha=\overset{2}{\psi}\cdot\overline{\tau}_\alpha$, $\alpha=1,2$, on S_0.

Replacing the boundary term in (5.3) by (5.5) we derive the following integral identity

$$\begin{aligned}(5.6)\quad &\sum_{i=1}^{2}\int_{\overset{i}{\Omega}_0}\left(\mu_i\overset{i}{H}_t\cdot\overset{i}{\psi}+\frac{1}{\sigma_i}\operatorname{rot}_\xi\overset{i}{H}\cdot\operatorname{rot}_\xi\overset{i}{\psi}\right)d\xi\\ &\qquad=\sum_{i=1}^{2}\int_{\overset{i}{\Omega}_0}\overset{i}{f}\cdot\overset{i}{\psi}\,d\xi+\sum_{\alpha=1}^{2}\int_{S_0}\mu_1\overline{\tau}_\alpha\times\overline{n}\cdot v\times\overset{1}{H}\psi\cdot\overline{\tau}_\alpha\,dS_0.\end{aligned}$$

To show the existence of weak solutions to problem (5.1) satisfying the integral identity (5.6) we need the existence of a fundamental basis. For this purpose we consider the elliptic problem

$$\begin{aligned}&\mu_1\overset{1}{\psi}+\operatorname{rot}\left(\frac{1}{\sigma_1}\operatorname{rot}\overset{1}{\psi}\right)=\overset{1}{f} &&\text{in } \overset{1}{\Omega}_0,\\ &\mu_2\overset{2}{\psi}+\operatorname{rot}\left(\frac{1}{\sigma_2}\operatorname{rot}\overset{2}{\psi}\right)=\overset{2}{f} &&\text{in } \overset{2}{\Omega}_0,\\ (5.7)\quad &\left(\frac{1}{\sigma_1}\operatorname{rot}\overset{1}{\psi}\right)\cdot\overline{\tau}_\alpha=\frac{1}{\sigma_2}\operatorname{rot}\overset{2}{\psi}\cdot\overline{\tau}_\alpha+g_\alpha\equiv g\cdot\overline{\tau}_\alpha,\quad \alpha=1,2, &&\text{on } S_0,\\ &\overset{1}{\psi}\cdot\overline{n}\times\overline{\tau}_\alpha=\overset{2}{\psi}\cdot\overline{n}\times\overline{\tau}_\alpha,\quad\alpha=1,2,\quad \overset{1}{\psi}\cdot\overline{n}=\overset{2}{\psi}\cdot\overline{n} &&\text{on } S_0,\\ &\overset{2}{\psi}|_B=0.\end{aligned}$$

From $(5.7)_{1,2}$ we have $\mu_i\operatorname{div}\overset{i}{\psi}=\operatorname{div}\overset{i}{f}$, $i=1,2$. Hence, for $\overset{i}{f}$ is divergence free, we get that $\overset{i}{\psi}$ is also divergence free.

Using the Fredholm theorems we are going to construct eigenfunctions to the eigenvalue problem for (5.7). In this way we find a fundamental base necessary for showing the existence of weak solutions to the integral identity (5.6) by the Faedo–Galerkin method.

To derive the integral identity for solutions to problem (5.7) we multiply $(5.7)_1$ by $\overset{1}{\phi}$, $(5.7)_2$ by $\overset{2}{\phi}$, integrate the results over $\overset{i}{\Omega}_0$, $i=1,2$, respectively, add and use boundary conditions $(5.7)_{3,4}$. Assume that $\overset{i}{f}$ is divergence free $i=1,2$. Then we have

$$\begin{aligned}(5.8)\quad \mathcal{Z}(\psi,\phi)&\equiv\sum_{i=1}^{2}\int_{\overset{i}{\Omega}_0}\left(\mu_i\overset{i}{\psi}\cdot\overset{i}{\phi}+\frac{1}{\sigma_i}\operatorname{rot}\overset{i}{\psi}\cdot\operatorname{rot}\overset{i}{\phi}\right)dx\\ &=\sum_{i=1}^{2}\int_{\overset{i}{\Omega}_0}\overset{i}{f}\cdot\overset{i}{\phi}\,dx+\sum_{\alpha=1}^{2}\int_{S_0}\overline{\tau}_\alpha\times\overline{n}\cdot g\varphi\cdot\overline{\tau}_\alpha\,dS_0,\end{aligned}$$

