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NONLINEAR UNILATERAL PARABOLIC PROBLEMS IN MUSIELAK–ORLICZ SPACES WITH L^1 DATA

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Abstract. We study, in Musielak–Orlicz spaces, the existence of solutions for some strongly nonlinear parabolic unilateral problem with L^1 data and without sign condition on nonlinearity.

1. Introduction

Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N$ $(N \geq 2)$ and let $\mathrm{Q} = \Omega \times (0,T)$, $T > 0$. Consider the following nonlinear parabolic problem:

$$
\begin{cases}
\dfrac{\partial u}{\partial t} + A(u) + g(x,t,u,\nabla u) = f & \text{in Q}, \\
u = 0 & \text{on } \partial\Omega \times (0,T), \\
u(x,0) = u_0(x) & \text{in } \Omega,
\end{cases} \tag{1.1}
$$

where $A(u) = -\operatorname{div} a(x,t,u,\nabla u)$ is a Leray–Lions operator defined on $D(A) \subset W_0^{1,x}L_\varphi(\mathrm{Q}) \to W^{-1,x}L_{\overline{\varphi}}(\mathrm{Q})$ with φ and $\overline{\varphi}$ two complementary Musielak–Orlicz functions, and g is a nonlinearity satisfying the growth condition

$$
|g(x,t,s,\xi)| \leq c'(x,t) + b(s)\varphi(x,|\xi|),
$$

where $b\colon \mathbb{R} \to \mathbb{R}^+$ is a continuous nondecreasing function in $L^1(\mathbb{R})$ and $c'(\,\cdot\,,\,\cdot\,)$ is a given nonnegative function in $L^1(\mathrm{Q})$.

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On Orlicz spaces, it is well known that the solvability of (1.1) was established by Donaldson in [12] for $g = 0$ and by Robert [24] for $g = g(x,t,u)$ and also by Elmahi [13] for $g = g(x,t,u,\nabla u)$ with some conditions on the operator A and on the N-function.

Without assuming any restriction on the N-function, Elmahi and Meskine [14] studied problem (1.1) when $g = 0$. If $g = g(x,t,u,\nabla u)$ the same authors showed the existence of solutions for problem (1.1) in the variational case in [15] and then in [16] when $f \in L^1(\mathrm{Q})$.

The unilateral problem corresponding to (1.1) was studied in [6] by Azroul et al. in the case $f \in L^1(\mathrm{Q})$ and without assuming the sign condition on the nonlinearity g (see also [22]).

In the framework of variable exponent Sobolev spaces, Bendahmane et al. [7] proved the existence and uniqueness of renormalized solution for some nonlinear parabolic problem involving the $p(x)$-Laplacien. Also an existence result of solutions for a class of doubly nonlinear parabolic equations with variable exponents was established in [5].

In Musielak–Orlicz spaces, Ahmed oubeid, Benkirane and Sidi El vally [3] showed the existence of solutions for the nonlinear parabolic problem (1.1) where the second member f is taken in $W^{-1,x}E_{\overline{\varphi}}(\mathrm{Q})$. The same problem was studied in [2] by Ahmed et al. but without assuming the sign condition on the nonlinearity.

Our main goal in this paper is to prove, in Musielak–Orlicz spaces, the existence of solution for the unilateral problem associated to (1.1) under the assumption that the Musielak–Orlicz function φ depends only on $N-1$ coordinates of the spatial variable x, this assumption allows us to use a Poincaré inequality in this type of spaces (see Lemma 2.6). A Poincaré-type inequality was proved again by Fan [17] but the conditions assumed by the author there are different from those assumed in this paper (see [17, Theorem 1.2]).

The study of nonlinear partial differential equations in this type of spaces is strongly motivated by numerous phenomena of physics, namely the problems related to non-Newtonian fluids of strongly inhomogeneous behavior with a high ability of increasing their viscosity under a different stimulus, like shear rate, magnetic or electric field [18].

Further works concerning the Musielak spaces can be found in [17]–[19], [25].

2. Preliminaries

2.1. Musielak–Orlicz function. Let Ω be an open subset of $\mathbb{R}^N$ $(N \geq 2)$ and let φ be a real-valued function defined in $\Omega \times \mathbb{R}_+$ and satisfying the following conditions:

(i) $\varphi(x, \cdot)$ is an N-function for all $x \in \Omega$ $\Big($i.e. convex, nondecreasing, continuous, $\varphi(x,0) = 0$, $\varphi(x,t) > 0$ for all $t > 0$, $\lim\limits_{t\to 0} \sup\limits_{x\in\Omega} \varphi(x,t)/t = 0$ and $\lim\limits_{t\to\infty} \inf\limits_{x\in\Omega} \varphi(x,t)/t = \infty\Big)$;

(ii) $\varphi(\cdot, t)$ is a measurable function for all $t \geq 0$.

The function φ is called a Musielak–Orlicz function.

For a Musielak–Orlicz function φ we put $\varphi_x(t) = \varphi(x,t)$ and we associate its nonnegative reciprocal function φ_x^{-1}, with respect to t, that is

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x, \varphi_x^{-1}(t)) = t.$$

For a Musielak function φ we put

$$\overline{\varphi}(x,s) = \sup_{t\geq 0} \, (st - \varphi(x,t)).$$

$\overline{\varphi}$ is called the Musielak function complementary to φ (or conjugate of φ).

The Musielak–Orlicz function φ is said to satisfy the Δ_2-condition if for some $c_0 > 0$ and a nonnegative function h, integrable in Ω, we have

$$\varphi(x,2t) \leq \; c_0 \, \varphi(x,t) + h(x) \quad \text{for all } x \in \Omega \text{ and all } t \geq 0.$$

When the above inequality holds only for $t \geq t_0 > 0$, φ is said to satisfy the Δ_2-condition near infinity.

Let φ and γ be two Musielak–Orlicz functions, we say that φ dominates γ and we write $\gamma \prec \varphi$, near infinity (resp. globally) if there exist two positive constants c and t_0 such that $\gamma(x,t) \leq \varphi(x,ct)$ for almost all $x \in \Omega$ and for all $t \geq t_0$ (resp. for all $t \geq 0$, i.e. $t_0 = 0$).

The notation $\gamma \prec\prec \varphi$ means that γ grows essentially less rapidly than φ, i.e. for each $c > 0$, we have

$$\sup_{x\in\Omega} \frac{\gamma(x,ct)}{\varphi(x,t)} \to 0 \text{ as } t \to \infty.$$

REMARK 2.1 ([10]). If $\gamma \prec\prec \varphi$, then for all $\varepsilon > 0$ there exists $k(\varepsilon) > 0$ such that for almost all $x \in \Omega$ we have $\gamma(x,t) \leq k(\varepsilon) \, \varphi(x, \varepsilon t)$ for all $t \geq 0$.

2.2. Musielak–Orlicz space. The Musielak–Orlicz class $K_\varphi(\Omega)$ (respectively the Musielak–Orlicz space $L_\varphi(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that

$$\varrho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|) \, dx < +\infty$$

$$\left(\text{respectively, } \int_\Omega \varphi\left(x, \frac{|u(x)|}{\lambda}\right) dx < +\infty \quad \text{for some } \lambda > 0\right).$$

We define, in the space $L_\varphi(\Omega)$, the Luxemburg norm by

$$\|u\|_{\varphi,\Omega} = \inf \left\{ \lambda > 0 : \int_\Omega \varphi\left(x, \frac{|u(x)|}{\lambda}\right) dx \leq 1 \right\}$$

and the Orlicz norm by

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_{\overline{\varphi}} \leq 1} e \int_\Omega |u(x)\, v(x)|\, dx.$$

These two norms are equivalent [21].

The closure in $L_\varphi(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$. It is a separable space and $(E_{\overline{\varphi}}(\Omega))^* = L_\varphi(\Omega)$ [21]. We have $E_\varphi(\Omega) = K_\varphi(\Omega)$ if and only if $K_\varphi(\Omega) = L_\varphi(\Omega)$ if and only if φ satisfies the Δ_2-condition for large values of t or for all values of t, according to whether Ω has finite measure or not.

We define $W^1L_\varphi(\Omega) = \{u \in L_\varphi(\Omega) : D^\alpha u \in L_\varphi(\Omega)$, for all $|\alpha| \leq 1\}$ and $W^1E_\varphi(\Omega) = \{u \in E_\varphi(\Omega) : D^\alpha u \in E_\varphi(\Omega)$, for all $|\alpha| \leq 1\}$, where $\alpha = (\alpha_1, \ldots, \alpha_N)$, $|\alpha| = |\alpha_1| + \ldots + |\alpha_N|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^1L_\varphi(\Omega)$ is called the Musielak–Orlicz–Sobolev space.

Let $\overline{\varrho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq 1} \varrho_{\varphi,\Omega}(D^\alpha u)$ and

$$\|u\|^1_{\varphi,\Omega} = \inf\left\{\lambda > 0 : \overline{\varrho}_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) \leq 1\right\} \quad \text{for } u \in W^1L_\varphi(\Omega).$$

The functional $\|u\|^1_{\varphi,\Omega}$ is a norm on $W^1L_\varphi(\Omega)$ and the pair $\langle W^1L_\varphi(\Omega), \|u\|^1_{\varphi,\Omega}\rangle$ is a Banach space if φ satisfies the following condition [21]:

$$\text{there exists a constant } c > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x,1) \geq c. \tag{2.1}$$

The space $W^1L_\varphi(\Omega)$ is identified to a subspace of the product $\prod_{|\alpha|\leq 1} L_\varphi(\Omega) = \Pi L_\varphi$, this subspace is $\sigma(\Pi L_\varphi, \Pi E_{\overline{\varphi}})$-closed.