where $\psi=(\overset{1}{\psi},\overset{2}{\psi})$, $\phi=(\overset{1}{\phi},\overset{2}{\phi})$, $\overset{i}{\psi}$, $\overset{i}{\phi}$, $i=1,2$, $\overset{1}{\psi}\big|_{S_0}=\overset{2}{\psi}\big|_{S_0}$, $\overset{1}{\phi}\big|_{S_0}=\overset{2}{\phi}|_{S_0}=\varphi$ are divergence free and satisfy boundary conditions $(5.7)_{3,4}$. Let us introduce the notation

$$(\psi,\phi)_{L_2(\overset{i}{\Omega}_0)} = \int_{\overset{i}{\Omega}_0} \psi\cdot\phi\,dx, \qquad (\psi,\phi)_{L_2(\Omega)} = \sum_{i=1}^2 \int_{\overset{i}{\Omega}_0} \overset{i}{\psi}(x)\cdot\overset{i}{\phi}(x)\,dx, \tag{5.9}$$

where $\Omega = \overset{1}{\Omega}_0 \times \overset{2}{\Omega}_0$. Moreover, we define

$$(\psi,\phi)_{\overline{H}(\Omega)} = \sum_{i=1}^2 \mu_i\big(\overset{i}{\psi},\overset{i}{\phi}\big)_{L_2(\overset{i}{\Omega}_0)} + \frac{1}{\sigma_i}\big(\operatorname{rot}\overset{i}{\psi},\operatorname{rot}\overset{i}{\phi}\big)_{L_2(\overset{i}{\Omega}_0)}, \tag{5.10}$$

where ψ, ϕ satisfy (2.6).

In view of the above notation, identity (5.8) can be described in the following short form

$$\mathcal{Z}(\psi,\phi) \equiv (\psi,\phi)_{\overline{H}(\Omega)} = (f,\phi)_{L_2(\Omega)} + (g,\phi)_{L_2(S_0)}, \tag{5.11}$$

where the last term on the r.h.s. of (5.11) has the form of the last term on the r.h.s. of (5.8).

To apply the Fredholm theorems we need to have that $(\psi,\psi)_{\overline{H}(\Omega)}$ is equivalent to the norm

$$\begin{aligned} \|\psi\|^2_{H(\Omega)} &= (\psi,\psi)_{H(\Omega)} \\ &= \sum_{i=1}^2 \big\|\overset{i}{\psi}\big\|^2_{H^1(\overset{i}{\Omega}_0)} = \sum_{i=1}^2 \big(\overset{i}{\psi},\overset{i}{\psi}\big)_{L_2(\overset{i}{\Omega}_0)} + \big(\nabla\overset{i}{\psi},\nabla\overset{i}{\psi}\big)_{L_2(\overset{i}{\Omega})}, \end{aligned} \tag{5.12}$$

where ψ satisfies (2.6). $H(\Omega)$ is the closure of smooth functions satisfying (2.6) in the norm

$$\|\psi\|_{H(\Omega)} = \sum_{i=1}^2 \bigg(\int_{\overset{i}{\Omega}_0} \big(|\overset{i}{\psi}|^2 + |\nabla\overset{i}{\psi}|^2\big)\,dx\bigg)^{1/2}$$

In the next lemma we derive a weak formulation of problem (5.7).

Lemma 5.1. *Assume that* $\overset{i}{f}$ *is divergence free,* $\operatorname{rot}\overset{i}{\psi} \in L_2(\overset{i}{\Omega}_0)$, $i=1,2$ *and satisfy boundary and transmission conditions* $\overset{1}{\psi} = \overset{2}{\psi}$ *on* S_0, $\overset{2}{\psi}|_B = 0$. *Then the norms* $(\psi,\psi)_{\overline{H}(\Omega)}$ *and* $(\psi,\psi)_{H(\Omega)}$ *are equivalent. Moreover, the integral identities* (5.11) *and* $(\psi,\phi)_{H(\Omega)} = (f,\phi)_{L_2(\Omega)} + (g,\phi)_{L_2(S_0)}$ *are equivalent.*