The space $W^1_0L_\varphi(\Omega)$ is defined as the $\sigma(\Pi L_\varphi, \Pi E_{\overline{\varphi}})$-closure of the Schwartz space $\mathfrak{D}(\Omega)$ in $W^1L_\varphi(\Omega)$ and the space $W^1_0E_\varphi(\Omega)$ as the (norm) closure of $\mathfrak{D}(\Omega)$ in $W^1L_\varphi(\Omega)$.

For two complementary Musielak–Orlicz functions φ and $\overline{\varphi}$, we have [21]:

(i) The Young inequality: $rs \leq \varphi(x,r) + \overline{\varphi}(x,s)$ for all $r, s \geq 0$, $x \in \Omega$.

(ii) The Hölder inequality:

$$\left|\int_\Omega u(x)\ v(x)\, dx\right| \leq 2\|u\|_{\varphi,\Omega}\, \|v\|_{\overline{\varphi},\Omega}, \quad \text{for all } u \in L_\varphi(\Omega),\ v \in L_{\overline{\varphi}}(\Omega).$$

We say that a sequence $u_n \subset L_\varphi(\Omega)$ converges to $u \in L_\varphi(\Omega)$ for the modular convergence if there exists a constant $\lambda > 0$ such that

$$\lim_{n\to\infty} \varrho_{\varphi,\Omega}\left(\frac{u_n - u}{\lambda}\right) = 0.$$

This implies the convergence for $\sigma(\Pi L_\varphi, \Pi L_{\overline{\varphi}})$ (Lemma 4.7 of [10]).

We say that a sequence of functions u_n converges to u for the modular convergence in $W^1L_\varphi(\Omega)$ (respectively, in $W_0^1L_\varphi(\Omega)$) if, for some $\lambda > 0$,

$$\lim_{n\to\infty} \overline{\varrho}_{\varphi,\Omega}\left(\frac{u_n - u}{\lambda}\right) = 0.$$

Let $W^{-1}L_{\overline{\varphi}}(\Omega)$ (respectively, $W^{-1}E_{\overline{\varphi}}(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{\varphi}}$ (respectively, $E_{\overline{\varphi}}$).

LEMMA 2.2 ([4]). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let φ and $\overline{\varphi}$ be two complementary Musielak–Orlicz functions which satisfy the following conditions*:

(a) *There exists a constant $c > 0$ such that $\inf_{x\in\Omega} \varphi(x,1) \geq c$.*

(b) *There exists a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x-y| \leq 1/2$ we have*

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leq t^{(A/\log(1/|x-y|))} \quad \textit{for all } t \geq 1. \tag{2.2}$$

(c)

$$\int_\Omega \varphi(x,1)\,dx < \infty. \tag{2.3}$$

(d) *There exists a constant $C > 0$ such that*

$$\overline{\varphi}(x,1) \leq C \quad \textit{a.e. in } \Omega. \tag{2.4}$$

Under these assumptions, $\mathfrak{D}(\Omega)$ is dense in $L_\varphi(\Omega)$, $\mathfrak{D}(\Omega)$ is dense in $W_0^1L_\varphi(\Omega)$ and $\mathfrak{D}(\overline{\Omega})$ is dense in $W^1L_\varphi(\Omega)$ for the modular convergence.

LEMMA 2.3 ([10]). *Let $F\colon \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let φ be a Musielak–Orlicz function and let $u \in W_0^1L_\varphi(\Omega)$. Then $F(u) \in W_0^1L_\varphi(\Omega)$. Moreover, if the set D of discontinuity points of F' is finite, we have*

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\,\dfrac{\partial u}{\partial x_i} & \textit{a.e. in } \{x \in \Omega : u(x) \notin D\}, \\ 0 & \textit{a.e. in } \{x \in \Omega : u(x) \in D\}. \end{cases}$$

LEMMA 2.4 ([10]). *Let $F\colon \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let φ be a Musielak–Orlicz function, then the mapping $T_F\colon W^1L_\varphi(\Omega) \to W^1L_\varphi(\Omega)$ defined by $T_F(u) = F(u)$ is sequentially continuous with respect to the weak* topology $\sigma(\Pi L_\varphi, \Pi E_{\overline{\varphi}})$.*

LEMMA 2.5. *Let $f_n, f \in L^1(\Omega)$ be such that $f_n \geq 0$ almost everywhere in Ω, $f_n \to f$ almost everywhere in Ω and*

$$\int_\Omega f_n(x)\,dx \to \int_\Omega f(x)\,dx.$$

Then $f_n \to f$ strongly in $L^1(\Omega)$.

LEMMA 2.6 ([4]). *Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N$ and let φ be a Musielak–Orlicz function satisfying the conditions of Lemma* 2.2. *Assume also that the function φ depends only on $N-1$ coordinates of x. Then there exists a constant $\lambda > 0$ depending only on Ω such that*

$$\int_\Omega \varphi(x,|v|)\,dx \leq \int_\Omega \varphi(x,\lambda|\nabla v|)\,dx \quad \textit{for all } v \in W_0^1 L_\varphi(\Omega).$$

COROLLARY 2.7 (Poincaré Inequality, [4]). *Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N$ and let φ be a Musielak–Orlicz function satisfying the same conditions of Lemma* 2.6. *Then there exists a constant $C > 0$ such that*

$$\|v\|_\varphi \leq C\,\|\nabla v\|_\varphi \quad \textit{for all } v \in W_0^1 L_\varphi(\Omega).$$

REMARK 2.8 ([2]). The result of Lemma 2.6 remains valid if we assume that the Musielak function φ decreases with respect to one of coordinates of x.

The following example shows that the integral form of the Poincaré inequality cannot, in general, hold.

EXAMPLE 2.9 ([11]). Let $p\colon (-2,2) \to [2,3]$ be a Lipschitz continuous exponent that equals 3 in $(-2,-1)\cup(1,2)$, 2 in $(-1/2,1/2)$ and is linear elsewhere. Let u_λ be a Lipschitz function such that $u_\lambda(\pm 2) = 0$, $u_\lambda = \lambda$ in $(-1,1)$ and $|u'_\lambda| = \lambda$ in $(-2,-1)\cup(1,2)$. Then

$$\frac{\overline{\varrho}_{p(\,\cdot\,)}(u_\lambda)}{\overline{\varrho}_{p(\,\cdot\,)}(u'_\lambda)} = \frac{\displaystyle\int_{-2}^{2} |u_\lambda|^{p(x)}\,dx}{\displaystyle\int_{-2}^{2} |u'_\lambda|^{p(x)}\,dx} \geq \frac{\displaystyle\int_{-1/2}^{1/2} \lambda^2\,dx}{\displaystyle 2\int_{-2}^{-1} |\lambda|^3\,dx} = \frac{1}{2\lambda} \to \infty \quad \text{as } \lambda \to 0^+.$$

2.3. Inhomogeneous Musielak–Orlicz spaces. Let Ω be a bounded domain of $\mathbb{R}^N$ and $\mathrm{Q} = \Omega \times [0,T]$, $T > 0$. Let φ be a Musielak–Orlicz function. For each $\alpha \in \mathbb{N}^N$, denote by D_x^α the distributional derivative on Q of order α with respect to the variable $x \in \mathbb{N}^N$.

The inhomogeneous Musielak–Orlicz-Sobolev spaces of order 1 are defined as follows:

$$W^{1,x}L_\varphi(\mathrm{Q}) = \{u \in L_\varphi(\mathrm{Q}) : D_x^\alpha u \in L_\varphi(\mathrm{Q}) \text{ for all } |\alpha| \leq 1\},$$
$$W^{1,x}E_\varphi(\mathrm{Q}) = \{u \in E_\varphi(\mathrm{Q}) : D_x^\alpha u \in E_\varphi(\mathrm{Q}) \text{ for all } |\alpha| \leq 1\}.$$

The last space is a subspace of the first one and both are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha|\leq 1} \|D_x^\alpha u\|_{\varphi,\mathrm{Q}}.$$

We can easily show that they form a complementary system when Ω is a Lipschitz domain. These spaces are considered as subspaces of the product space $\Pi L_\varphi(\mathrm{Q})$ which has $N+1$ copies. We shall also consider the weak topologies $\sigma(\Pi L_\varphi, \Pi E_{\overline{\varphi}})$

and $\sigma(\Pi L_\varphi, \Pi L_{\overline{\varphi}})$. If $u \in W^{1,x}L_\varphi(\mathrm{Q})$ then the function: $t \mapsto u(t) = u(t, \cdot)$ is defined on $(0,T)$ with values in $W^1L_\varphi(\Omega)$, and if, further, $u \in W^{1,x}E_\varphi(\mathrm{Q})$ then this function is a $W^1E_\varphi(\Omega)$-valued and is strongly measurable. Furthermore, the following imbedding holds: $W^{1,x}E_\varphi(\mathrm{Q}) \subset L^1(0,T,W^1E_\varphi(\Omega))$. The space $W^{1,x}L_\varphi(\mathrm{Q})$ is not in general separable, if $u \in W^{1,x}L_\varphi(\mathrm{Q})$, we can not conclude that the function $u(t)$ is measurable on $(0,T)$. However, the scalar function $t \mapsto \|u(t)\|_{\varphi,\Omega}$ is in $L^1(0,T)$.

The space $W_0^{1,x}E_\varphi(\mathrm{Q})$ is defined as the (norm) closure in $W^{1,x}E_\varphi(\mathrm{Q})$ of $\mathfrak{D}(\mathrm{Q})$. We can show as in [9] that when Ω is a Lipschitz domain, then each element u of the closure of $\mathfrak{D}(\mathrm{Q})$ with respect to the weak* topology $\sigma(\Pi L_\varphi, \Pi E_{\overline{\varphi}})$ is limit, in $W^{1,x}L_\varphi(\mathrm{Q})$, of some subsequence $(u_n) \subset \mathfrak{D}(\mathrm{Q})$ for the modular convergence, i.e. there exists $\lambda > 0$ such that for all $|\alpha| \leq 1$, we have

$$\int_{\mathrm{Q}} \varphi\bigg(x, \frac{D_x^\alpha u_n - D_x^\alpha u}{\lambda}\bigg)\,dx\,dt \to 0 \quad \text{as } n \to \infty.$$

This implies that (u_n) converges to u in $W^{1,x}L_\varphi(\mathrm{Q})$ for the weak topology $\sigma(\Pi L_\varphi, \Pi L_{\overline{\varphi}})$. Consequently

$$(\overline{\mathfrak{D}(\mathrm{Q})})^{\sigma(\Pi L_\varphi, \Pi E_{\overline{\varphi}})} = (\overline{\mathfrak{D}(\mathrm{Q})})^{\sigma(\Pi L_\varphi, \Pi L_{\overline{\varphi}})}.$$

This space will be denoted by $W_0^{1,x}L_{\overline{\varphi}}(\mathrm{Q})$. Furthermore,

$$W_0^{1,x}E_\varphi(\mathrm{Q}) = W_0^{1,x}L_\varphi(\mathrm{Q}) \cap \Pi E_\varphi.$$

By using Corollary 2.7, there is a constant $c > 0$ such that for all $u \in W_0^{1,x}L_\varphi(\mathrm{Q})$ one has

$$\sum_{|\alpha| \leq 1} \|D_x^\alpha u\|_{\varphi,\mathrm{Q}} \leq c \sum_{|\alpha| = 1} \|D_x^\alpha u\|_{\varphi,\mathrm{Q}}.$$

Thus both sides of the last inequality are equivalent norms on $W_0^{1,x}L_\varphi(\mathrm{Q})$. We have then the following complementary system:

$$\begin{pmatrix} W_0^{1,x}L_\varphi(\mathrm{Q}) & F \\ W_0^{1,x}E_\varphi(\mathrm{Q}) & F_0 \end{pmatrix},$$

F being the dual space of $W_0^{1,x}E_\varphi(\mathrm{Q})$. It is also, except for an isomorphism, the quotient of $\Pi L_{\overline{\varphi}}$ by the polar set $(W_0^{1,x}E_\varphi(\mathrm{Q}))^\perp$, and will be denoted by $F = W^{-1,x}L_{\overline{\varphi}}(\mathrm{Q})$ and it is shown that

$$W^{-1,x}L_{\overline{\varphi}}(\mathrm{Q}) = \bigg\{ f = \sum_{|\alpha| \leq 1} D_x^\alpha f_\alpha : f_\alpha \in L_{\overline{\varphi}}(\mathrm{Q}) \bigg\}.$$

This space will be equipped with the usual quotient norm

$$\|f\| = \inf \sum_{|\alpha| \leq 1} \|f_\alpha\|_{\overline{\varphi},\mathrm{Q}}$$

where the inf is taken on all possible decompositions

$$f = \Big\{ \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha : f_\alpha \in L_{\overline{\varphi}}(\mathrm{Q}) \Big\}.$$

The space F_0 is then given by

$$F_0 = \Big\{ f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha : f_\alpha \in E_{\overline{\varphi}}(\mathrm{Q}) \Big\}$$

and denoted by $W^{-1,x}E_{\overline{\varphi}}(\mathrm{Q})$.

Definition 2.10. We say that $u_n \to u$ in $W^{-1,x}L_{\overline{\varphi}}(\mathrm{Q}) + L^1(\mathrm{Q})$ for the modular convergence if we can write

$$u_n = \sum_{|\alpha|\leq 1} D_x^\alpha u_n^\alpha + u_n^0 \quad \text{and} \quad u = \sum_{|\alpha|\leq 1} D_x^\alpha u^\alpha + u^0$$

with $u_n^\alpha \to u^\alpha$ in $L_{\overline{\varphi}}(\mathrm{Q})$ for the modular convergence for all $|\alpha| \leq 1$ and $u_n^0 \to u^0$ strongly in $L^1(\mathrm{Q})$.

Remark 2.11. We can easily check, using Lemma 2.3, that each uniformly Lipschitzian mapping F, with $F(0) = 0$, acts in inhomogeneous Musielak–Orlicz–Sobolev spaces of order 1: $W^{1,x}L_\varphi(\mathrm{Q})$ and $W_0^{1,x}L_\varphi(\mathrm{Q})$.

Let us define, for all $\mu > 0$ and all $(x,t) \in \mathrm{Q}$, the time mollification of a function $u \in L_\varphi(\mathrm{Q})$, by

$$u_\mu(x,t) = \mu \int_{-\infty}^{t} \widetilde{u}(x,s) \exp(\mu(s-t))\, ds, \tag{2.5}$$

where $\widetilde{u}(x,s) = u(x,s)\chi_{(0,T)}(s)$ is the zero extension of u.

Lemma 2.12. [3]

(a) *If $u \in L_\varphi(\mathrm{Q})$, then $u_\mu \to u$ in $L_\varphi(\mathrm{Q})$ for the modular convergence, as $\mu \to +\infty$.*

(b) *If $u \in W^{1,x}L_\varphi(\mathrm{Q})$, then $u_\mu \to u$ in $W^{1,x}L_\varphi(\mathrm{Q})$ for the modular convergence, as $\mu \to +\infty$, and $\partial u_\mu/\partial t = \mu(u - u_\mu)$.*

Lemma 2.13 ([3]). *Let φ be a Musielak–Orlicz function and let u_n be a sequence in $W^{1,x}L_\varphi(\mathrm{Q})$ such that u_n converges to u weakly in $W^{1,x}L_\varphi(\mathrm{Q})$ for $\sigma(\Pi L_\varphi, \Pi E_{\overline{\varphi}})$ and $\partial u_n/\partial t = h_n + k_n$ in $\mathfrak{D}'(\mathrm{Q})$ with $(h_n)_n$ bounded in $W^{-1,x}L_{\overline{\varphi}}(\mathrm{Q})$ and $(k_n)_n$ bounded in the space $\mathcal{M}(\mathrm{Q})$. Then, u_n converges to u strongly in $L^1_{\mathrm{loc}}(\mathrm{Q})$. If further, $u_n \in W_0^{1,x}L_{\overline{\varphi}}(\mathrm{Q})$, then u_n converges to u strongly in $L^1(\mathrm{Q})$.*

Lemma 2.14. *Let Ω be a bounded open Lipschitz domain of $\mathbb{R}^N$, then*

$$\Big\{ u \in W_0^{1,x}L_{\overline{\varphi}}(\mathrm{Q}) : \frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{\varphi}}(\Omega) + L^1(\mathrm{Q}) \Big\} \subset \mathcal{C}([0,T], L^1(\Omega)).$$

Proof. It is easily adapted from that given in Lemma 5 of [16]. $\square$

Proposition 2.15 ([3]). *Let* $(u_n)_n$ *be a bounded sequence in* $W_0^{1,x}L_\varphi(\mathrm{Q})$ *such that* $\partial u_n/\partial t$ *is bounded in* $W^{-1,x}L_{\overline{\varphi}}(\Omega)+L^1(\mathrm{Q})$. *Then* u_n *is relatively compact in* $L^1(\mathrm{Q})$.

Lemma 2.16 ([3]). *Let* $u \in W_0^{1,x}L_\varphi(\mathrm{Q})$ *be such that* $\partial u/\partial t \in W^{-1,x}L_{\overline{\varphi}}(\Omega)+L^1(\mathrm{Q})$ *and* $u \geq \psi$ *with* $\psi \in W_0^{1,x}E_\varphi(\mathrm{Q}) \cap L^\infty(\mathrm{Q})$. *Then, there exists a smooth function* (u_j) *such that* $u_j \geq \psi$, $u_j \to u$ *for the modular convergence in* $W_0^{1,x}L_\varphi(\mathrm{Q})$ *and* $\partial u_j/\partial t \to \partial u/\partial t$ *for the modular convergence in* $W^{-1,x}L_{\overline{\varphi}}(\mathrm{Q})+L^1(\mathrm{Q})$.

3. Assumptions and main result

Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N$ $(N \geq 2)$ and $\mathrm{Q} = \Omega \times (0,T)$, $T>0$. Let φ and γ be two Musielak–Orlicz functions such that $\gamma \prec\prec \varphi$ and φ satisfies the conditions of Lemma 2.6.