Proof. The first part of assertion is obvious and based on Lemma 2.3. Therefore we will concentrate on the proof of the second part. To prove the lemma we are looking for functions $\psi = \big(\overset{1}{\psi},\overset{2}{\psi}\big)$ as weak solutions to the problem

$$\mu_i\overset{i}{\psi} + \frac{1}{\sigma_i}\operatorname{rot}^2\overset{i}{\psi} = \overset{i}{f}, \qquad \text{in } \overset{i}{\Omega}_0,\ i=1,2, \tag{5.13}$$

$$\frac{1}{\sigma_1}\operatorname{rot}\overset{1}{\psi}\cdot\overline{\tau}_\alpha = \frac{1}{\sigma_2}\operatorname{rot}\overset{2}{\psi}\cdot\overline{\tau}_\alpha + g\cdot\overline{\tau}_\alpha, \quad \alpha=1,2, \qquad \text{on } S_0, \tag{5.14}$$

$$\overset{1}{\psi}\cdot\overline{n}\times\overline{\tau}_\alpha=\overset{2}{\psi}\cdot\overline{n}\times\overline{\tau}_\alpha,\quad \alpha=1,2,\quad \overset{1}{\psi}\cdot\overline{n}=\overset{2}{\psi}\cdot\overline{n}\quad\text{on } S_0, \tag{5.15}$$

$$\overset{2}{\psi}|_B=0, \tag{5.16}$$

where $\operatorname{rot}^2=\operatorname{rot}\operatorname{rot}$.

Multiplying (5.13) by $\overset{i}{\psi}$, integrating over $\overset{i}{\Omega}_0$, $i=1,2$, adding the results and integrating by parts, we obtain

$$\begin{aligned}\sum_{i=1}^2\bigg(\mu_i\int_{\overset{i}{\Omega}_0}|\overset{i}{\psi}|^2\,dx+\frac{1}{\sigma_i}\int_{\overset{i}{\Omega}_0}|\operatorname{rot}\overset{i}{\psi}|^2\,dx\bigg)\\ +\sum_{\alpha=1}^2\int_{S_0}\bigg(\frac{1}{\sigma_1}\operatorname{rot}\overset{1}{\psi}\cdot\overline{\tau}_\alpha\overset{1}{\psi}\cdot(\overline{n}\times\overline{\tau}_\alpha)-\frac{1}{\sigma_2}\operatorname{rot}\overset{2}{\psi}\cdot\overline{\tau}_\alpha\overset{2}{\psi}\cdot\overline{n}\times\overline{\tau}_\alpha\bigg)\,dS_0\\ =\sum_{i=1}^2\int_{\overset{i}{\Omega}_0}\overset{i}{f}\cdot\overset{i}{\psi}\,dx,\end{aligned} \tag{5.17}$$

where boundary conditions on B were used.

In view of transmission conditions (5.14)–(5.15) equality (5.17) yields

$$\begin{aligned}\sum_{i=1}^2\bigg(\mu_i\int_{\overset{i}{\Omega}_0}|\overset{i}{\psi}|^2\,dx+\frac{1}{\sigma_i}\int_{\overset{i}{\Omega}_0}|\operatorname{rot}\overset{i}{\psi}|^2\,dx\bigg)\\ =\sum_{i=1}^2\int_{\overset{i}{\Omega}_0}\overset{i}{f}\cdot\overset{i}{\psi}\,dx+\sum_{\alpha=1}^2\int_{S_0}g\cdot\overline{\tau}_\alpha\psi\cdot\overline{n}\times\overline{\tau}_\alpha\,dS_0.\end{aligned} \tag{5.18}$$

Applying the Hölder and Young inequalities to the r.h.s. of (5.18) implies

$$\begin{aligned}\sum_{i=1}^2\bigg(\mu_i\int_{\overset{i}{\Omega}_0}|\overset{i}{\psi}|^2\,dx+\frac{1}{\sigma_i}\int_{\overset{i}{\Omega}_0}|\operatorname{rot}\overset{i}{\psi}|^2\,dx\bigg)\\ \le\sum_{i=1}^2\Big(\varepsilon\|\overset{i}{\psi}\|^2_{H^1(\overset{i}{\Omega}_0)}+c/\varepsilon\|\overset{i}{f}\|^2_{L_2(\overset{i}{\Omega}_0)}+c/\varepsilon\|g\cdot\overline{\tau}_i\|^2_{L_2(S_0)}\Big).\end{aligned} \tag{5.19}$$