Let $A\colon D(A) \subset W_0^{1,x}L_\varphi(\mathrm{Q}) \to W^{-1,x}L_{\overline{\varphi}}(\mathrm{Q})$ be a mapping given by $A(u) = -\operatorname{div} a(x,t,u,\nabla u)$ where $a\colon \mathrm{Q}\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying for almost every $(x,t) \in Q$ and all $s \in \mathbb{R}, \xi \neq \xi^* \in \mathbb{R}^N$

$$|a(x,t,s,\xi)| \leq k_1(c(x,t) + \overline{\varphi}_x^{-1}\gamma(x,k_2|s|) + \overline{\varphi}_x^{-1}\varphi(x,k_3|\xi|)), \tag{3.1}$$

$$(a(x,t,s,\xi) - a(x,t,s,\xi_*))\,(\xi-\xi_*) > 0, \tag{3.2}$$

$$a(x,t,s,\xi)\cdot\xi \geq \alpha\,\varphi(x,|\xi|), \tag{3.3}$$

where $c(\,\cdot\,,\,\cdot\,)$ belongs to $E_{\overline{\varphi}}(\Omega)$, $c \geq 0$, $k_i > 0$ $(i=1,2,3)$ and $\alpha \in \mathbb{R}_+^*$.

Let $g\colon \mathrm{Q}\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that, for almost every $(x,t) \in \mathrm{Q}$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$,

$$|g(x,t,s,\xi)| \leq b(|s|)(c'(x,t) + \varphi(x,|\xi|)), \tag{3.4}$$

where $b\colon \mathbb{R} \to \mathbb{R}^+$ is a continuous nondecreasing function in $L^1(\mathbb{R})$ and $c'(\,\cdot\,,\,\cdot\,)$ is a given nonnegative function in $L^1(\mathrm{Q})$. Finally, assume that

$$f \in L^1(\mathrm{Q}) \quad \text{and} \quad u_0 \in L^1(\Omega). \tag{3.5}$$

For all $s \in \mathbb{R}$ and $k \geq 0$, we define the truncation at height k by

$$T_k(s) = \max(-k, \min(k,s)).$$

We shall prove the following existence theorem

Theorem 3.1. *Assume that* (3.1)–(3.5) *hold true, then there exists at least one solution of the unilateral problem corresponding to* (1.1) *in the following* *sense*:

$$
(3.6)\qquad
\begin{cases}
u \geq \psi \quad \text{a.e. in } \mathrm{Q}, \quad T_k(u) \in W_0^{1,x}L_\varphi(\mathrm{Q}), \quad g(x,t,u,\nabla u) \in L^1(\mathrm{Q}),\\
\displaystyle\int_\Omega S_k(u(\tau)-v(\tau))\,dx + \int_0^\tau \left\langle \frac{\partial v}{\partial t}, T_k(u-v)\right\rangle dt\\
\qquad \displaystyle + \int_{\mathrm{Q}_\tau} a(x,t,u,\nabla u)\cdot \nabla T_k(u-v)\,dx\,dt\\
\qquad \displaystyle + \int_{\mathrm{Q}_\tau} g(x,t,u,\nabla u) T_k(u-v)\,dx\,dt\\
\quad \displaystyle \leq \int_{\mathrm{Q}_\tau} f\ T_k(u-v)\,dx\,dt + \int_\Omega S_k(u_0 - v(0))\,dx,\\
\text{for all } v \in W_0^{1,x}L_\varphi(\mathrm{Q}) \cap L^\infty(\mathrm{Q}) \text{ such that}\\
\displaystyle\frac{\partial v}{\partial t} \in W^{-1,x}L_{\overline{\varphi}}(\mathrm{Q}) + L^1(\mathrm{Q})\\
\text{and } \ v \geq \psi \text{ almost everywhere in } \mathrm{Q}\\
\text{and for all } k > 0,\ \ \tau \in (0,T),
\end{cases}
$$

where $\mathrm{Q}_\tau = \Omega \times (0,\tau)$, $\psi \in W_0^{1,x}E_\varphi(\mathrm{Q}) \cap L^\infty(\mathrm{Q})$ *and* $S_k(r) = \int_0^r T_k(s)\,ds$.

Proof. *Step 1. A priori estimates.* Consider the following approximate problem:

$$
(\mathcal{P}_n)\qquad
\begin{cases}
u_n \in W_0^{1,x}L_\varphi(\mathrm{Q}), \quad u_n(x,0) = u_{0n} \quad \text{a.e. in } \Omega,\\
\displaystyle\int_0^T \left\langle \frac{\partial u_n}{\partial t}, u_n - v\right\rangle dt + \int_{\mathrm{Q}} a(x,t,u_n,\nabla u_n)\cdot\nabla(u_n - v)\,dx\,dt\\
\qquad \displaystyle + \int_{\mathrm{Q}} g_n(x,t,u_n,\nabla u_n)(u_n - v)\,dx\,dt\\
\quad \displaystyle -n\int_{\mathrm{Q}} T_n(u_n-\psi)^-(u_n - v)\,dx\,dt\\
\quad \displaystyle = \int_{\mathrm{Q}} f_n(u_n - v)\,dx\,dt \quad \text{for all } \ v \in W_0^{1,x}L_\varphi(\mathrm{Q}) \cap L^\infty(\mathrm{Q}),
\end{cases}
$$

where $(f_n) \subset \mathfrak{D}(\mathrm{Q})$ such that $f_n \to f$ strongly in $L^1(\mathrm{Q})$, $u_{0n} \subset \mathfrak{D}(\Omega)$ such that $u_{0n} \to u_0$ strongly in $L^1(\Omega)$ and $g_n(x,t,s,\xi) = g(x,t,s,\xi)/(1 + |g(x,t,s,\xi)|/n)$. Since $|g_n(x,t,s,\xi)| \leq |g(x,t,s,\xi)|$ and $|g_n(x,t,s,\xi)| \leq n$, then there exists, thanks to [3], at least one solution of the approximate problem $(\mathcal{P}_n)$.

For $h \geq \|\psi\|_\infty$ and $G(r) = \int_0^r (b(s)/\alpha)\,ds$, where α is the constant defined in formula (3.3), taking $v = u_n - \exp(G(u_n))T_k(u_n - T_h(u_n))$ as a test function in the problem $(\mathcal{P}_n)$, we get

$$
\begin{aligned}
&\int_0^T \left\langle \frac{\partial u_n}{\partial t}, \exp(G(u_n))T_k(u_n - T_h(u_n))\right\rangle dt\\
&\quad + \int_{\mathrm{Q}} a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(u_n - T_h(u_n))\exp(G(u_n))\,dx\,dt\\
&\quad + \int_{\mathrm{Q}} a(x,t,u_n,\nabla u_n)\cdot\nabla(\exp(G(u_n)))T_k(u_n - T_h(u_n))\,dx\,dt
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_Q g_n(x,t,u_n,\nabla u_n)\exp(G(u_n))T_k(u_n - T_h(u_n))\,dx\,dt \\
&- n\int_Q m(T_n(u_n-\psi)^-)\exp(G(u_n))T_k(u_n - T_h(u_n))\,dx\,dt \\
&= \int_Q f_n \exp(G(u_n))T_k(u_n - T_h(u_n))\,dx\,dt
\end{aligned}
$$

which gives

$$
\begin{aligned}
&\int_0^T \left\langle \frac{\partial u_n}{\partial t}, \exp(G(u_n))T_k(u_n - T_h(u_n))\right\rangle dt \\
&\quad + \int_{\{h\le|u_n|\le h+k\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n \exp(G(u_n)\,dx\,dt \\
&\quad + \int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla u_n \frac{b(u_n)}{\alpha}\exp(G(u_n))T_k(u_n - T_h(u_n))\,dx\,dt \\
&\quad + \int_Q g_n(x,t,u_n,\nabla u_n)\exp(G(u_n))T_k(u_n - T_h(u_n))\,dx\ dt \\
&\quad - n\int_Q T_n(u_n-\psi)^-\exp(G(u_n))T_k(u_n - T_h(u_n))\,dx\,dt \\
&= \int_Q f_n \exp(G(u_n))T_k(u_n - T_h(u_n))\,dx\,dt.
\end{aligned}
$$

Using the coercivity conditions (3.3) and (3.4), we obtain

$$
\begin{aligned}
&\int_0^T \left\langle \frac{\partial u_n}{\partial t}, \exp(G(u_n))T_k(u_n - T_h(u_n))\right\rangle dt \\
&\quad + \int_{\{h\le|u_n|\le h+k\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n \exp(G(u_n))\,dx\,dt \\
&\quad - n\int_Q T_n(u_n-\psi)^-\exp(G(u_n)T_k(u_n - T_h(u_n))\,dx\,dt \\
&\le \int_Q (|f_n| + c'(x,t))\exp(G(u_n))T_k(u_n - T_h(u_n))\,dx\,dt.
\end{aligned} \tag{3.7}
$$

On the other hand, we have

$$
\begin{aligned}
&\int_0^T \left\langle \frac{\partial u_n}{\partial t}, \exp(G(u_n))T_k(u_n - T_h(u_n))\right\rangle dt \\
&\qquad\qquad = \int_\Omega B_{k,h}(u_n(T))\,dx - \int_\Omega B_{k,h}(u_{0n})\,dx,
\end{aligned}
$$

where

$$
B_{k,h}(r) = \int_0^r T_k(s - T_h(s))\exp(G(s))\,ds.
$$

Since $|G(u_n)| \le \|b\|_{L^1(\mathbb{R})}/\alpha$, we have

$$
0 \le \int_\Omega B_{k,h}(u_{0n})\,dx \le k\exp\left(\frac{\|b\|_{L^1(\mathbb{R})}}{\alpha}\right)\|u_{0n}\|_{L^1(\Omega)} = C\,k.
$$

Using the fact that $\int_\Omega B_{k,h}(u_n(T))\,dx \geq 0$ and the coercivity condition (3.3), we get from (3.7)

$$\alpha \int_{\{h\leq |u_n|\leq h+k\}} \varphi(x,|\nabla u_n|)\exp(G(u_n))\,dx\,dt \\ - n\int_Q T_n(u_n-\psi)^-\exp(G(u_n)T_k(u_n-T_h(u_n))\,dx\,dt \leq C\,k,$$

thus

$$-n\int_Q T_n(u_n-\psi)^-\exp(G(u_n))\frac{T_k(u_n-T_h(u_n))}{k}\,dx\,dt \leq C.$$

Since

$$-n\int_Q T_n(u_n-\psi)^-\exp(G(u_n))T_k(u_n-T_h(u_n))\,dx\,dt \geq 0$$

and

$$\exp(G(-\infty)) \leq \exp(G(u_n)) \leq \exp(G(+\infty)),$$
$$\exp(|G(\pm\infty)|) \leq \exp\bigg(\frac{\|b\|_{L^1(\mathbb{R})}}{\alpha}\bigg), \tag{3.8}$$

then, by letting k tend to infinity and using Fatou's Lemma, we obtain

$$n\int_Q T_n(u_n-\psi)^-\,dx\,dt \leq C. \tag{3.9}$$

Now, taking $v = u_n - \exp(G(u_n))T_k(u_n)^+\chi_{(0,\tau)}$ as a test function in the approximate problem $(\mathcal{P}_n)$ with $\tau\in(0,T)$, we get

$$\int_0^T \bigg\langle \frac{\partial u_n}{\partial t}, \exp(G(u_n))T_k(u_n)^+\chi_{(0,\tau)}\bigg\rangle\,dt \\ + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\cdot\nabla(\exp(G(u_n))T_k(u_n)^+)\,dx\,dt \\ + \int_{Q_\tau} g_n(x,t,u_n,\nabla u_n)\exp(G(u_n))T_k(u_n)^+\,dx\,dt \\ - n\int_{Q_\tau} T_n(u_n-\psi)^-\exp(G(u_n))T_k(u_n)^+\,dx\,dt \\ = \int_{Q_\tau} f_n\exp(G(u_n))T_k(u_n)^+\,dx\,dt,$$

then

$$\int_0^T \bigg\langle \frac{\partial u_n}{\partial t}, \exp(G(u_n))T_k(u_n)^+\chi_{(0,\tau)}\bigg\rangle\,dt \\ + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(u_n)^+\exp(G(u_n))\,dx\,dt \\ + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\frac{b(u_n)}{\alpha}\exp(G(u_n))T_k(u_n)^+\,dx\,dt$$

$$- n\int_{Q_\tau} T_n(u_n-\psi)^- \exp(G(u_n))T_k(u_n)^+\,dx\,dt$$
$$\le \int_{Q_\tau} |g_n(x,t,u_n,\nabla u_n)|\exp(G(u_n))T_k(u_n)^+\,dx\,dt$$
$$+ \int_{Q_\tau} |f_n|\exp(G(u_n))T_k(u_n)^+\,dx\,dt.$$

Let $B_k(r)=\int_0^r T_k(s)^+\exp(G(s))\,ds$, we have

$$|B_k(r)|\le k\exp\left(\frac{\|b\|_{L^1(\mathbb{R})}}{\alpha}\right)r,$$

and

$$\int_0^\tau \left\langle \frac{\partial u_n}{\partial t}, \exp(G(u_n))T_k(u_n)^+\chi_{(0,\tau)}\right\rangle dt = \int_\Omega B_k(u_n(\tau))\,dx - \int_\Omega B_k(u_{0n})\,dx.$$

Then

$$\int_0^T \left\langle \frac{\partial u_n}{\partial t}, \exp(G(u_n))T_k(u_n)^+\chi_{(0,\tau)}\right\rangle dt$$
$$\ge \int_\Omega B_k(u_n(\tau))\,dx - k\exp\left(\frac{\|b\|_{L^1(\mathbb{R})}}{\alpha}\right)\|u_{0n}\|_{L^1(\Omega)},$$

which gives

$$\begin{aligned}
(3.10)\quad & \int_\Omega B_k(u_n(\tau))\,dx + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(u_n)^+\exp(G(u_n))\,dx\,dt\\
&+\int_{Q_\tau} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\frac{b(u_n)}{\alpha}\exp(G(u_n))T_k(u_n)^+\,dx\,dt\\
&-n\int_{Q_\tau} T_n(u_n-\psi)^-\exp(G(u_n))T_k(u_n)^+\,dx\,dt\\
&\le \int_{Q_\tau} |g_n(x,t,u_n,\nabla u_n)|\exp(G(u_n))T_k(u_n)^+\,dx\,dt\\
&+\int_{Q_\tau} |f_n|\exp(G(u_n))T_k(u_n)^+\,dx\,dt + k\exp\left(\frac{\|b\|_{L^1(\mathbb{R})}}{\alpha}\right)\|u_{0n}\|_{L^1(\Omega)}.
\end{aligned}$$

Moreover, since $B_k(u_n(\tau))\ge 0$, we have

$$\begin{aligned}
&\int_{Q_\tau} a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(u_n)^+\exp(G(u_n))\,dx\,dt\\
&\qquad - n\int_{Q_\tau} T_n(u_n-\psi)^-\exp(G(u_n))T_k(u_n)^+\,dx\,dt\\
&\qquad \le \int_{Q_\tau} (|f_n|+c'(x,t))\exp(G(u_n))T_k(u_n)^+\,dx\,dt\\
&\qquad + k\exp\left(\frac{\|b\|_{L^1(\mathbb{R})}}{\alpha}\right)\|u_{0n}\|_{L^1(\Omega)},
\end{aligned}$$

then

$$
\begin{aligned}
(3.11)\quad \int_{\{0\le u_n\le k\}} & a(x,t,u_n,\nabla u_n)\cdot\nabla u_n \exp(G(u_n))\,dx\,dt \\
& - n\int_{Q_\tau} T_n(u_n-\psi)^- \exp(G(u_n))T_k(u_n)^+\,dx\,dt \\
& \le k\exp\bigg(\frac{\|b\|_{L^1(\mathbb{R})}}{\alpha}\bigg)(\|f\|_{L^1(Q)}+\|c'\|_{L^1(Q)}+\|u_{0n}\|_{L^1(\Omega)}) = Ck.
\end{aligned}
$$

Using (3.9), we have

$$
\begin{aligned}
\bigg| -n\int_{Q_\tau} & T_n(u_n-\psi)^- \exp(G(u_n))T_k(u_n)^+\,dx\,dt\bigg| \\
& \le k\exp\bigg(\frac{\|b\|_{L^1(\mathbb{R})}}{\alpha}\bigg)\, n\int_{Q_\tau} T_n(u_n-\psi)^-\,dx\,dt = Ck,
\end{aligned}
$$

therefore, inequality (3.11) becomes

$$
\int_{\{0\le u_n\le k\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n \exp(G(u_n)\,dx\,dt \le Ck
$$

and, by using (3.8), we obtain

$$
(3.12)\qquad \int_{\{0\le u_n\le k\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\,dx\,dt \le Ck.
$$

Hence, from (3.3), we get

$$
(3.13)\qquad \int_{\{0\le u_n\le k\}} \varphi(x,|\nabla u_n|)\,dx\,dt \le Ck.
$$

On the other hand, if we take $v = u_n + \exp(G(u_n))T_k(u_n)^-\chi_{(0,\tau)}$ as a test function in the approximate problem $(\mathcal{P}_n)$ and we argue as above, we obtain

$$
(3.14)\qquad \int_{\{-k\le u_n\le 0\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\,dx\,dt \le Ck
$$

and so

$$
(3.15)\qquad \int_{\{-k\le u_n\le 0\}} \varphi(x,|\nabla u_n|)\,dx\,dt \le Ck.
$$

Combining (3.13) and (3.15), we deduce that

$$
(3.16)\qquad \int_Q \varphi(x,|\nabla T_k(u_n)|)\,dx\,dt = \int_{\{|u_n|\le k\}} \varphi(x,|\nabla u_n|)\,dx\,dt \le Ck.
$$

Finally,

$$
(3.17)\qquad \|T_k(u_n)\|_{W_0^{1,x}L_\varphi(Q)} \le Ck \quad \text{for all } n\in\mathbb{N},
$$

where C is a constant independent of n.