To apply Lemma 2.3 we consider the set

$$\begin{aligned}\operatorname{div}\overset{i}{\psi}=0\qquad&\text{in }\overset{i}{\Omega}_0,\ i=1,2,\\ \overset{1}{\psi}_\tau=\overset{2}{\psi}_\tau,\quad \overset{1}{\psi}_n=\overset{2}{\psi}_n\quad&\text{on } S_0,\\ \overset{2}{\psi}\big|_B=0,&\end{aligned} \tag{5.20}$$

where $\operatorname{rot}\overset{i}{\psi}\in L_2(\overset{i}{\Omega}_0)$.

The second transmission condition in $(5.20)_2$ follows from local integration of equations in $(5.20)_1$. Then using (2.7) in (5.19) and utilizing that ε is sufficiently small we have

$$\sum_{i=1}^{2} \|\overset{i}{\psi}\|^2_{H^1(\overset{i}{\Omega}_0)} \le c \sum_{i=1}^{2} \Big(\|\overset{i}{f}\|^2_{L_2(\overset{i}{\Omega}_0)} + \|g \cdot \overline{\tau}_i\|^2_{L_2(S_0)} \Big). \tag{5.21}$$

Hence, the first Fredholm theorem implies existence of weak solutions to problem (5.13)–(5.16). Comparing (5.18) and (5.21) we have equivalence of norms $(\psi,\psi)_{\overline{H}(\Omega)}$ and $(\psi,\psi)_{H(\Omega)}$. □

Next, we express (5.11) in the form

$$(\psi,\phi)_{H(\Omega)} = (f,\phi)_{L_2(\Omega)} + (g,\phi)_{L_2(S_0)}. \tag{5.22}$$

By the Riesz theorem, the r.h.s. of (5.22) is expressed in the form

$$(f,\phi)_{L_2(\Omega)} + (g,\phi)_{L_2(S_0)} = (F,\phi)_{H(\Omega)}. \tag{5.23}$$

Therefore, (5.22) takes the form

$$(\psi,\phi)_{H(\Omega)} = (F,\phi)_{H(\Omega)}, \tag{5.24}$$

which holds for any $\phi \in H(\Omega)$. Hence, (5.24) implies the following functional equation

$$\psi = F. \tag{5.25}$$

Since we are interested in a construction of eigenfunctions to problem (5.7) we consider the following eigenvalue problem

$$\mathcal{Z}\psi = \lambda\psi, \tag{5.26}$$

where λ is a parameter and $\mathcal{Z}$ describes the equations in $(5.7)_{1,2}$. Then, instead of (5.8), we have

$$\mathcal{Z}(\psi,\phi) \equiv \sum_{i=1}^{2} \int_{\overset{i}{\Omega}_0} \Big(\mu_i \overset{i}{\psi}\overset{i}{\phi} + \frac{1}{\sigma_i} \operatorname{rot} \overset{i}{\psi} \cdot \operatorname{rot} \overset{i}{\phi} \Big)\, dx = -\lambda \sum_{i=1}^{2} \int_{\overset{i}{\Omega}_0} \overset{i}{\psi} \cdot \overset{i}{\phi}\, dx. \tag{5.27}$$

Similarly as (5.25) we obtain the functional equation

$$\psi = \lambda B\psi, \tag{5.28}$$

where

$$(B\psi,\phi)_{H(\Omega)} = -\sum_{i=1}^{2} \int_{\overset{i}{\Omega}_0} \overset{i}{\psi} \cdot \overset{i}{\phi}\, dx.$$

Operator B is self-symmetric and negative because

$$(B\psi,\phi)_{H(\Omega)} = (\psi, B\phi)_{H(\Omega)} \quad \text{and} (B\psi,\psi)_{H(\Omega)} < 0.$$

Operator B is compactly continuous symmetric operator. Hence, for any eigenvalue λ, corresponds at least one nontrivial solution to the homogeneous problem

$$\psi = \lambda B\psi. \tag{5.29}$$

Hence, we have

LEMMA 5.2. *The eigenfunctions to problem* (5.29) *compose a dense set of functions in* $H(\Omega)$. *The functions can be chosen as basic functions for the Faedo–Galerkin method.*

LEMMA 5.3. *For given r.h.s. of* (5.6) *and in view of Lemmas* 5.1 *and* 5.2 *the existence of solutions to* (5.6) *can be made by the Faedo–Galerkin method.*

REMARK 5.4. The considerations in this Section were performed in the set of divergence free functions. This was forced by Lemma 2.3, where (2.6) was examined. However, in each step of the method of successive approximations we can not expect divergence free functions (velocity, magnetic field in the Lagrangian coordinates). Therefore, (2.6) must be replaced by the following one. (The below considerations hold for $\mu_1 \neq \mu_2$ but according to Lemma 2.3 we need only that $\mu_1 = \mu_2$.)

$$
\begin{aligned}
&\operatorname{div} \overset{i}{u} = \overset{i}{h} \qquad && \text{in } \overset{i}{\Omega}_0, \ i = 1, 2, \\
&\overset{1}{u}_\tau = \overset{2}{u}_\tau, \quad \mu_1 \overset{1}{u}_n = \mu_2 \overset{2}{u}_n && \text{on } S_0, \\
&\overset{2}{u}\big|_B = 0.
\end{aligned} \tag{5.30}
$$

To use Lemma 2.3 we construct functions $\overset{i}{\varphi}$, $i = 1, 2$, satisfying the problem

$$
\begin{aligned}
&\Delta \overset{i}{\varphi} = \overset{i}{h} && \text{in } \overset{i}{\Omega}_0, \ i = 1, 2, \\
&\overset{1}{\varphi} = \overset{2}{\varphi}, \quad \mu_1 \overline{n} \cdot \nabla \overset{1}{\varphi} = \mu_2 \overline{n} \cdot \nabla \overset{2}{\varphi} && \text{on } S_0, \\
&\overline{n} \cdot \nabla \overset{2}{\varphi}\big|_B = 0, \\
&\int_{\overset{i}{\Omega}_0} \overset{i}{\varphi} \, dx = 0, && i = 1, 2.
\end{aligned} \tag{5.31}
$$

To show the existence of solutions to problem (5.31) we have to find an energy estimate and apply the Fredholm theorem. Therefore, we multiply $(5.31)_1$ by $\overset{i}{\varphi}$ and integrate over $\overset{i}{\Omega}_t$. After adding we get

$$
\sum_{i=1}^{2} \int_{\overset{i}{\Omega}_0} \mu_i \Delta \overset{i}{\varphi} \overset{i}{\varphi} \, dx = \sum_{i=1}^{2} \int_{\overset{i}{\Omega}_0} \mu_i \overset{i}{h} \overset{i}{\varphi} \, dx.
$$

Integration by parts yields

$$
\begin{aligned}
\sum_{i=1}^{2} \int_{\overset{i}{\Omega}_0} \mu_i \big|\nabla \overset{i}{\varphi}\big|^2 dx + \int_{S_0} \Big(\mu_1 \overline{n} \cdot \nabla \overset{1}{\varphi} \overset{1}{\varphi} - \mu_2 \overline{n} \cdot \nabla \overset{2}{\varphi} \overset{2}{\varphi}\Big) \, dS_0 \\
= - \sum_{i=1}^{2} \int_{\overset{i}{\Omega}_0} \mu_i \overset{i}{h} \overset{i}{\varphi} \, dx.
\end{aligned} \tag{5.32}
$$

In view of $(5.31)_2$ the boundary term in (5.32) vanishes. Then $(5.31)_4$ and the Poincaré inequality give

$$\sum_{i=1}^{2} \|\overset{i}{\varphi}\|_{H^1(\overset{i}{\Omega}_0)} \le c \sum_{i=1}^{2} \|\overset{i}{h}\|_{L_2(\overset{i}{\Omega}_0)}. \tag{5.33}$$

Then the Fredholm theorem gives existence of solutions to problem (5.31) under assumption that $\overset{i}{h} \in L_2(\overset{i}{\Omega}_0)$, $i = 1, 2$.