Now, we shall prove that for all $k > h > 0$, there exists a constant C such that

$$\int_Q \varphi(x, |\nabla T_k(u_n - T_h(u_n))|)\,dx\,dt \le Ck. \tag{3.18}$$

For $k > h > 0$ and a real positive η small enough, testing $(\mathcal{P}_n)$ by the admissible function $v = u_n - \eta \exp(G(u_n))T_k(u_n - T_h(u_n))^+\chi_{(0,\tau)}$, we get

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial u_n}{\partial t}, \exp(G(u_n))T_k(u_n - T_h(u_n))^+\chi_{(0,\tau)} \right\rangle dt \\
&\quad + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\cdot \nabla T_k(u_n - T_h(u_n))^+ \exp(G(u_n))\,dx\,dt \\
&\quad + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\cdot \nabla u_n \frac{b(u_n)}{\alpha} \exp(G(u_n))T_k(u_n - T_h(u_n))^+\,dx\,dt \\
&\quad - n\int_{Q_\tau} T_n(u_n - \psi)^- \exp(G(u_n))T_k(u_n - T_h(u_n))^+\,dx\,dt \\
&\le \int_{Q_\tau} |g_n(x,t,u_n,\nabla u_n)| \exp(G(u_n))T_k(u_n - T_h(u_n))^+\,dx\,dt \\
&\quad + \int_{Q_\tau} |f_n| \exp(G(u_n))T_k(u_n - T_h(u_n))^+\,dx\,dt.
\end{aligned}$$

Let $B^+_{k,h}(r) = \int_0^r T_k(s - T_h(s))^+ \exp(G(s))\,ds$. Using (3.4) and (3.9), yields

$$\begin{aligned}
&\int_\Omega B^+_{k,h}(u_n(\tau))\,dx + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\cdot \nabla T_k(u_n - T_h(u_n))^+ \exp(G(u_n))\,dx\,dt \\
&\qquad \le \int_{Q_\tau} (|f_n| + c'(x,t)) \exp(G(u_n))T_k(u_n - T_h(u_n))^+\,dx\,dt \\
&\qquad\quad + k\exp\left(\frac{b(u_n)}{\alpha}\right)\left(\|u_{0n}\|_{L^1(\Omega)} + C\right),
\end{aligned}$$

then, by using the fact that $B^+_{k,h} > 0$, we get

$$\begin{aligned}
&\int_{\{h \le u_n \le h+k\}} a(x,t,u_n,\nabla u_n)\cdot \nabla u_n \exp(G(u_n))\,dx\,dt \\
&\qquad \le k\exp\left(\frac{\|b\|_{L^1(\mathbb{R})}}{\alpha}\right)(\|f\|_{L^1(Q)} + \|c'\|_{L^1(Q)} + \|u_{0n}\|_{L^1(\Omega)} + C) = Ck
\end{aligned}$$

which gives by (3.3),

$$\int_{\{h \le u_n \le h+k\}} \varphi(x, |\nabla u_n|)\,dx\,dt \le Ck, \tag{3.19}$$

where C is a positive constant not depending on n, k and h.

Similarly, taking $v = u_n + \eta\exp(-G(u_n))T_k(u_n - T_h(u_n))^-\chi_{(0,\tau)}$ as a test function in $(\mathcal{P}_n)$, we obtain

$$\int_{\{-h-k \le u_n \le -h\}} \varphi(x, |\nabla u_n|)\,dx\,dt \le Ck. \tag{3.20}$$

Combining (3.19) and (3.20), we obtain the estimate (3.18).

On the other hand, let $k > h > 0$ be large enough. By Lemma 2.6, there exists a positive constant λ such that

$$\int_{\mathrm{Q}} \varphi\bigg(x, \frac{|T_k(u_n - T_h(u_n))|}{\lambda}\bigg)\, dx\, dt \le \int_{\mathrm{Q}} \varphi(x, |\nabla T_k(u_n - T_h(u_n))|)\, dx\, dt \le Ck,$$

which implies that

$$\begin{aligned} &\operatorname{meas}\{|u_n - T_h(u_n)| > k\} \\ &\quad \le \frac{1}{\inf\limits_{x\in\Omega} \varphi(x, k/\lambda)} \int_{\{|u_n - T_h(u_n)| > k\}} \varphi\bigg(x, \frac{k}{\lambda}\bigg)\, dx\, dt \\ &\quad \le \frac{1}{\inf\limits_{x\in\Omega} \varphi(x, k/\lambda)} \int_{\mathrm{Q}} \varphi\bigg(x, \frac{1}{\lambda}|T_k(u_n - T_h(u_n))|\bigg)\, dx\, dt \le \frac{C, k}{\inf\limits_{x\in\Omega} \varphi(x, k/\lambda)} \end{aligned}$$

for all n and for all $k > h > 0$. We have for all $n \in \mathbb{N}$ and for all $k > h > 0$

$$\operatorname{meas}\{|u_n| > k\} \le \operatorname{meas}\{|u_n - T_h(u_n)| > k - h\} \le \frac{C(k-h)}{\inf\limits_{x\in\Omega} \varphi(x, (k-h)/\lambda)}.$$

Letting k to infinity, we deduce that $\operatorname{meas}\{|u_n| > k\} \to 0$ as $k \to \infty$. For any $\mu > 0$, we have

$$\begin{aligned} \text{(3.21)}\quad \operatorname{meas}\{|u_n - u_m| > \mu\} &\le \operatorname{meas}\{|u_n| > k\} \\ &\quad + \operatorname{meas}\{|u_m| > k\} + \operatorname{meas}\{|T_k(u_n) - T_k(u_m)| > \mu\}. \end{aligned}$$

Since $T_k(u_n)$ is bounded in $W_0^{1,x}L_\varphi(\mathrm{Q})$, there exists some $v_k \in W_0^{1,x}L_\varphi(\mathrm{Q})$ such that $T_k(u_n) \rightharpoonup v_k$ weakly in $W_0^{1,x}L_\varphi(\mathrm{Q})$ for $\sigma(\Pi L_\varphi, \Pi E_{\overline{\varphi}})$, strongly in $E_\varphi(\mathrm{Q})$ and a.e. in Q. Then, we can assume that $T_k(u_n)$ is a Cauchy sequence in measure in Q.

Let $\varepsilon > 0$. By (3.21) there exists $k(\varepsilon) > 0$ such that

$$\operatorname{meas}\{|u_n - u_m| > \mu\} \le \varepsilon, \quad \text{for all } n, m \ge h_0\ (k(\varepsilon), \mu).$$

This proves that $(u_n)_n$ is a Cauchy sequence in measure in Q, thus it converges almost everywhere to some measurable function u. Then

$$\begin{aligned} \text{(3.22)}\quad T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,x}L_\varphi(\mathrm{Q}) \text{ for } \sigma(\Pi L_\varphi, \Pi E_{\overline{\varphi}}),& \\ \text{strongly in } E_\varphi(\mathrm{Q}) \text{ and a.e. in Q.}& \end{aligned}$$

Step 2. Almost everywhere convergence of the gradients. For $k \in \mathrm{N}$, let $(v_j^k)_j \in \mathfrak{D}(\mathrm{Q})$ such that $v_j^k \to T_k(u)$ for the modular convergence in $W_0^{1,x}L_\varphi(\mathrm{Q})$. Denoting by $\chi_{j,s}$ and χ_s, respectively, the characteristic functions of the sets $\mathrm{Q}^{j,s} = \{(x,t) \in \mathrm{Q} : |\nabla T_k(v_j^k)| \le s\}$ and $\mathrm{Q}^s = \{(x,t) \in \mathrm{Q} : |\nabla T_k(u)| \le s\}$. We will introduce the following function:

$$h_m(s) = \begin{cases} 1 & \text{if } |s| \le m, \\ m+1-|s| & \text{if } m \le |s| \le m+1, \\ 0 & \text{if } |s| \ge m+1, \end{cases} \tag{3.23}$$

where m is a nonnegative real parameter with $m > k$.

First, we show boundedness of $(a(x,t,u_n,\nabla u_n))_n$ in $(L_{\overline{\varphi}}(\mathrm{Q}))^N$. Let $\vartheta \in (E_\varphi(\mathrm{Q}))^N$ be such that $\|\vartheta\|_{\varphi,\mathrm{Q}} = 1$. We have

$$\int_{\mathrm{Q}} \left(a(x,t,T_k(u_n),\nabla T_k(u_n)) - a\left(x,t,T_k(u_n),\frac{\vartheta}{k_3}\right)\right) \left(\nabla T_k(u_n) - \frac{\vartheta}{k_3}\right) dx \ge 0,$$

then

$$\begin{aligned} \int_{\mathrm{Q}} \frac{1}{k_3} a(x,t,&T_k(u_n),\nabla T_k(u_n))\,\vartheta\,dx \\ &\le \int_{\mathrm{Q}} a(x,t,T_k(u_n),\nabla T_k(u_n))\,\nabla T_k(u_n)\,dx \\ &\quad - \int_{\mathrm{Q}} a\left(x,t,T_k(u_n),\frac{\vartheta}{k_3}\right)\left(\nabla T_k(u_n) - \frac{\vartheta}{k_3}\right) dx \\ &\le Ck - \int_{\mathrm{Q}} a\left(x,t,T_k(u_n),\frac{\vartheta}{k_3}\right)\nabla T_k(u_n)\,dx \\ &\quad + \frac{1}{k_3}\int_{\mathrm{Q}} a\left(x,t,T_k(u_n),\frac{\vartheta}{k_3}\right)\vartheta\,dx. \end{aligned}$$

By using Young's inequality in the last two terms and (3.16) we get

$$\begin{aligned} &\int_{\mathrm{Q}} a(x,t,T_k(u_n),\nabla T_k(u_n))\vartheta\,dx \\ &\le Ckk_3 + 3k_1(1+k_3)\int_{\mathrm{Q}} \overline{\varphi}\left(x, \frac{|a(x,t,T_k(u_n),\vartheta/k_3)|}{3k_1}\right) dx \\ &\quad + 3k_1k_3\int_{\mathrm{Q}} \varphi(x,|\nabla T_k(u_n)|)\,dx + 3k_1\int_{\mathrm{Q}} \varphi(x,|\vartheta|)\,dx \\ &\le Ckk_3 + 3Ckk_1k_3 + 3k_1 + 3k_1(1+k_3)\int_{\mathrm{Q}} \overline{\varphi}\left(x, \frac{|a(x,t,T_k(u_n),\vartheta/k_3)|}{3k_1}\right) dx. \end{aligned}$$