Introducing the new functions

$$\overset{i}{v} = \overset{i}{u} - \nabla \overset{i}{\varphi}, \quad i = 1, 2, \tag{5.34}$$

we see that $\overset{1}{v}$, $\overset{2}{v}$ are functions satisfying (2.6).

Remark 5.5. We have to emphasize that considerations in Section 5 imply the existence of eigenvalues and eigenfunctions to problem (5.7) expressed in the Lagrangian coordinates and considered in the fixed initial domains $\overset{i}{\Omega}_0$ for $i = 1, 2$. In this section we do not care about the proof of existence of solutions to the free boundary problem (1.9), (1.10). The idea of such a proof is shown in Section 3. Since v and H expressed in the Lagrangian coordinates are not divergence free in these coordinates we need Lemma 2.3 and Remark 5.4 to show equivalence between norms of spaces $\overline{H}(\Omega)$ and $H(\Omega)$. The space $\overline{H}(\Omega)$ is natural for formulation of weak solutions to problem (5.7) (see (5.8)). But space $H(\Omega)$ is necessary to show that operator B is compact.

Remark 5.6. The request of one of the referees was to show the role of the Dirichlet boundary condition on B. For this purpose we consider the simplified problem

$$\begin{aligned} &\operatorname{rot}^2 \overset{i}{\psi} = \lambda \overset{i}{\psi}, && \text{in } \overset{i}{\Omega}_0,\ i = 1, 2, \\ &\text{transmission conditions} && \text{on } S_0, \\ &\overset{2}{\psi}\big|_B = 0. \end{aligned} \tag{5.35}$$

For $\lambda \ne 0$, (5.35) implies that $\operatorname{div} \overset{i}{\psi} = 0$, $i = 1, 2$. To examine the eigenvalue problem to (5.35) we need the weak formulation. Multiply $(5.35)_1$ by $\overset{i}{\psi}$, integrate over $\overset{i}{\Omega}$, sum over i and use transmission and boundary conditions. Then we have

$$\sum_{i=1}^{2} \|\operatorname{rot} \overset{i}{\psi}\|_{L_2(\overset{i}{\Omega}_0)} = \lambda \sum_{i=1}^{2} \|\overset{i}{\psi}\|_{L_2(\overset{i}{\Omega}_0)} \tag{5.36}$$

To apply the Fredholm theorem showing existence of eigenvalues and eigenvectors we need to prove that $\sum_{i=1}^{2} \|\operatorname{rot} \overset{i}{\psi}\|_{L_2(\overset{i}{\Omega}_0)}$ is equivalent to $\sum_{i=1}^{2} \|\overset{i}{\psi}\|_{H^1(\overset{i}{\Omega}_0)}$. For this we need Lemma 2.3. In the first part of the proof of Lemma 2.3 we have the equivalence

$$\sum_{i=1}^{2}\left\|\operatorname{rot}\overset{i}{\psi}\right\|^2_{L_2\left(\overset{i}{\Omega}_0\right)} \sim \sum_{i=1}^{2}\left\|\nabla\overset{i}{\psi}\right\|^2_{L_2\left(\overset{i}{\Omega}_0\right)}.$$

Applying the Poincaré inequality which is possible in the case of the homogeneous Dirichlet boundary conditions on B we show in the second part of the proof of Lemma 2.3 the equivalence

$$\sum_{i=1}^{2}\left\|\nabla\overset{i}{\psi}\right\|^2_{L_2\left(\overset{i}{\Omega}_0\right)} \sim \sum_{i=1}^{2}\left\|\overset{i}{\psi}\right\|^2_{H^1\left(\overset{i}{\Omega}_0\right)}.$$

Then (5.36) takes the form

$$\sum_{i=1}^{2}\left\|\overset{i}{\psi}\right\|^2_{H^1\left(\overset{i}{\Omega}_0\right)} \le c\lambda\sum_{i=1}^{2}\left\|\overset{i}{\psi}\right\|_{L_2\left(\overset{i}{\Omega}_0\right)}. \tag{5.37}$$

This estimate yields that operator B introduced in (5.28) is compact, so the Fredholm theorem works.

In Section 5 we have considered a more general system (5.7), where $\overset{i}{\psi}$, $i = 1,2$, are not divergence free. In this case in order to apply Lemma 2.3 we need Remark 5.4.

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