On the one hand, by using (3.1) and convexity of $\overline{\varphi}$ we have

$$\overline{\varphi}\left(x, \frac{|a(x,t,T_k(u_n),\vartheta/k_3)|}{3k_1}\right) \le \frac{1}{3}(\overline{\varphi}(x,c(x)) + \gamma(x,k_2|T_k(u_n)|) + \varphi(x,|\vartheta|)).$$

On the other hand, since γ grows essentially less rapidly than φ near infinity there exists $C(k) > 0$ such that $\gamma(x,k_2|T_k(u_n)|) \le \gamma(x,k_2\,k) \le C(k)\varphi(x,1)$ (see

Remark 2.1), then we obtain by integrating over Ω and using (2.3),

$$\int_{\mathrm{Q}} \overline{\varphi}\bigg(x, \frac{|a(x,t,T_k(u_n),\vartheta/k_3)|}{3k_1}\bigg)\,dx \\ \leq \frac{1}{3}\bigg(\int_{\mathrm{Q}} \overline{\varphi}(x,c(x))\,dx + C(k)\int_{\mathrm{Q}} \varphi(x,1)\,dx + \int_{\mathrm{Q}} \varphi(x,|\vartheta|)\,dx\bigg) \leq C_k,$$

where C_k is a constant depending on k. We deduce that

$$\int_{\mathrm{Q}} a(x,t,T_k(u_n),\nabla T_k(u_n))\vartheta\,dx \leq C_k, \quad \text{for all } \vartheta \in (E_\varphi(\mathrm{Q}))^N \text{ with } \|\vartheta\|_{\varphi,\Omega} = 1,$$

and thus

$$\|a(x,t,T_k(u_n),\nabla T_k(u_n))\|_{\overline{\varphi},\mathrm{Q}} \leq C_k,$$

which implies that $(a(x,t,T_k(u_n),\nabla T_k(u_n)))_n$ is bounded in $(L_{\overline{\varphi}}(\mathrm{Q}))^N$. Thus, up to a subsequence, there exists $l_k \in (L_{\overline{\varphi}}(\mathrm{Q}))^N$, such that

$$a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup l_k \quad \text{in } (L_{\overline{\varphi}}(\mathrm{Q}))^N \text{ for } \sigma(\Pi L_{\overline{\varphi}}, \Pi E_\varphi).$$

Now, we shall prove that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt = 0. \tag{3.24}$$

Taking $v = u_n + \exp(-G(u_n))T_1(u_n - T_m(u_n))^-$ as a test function in $(\mathcal{P}_n)$, we obtain

$$-\int_0^T \bigg\langle \frac{\partial u_n}{\partial t}, \exp(-G(u_n))T_1(u_n - T_m(u_n))^- \bigg\rangle\,dt \\ + \int_{\{-(m+1)\leq u_n\leq -m\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n \exp(-G(u_n))\,dx\,dt \\ + n\int_{\mathrm{Q}} T_n(u_n-\psi)^- \exp(-G(u_n))T_1(u_n - T_m(u_n))^-\,dx\,dt \\ \leq \int_{\mathrm{Q}} |f_n| \exp(G(u_n))T_1(u_n - T_m(u_n))^-\,dx\,dt \\ + \int_{\mathrm{Q}} |c'(x,t)| \exp(-G(u_n))T_1(u_n - T_m(u_n))\,dx\,dt.$$

Let

$$B_m(r) = -\int_0^r T_1(s - T_m(s))^- \exp(-G(s))\,ds$$

and using the fact that

$$n\int_{\mathrm{Q}} T_n(u_n-\psi)^- \exp(-G(u_n))T_1(u_n - T_m(u_n))^-\,dx\,dt \geq 0,$$

we obtain

$$\int_\Omega B_m(u_n(T))\,dx + \int_{\{-(m+1)\leq u_n\leq -m\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n \exp(-G(u_n))\,dx\,dt$$

$$\leq \exp\left(\frac{\|b\|_{L^1(\mathbb{R})}}{\alpha}\right)\bigg(\int_{\{|u_n|>m\}}|f_n|\,dx\,dt + \int_{\{|u_n|>m\}}|c'|\,dx\,dt + \int_{\{|u_{0n}|>m\}}|u_{0n}|\,dx\bigg).$$

Since $B_m(r)\geq 0$ and $c'(\,\cdot\,,\,\cdot\,)\in L^1(\mathrm{Q})$, we deduce, by Lebesgue's Theorem,

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{-(m+1)\leq u_n\leq -m\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt = 0. \tag{3.25}$$

As above, taking $v = u_n - \exp(G(u_n))T_1(u_n - T_m(u_n))^+$ as a test function in the approximate problem $(\mathcal{P}_n)$, where μ is the constant defined in (2.5), we obtain

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\leq u_n\leq m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt = 0. \tag{3.26}$$

Thus (3.24) follows from (3.25) and (3.26).

Now, taking $v = u_n - \exp(G(u_n))(T_k(u_n) - T_k(v_j^k)_\mu)^+ h_m(u_n)$ as a test function in the approximate problem $(\mathcal{P}_n)$, gives

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial u_n}{\partial t}, \exp(G(u_n))(T_k(u_n) - T_k(v_j^k)_\mu)^+ h_m(u_n)\right\rangle dt \\
&\quad + \int_{\{T_k(u_n)-T_k(v_j^k)_\mu\geq 0\}} a(x,t,u_n,\nabla u_n)\cdot\nabla(T_k(u_n) - T_k(v_j^k)_\mu) \\
&\quad \cdot \exp(G(u_n))\, h_m(u_n)\,dx\,dt \\
&\quad - \int_{\{m\leq u_n\leq m+1\}} \exp(G(u_n))\, a(x,t,u_n,\nabla u_n) \\
&\quad \cdot \nabla u_n (T_k(u_n) - T_k(v_j^k)_\mu)^+\,dx\,dt \\
&\quad - n\int_{\mathrm{Q}} m(T_n(u_n-\psi)^-) \\
&\quad \cdot \exp(G(u_n))(T_k(u_n) - T_k(v_j^k)_\mu)^+ h_m(u_n)\,dx\,dt \\
&\leq \int_{\mathrm{Q}} c'(x,t)\exp(G(u_n))(T_k(u_n) - T_k(v_j^k)_\mu)^+ h_m(u_n)\,dx\,dt \\
&\quad + \int_{\mathrm{Q}} f_n \exp(G(u_n))(T_k(u_n) - T_k(v_j^k)_\mu)^+ h_m(u_n)\,dx\,dt.
\end{aligned} \tag{3.27}$$

Note that

$$\begin{aligned}
-\int_{\{m\leq u_n\leq m+1\}} &\exp(G(u_n))\, a(x,t,u_n,\nabla u_n)\cdot\nabla u_n (T_k(u_n) - T_k(v_j^k)_\mu)^+\,dx\,dt \\
&\leq 2k\exp\left(\frac{\|b\|_{L^1(\mathbb{R})}}{\alpha}\right)\int_{\{m\leq u_n\leq m+1\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\,dx\,dt.
\end{aligned}$$

From (3.24), we deduce that the third term of (3.27) tends to zero as $n,m\to\infty$. On the other hand, since

$$(T_k(u_n) - T_k(v_j^k)_\mu)^+ \rightharpoonup (T_k(u) - T_k(v_j^k)_\mu)^+ \quad \text{weakly in } E_\varphi(\mathrm{Q})^N \text{as } n\to\infty,$$

$$
\begin{aligned}
(T_k(u)-T_k(v_j^k)_\mu)^+ &\rightharpoonup (T_k(u)-T_k(u)_\mu)^+ && \text{weakly in } E_\varphi(\mathrm{Q})^N \text{as } j\to\infty,\\
(T_k(u)-T_k(u)_\mu)^+ &\rightharpoonup 0 && \text{weakly in } E_\varphi(\mathrm{Q})^N \text{as } \mu\to\infty,
\end{aligned}
$$

then, by Lebesgue's Theorem, the right-hand side of (3.27) converges to zero and

$$
\bigg| -n\int_{\mathrm{Q}} T_n(u_n-\psi)^-\exp(G(u_n))(T_k(u_n)-T_k(v_j^k)_\mu)^+h_m(u_n)\,dx\,dt\bigg|\to 0,
$$

as n, j and $\mu\to\infty$.

Denote by $\varepsilon(n,m,j,\mu)$ various sequences of real numbers such that

$$
\lim_{\mu\to\infty}\lim_{j\to\infty}\lim_{m\to\infty}\lim_{n\to\infty}\varepsilon(n,m,j,\mu)=0.
$$

Consequently, (3.27) becomes

$$
\begin{aligned}
\int_0^T \Big\langle \frac{\partial u_n}{\partial t}, \exp(G(u_n))(T_k(u_n)-T_k(v_j^k)_\mu)^+h_m(u_n)\Big\rangle\,dt \\
+\int_{\{T_k(u_n)-T_k(v_j^k)_\mu\geq 0\}} \exp(G(u_n))\ a(x,t,u_n,\nabla u_n) \\
\cdot\nabla(T_k(u_n)-T_k(v_j^k)_\mu)\ h_m(u_n)\,dx\,dt\leq \varepsilon(n,m,j,\mu).
\end{aligned}
\tag{3.28}
$$

By Lemma 3.2 of [23], then

$$
\int_0^T \Big\langle \frac{\partial u_n}{\partial t}, \exp(G(u_n))(T_k(u_n)-T_k(v_j^k)_\mu)^+h_m(u_n)\Big\rangle\,dt\geq \varepsilon(n,j,\mu).
$$

Concerning the second term of left-hand side of (3.28), we have

$$
\begin{aligned}
\int_{\{T_k(u_n)-T_k(v_j^k)_\mu\}\geq 0} &\exp(G(u_n))\ a(x,t,u_n,\nabla u_n) \\
&\cdot\nabla(T_k(u_n)-T_k(v_j^k)_\mu)\ h_m(u_n)\,dx\,dt \\
&=\int_{\{T_k(u_n)-T_k(v_j^k)_\mu\}\geq 0} \exp(G(u_n))\ a(x,t,T_k(u_n),\nabla T_k(u_n)) \\
&\cdot\nabla(T_k(u_n)-T_k(v_j^k)_\mu)\ h_m(u_n)\,dx\,dt \\
&-\int_{\{T_k(u_n)-T_k(v_j^k)_\mu\geq 0;\ |u_n|>k\}} \exp(G(u_n))\ a(x,t,u_n,\nabla u_n) \\
&\cdot\nabla T_k(v_j^k)_\mu\ h_m(u_n)\,dx\,dt.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\bigg|\int_{\{T_k(u_n)-T_k(v_j^k)_\mu\geq 0;\ |u_n|>k\}} \exp(G(u_n))a(x,t,u_n,\nabla u_n) \\
\cdot\nabla T_k(v_j^k)_\mu\ h_m(u_n)\,dx\,dt\bigg|
\end{aligned}
\tag{3.29}
$$

$$\leq C \int_{\{T_k(u_n)-T_k(v_j^k)_\mu \geq 0;\ |u_n|>k\}} |a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))| \cdot |\nabla T_k(v_j^k)_\mu|\,dx\,dt.$$

It is easy to see that

$$\int_{\{|u_n|>k\}} |a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))||\nabla T_k(v_j^k)_\mu|\,dx\,dt = \varepsilon(n,j,\mu),$$

then

$$\int_{\{T_k(u_n)-T_k(v_j^k)_\mu \geq 0\}} \exp(G(u_n))\ a(x,t,T_k(u_n),\nabla T_k(u_n)) \cdot \nabla(T_k(u_n)-T_k(v_j^k)_\mu)\ h_m(u_n)\,dx\ dt \leq \varepsilon(n,j,\mu). \tag{3.30}$$

On the other hand, we have

$$\begin{aligned}\int_{\{T_k(u_n)-T_k(v_j^k)_\mu \geq 0\}} &\exp(G(u_n))\ a(x,t,T_k(u_n),\nabla T_k(u_n)) \cdot \nabla(T_k(u_n)-T_k(v_j^k)_\mu)\ h_m(u_n)\,dx\,dt \\ \geq \int_{\{T_k(u_n)-T_k(v_j^k)_\mu \geq 0\}} &[a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(v_j^k)_\mu \chi_s^j)] \\ &\cdot [\nabla T_k(u_n)-\nabla T_k(v_j^k)_\mu \chi_s^j] \exp(G(u_n))\ h_m(u_n)\,dx\,dt \\ + \int_{\{T_k(u_n)-T_k(v_j^k)_\mu \geq 0\}} &a(x,t,T_k(u_n),\nabla T_k(v_j^k)_\mu \chi_s^j) \\ &\cdot [\nabla T_k(u_n)-\nabla T_k(v_j^k)_\mu \chi_s^j] \cdot \exp(G(u_n))\ h_m(u_n)\,dx\,dt \\ - C \int_{Q\setminus Q^{j,s}} &|a(x,t,T_k(u_n),\nabla T_k(u_n))||\nabla T_k(v_j^k)_\mu|\ h_m(u_n)\,dx\,dt.\end{aligned} \tag{3.31}$$

Tending n, j, m and μ to infinity in the third term on the right-hand side of (3.31) one easily has

$$- C \int_{Q\setminus Q^{j,s}} |a(x,t,T_k(u_n),\nabla T_k(u_n))||\nabla T_k(v_j^k)_\mu|\ h_m(u_n)\,dx\,dt = -C\int_{Q\setminus Q^{j,s}} l_k\ |\nabla T_k(u)|\,dx\,dt + \varepsilon(n,j,m,\mu). \tag{3.32}$$

The second term on the right-hand side of (3.31) reads as

$$\int_{\{T_k(u_n)-T_k(v_j^k)_\mu \geq 0\}} a(x,t,T_k(u_n),\nabla T_k(v_j^k)_\mu \chi_s^j) \cdot [\nabla T_k(u_n)-\nabla T_k(v_j^k)_\mu \chi_s^j] \cdot \exp(G(u_n))\ h_m(u_n)\,dx\,dt \tag{3.33}$$

$$
\begin{aligned}
&= \int_{\{T_k(u_n)-T_k(v_j^k)_\mu \geq 0\}} a(x,t,T_k(u_n),\nabla T_k(v_j^k)_\mu \chi_s^j)\nabla T_k(u_n) \\
&\qquad\qquad \cdot \exp(G(u_n))\; h_m(u_n)\,dx\,dt \\
&\quad - \int_{\{T_k(u_n)-T_k(v_j^k)_\mu \geq 0\}} a(x,t,T_k(u_n),\nabla T_k(v_j^k)_\mu \chi_s^j)\nabla T_k(v_j^k)_\mu \chi_s^j \\
&\qquad\qquad \cdot \exp(G(u_n))\; h_m(u_n)\,dx\;dt.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\exp(G(u_n))\,a(x,t,T_k(u_n),\nabla T_k(v_j^k)_\mu\,\chi_s^j)\;h_m(u_n)\,\chi_{\{T_k(u_n)-T_k(v_j^k)_\mu\}\geq 0\}} \\
&\qquad \to \exp(G(u))\,a(x,t,T_k(u),\nabla T_k(v_j^k)_\mu\,\chi_s^j)\;h_m(u)\,\chi_{\{T_k(u)-T_k(v_j^k)_\mu\}\geq 0\}}
\end{aligned}
$$

strongly in $E_{\overline{\varphi}}(\mathrm{Q})^N$ as n tends to infinity and $\nabla T_k(u_n)$ converges to $\nabla T_k(u)$ weakly in $L_\varphi(\mathrm{Q})^N$ for $\sigma(\Pi L_\varphi, \Pi E_{\overline{\varphi}})$, then

$$
\begin{aligned}
&\int_{\{T_k(u_n)-T_k(v_j^k)_\mu \geq 0\}} \exp(G(u_n)) \\
&\qquad\qquad \cdot a(x,t,T_k(u_n),\nabla T_k(v_j^k)_\mu \chi_s^j)\nabla T_k(u_n)\,h_m(u_n)\,dx\,dt \\
&= \int_{\{T_k(u)-T_k(v_j^k)_\mu \geq 0\}} \exp(G(u)) \\
&\qquad\qquad \cdot a(x,t,T_k(u),\nabla T_k(v_j^k)_\mu \chi_s^j)\nabla T_k(u)\,h_m(u)\,dx\,dt + \varepsilon(n)
\end{aligned}
$$

and by letting j, s and μ tend to infinity, one easily has

$$
\begin{aligned}
&\int_{\{T_k(u)-T_k(v_j^k)_\mu \geq 0\}} \exp(G(u))\,a(x,t,T_k(u),\nabla T_k(v_j^k)_\mu \chi_s^j)\nabla T_k(u)\,h_m(u)\,dx\,dt \\
&\qquad = \int_{\mathrm{Q}} \exp(G(u))\; a(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u)\,h_m(u)\,dx\,dt + \varepsilon(n,j,s,\mu).
\end{aligned}
$$

Also, for the second term on the right-hand side of (3.33) it is easy to see that

$$
\begin{aligned}
&-\int_{\{T_k(u_n)-T_k(v_j^k)_\mu \geq 0\}} \exp(G(u_n)) \\
&\qquad\qquad a(x,t,T_k(u_n),\nabla T_k(v_j^k)_\mu \chi_s^j)\nabla T_k(v_j^k)_\mu \chi_s^j\,h_m(u_n)\,dx\,dt \\
&\quad = -\int_{\mathrm{Q}} \exp(G(u))\,a(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u)\,h_m(u)\,dx\,dt + \varepsilon(n,j,s,\mu),
\end{aligned}
$$

which gives by adding the two last equalities

$$
\begin{aligned}
&\int_{\{T_k(u_n)-T_k(v_j^k)_\mu \geq 0\}} \exp(G(u_n))\; a(x,t,T_k(u_n),\nabla T_k(v_j^k)_\mu \chi_s^j) \\
&\qquad\qquad \cdot [\nabla T_k(u_n) - \nabla T_k(v_j^k)_\mu \chi_s^j]\; h_m(u_n)\,dx\,dt = \varepsilon(n,j,s,\mu).
\end{aligned}
\tag{3.34}
$$

Combining (3.30)–(3.32) and (3.34) we find

$$\begin{aligned}(3.35)\qquad \exp(G(-\infty))\int_{\{T_k(u_n)-T_k(v_j^k)_\mu\geq 0\}}&[a(x,t,T_k(u_n),\nabla T_k(u_n))\\ &-a(x,t,T_k(u_n),\nabla T_k(v_j^k)_\mu\chi_s^j)]\\ &\cdot[\nabla T_k(u_n)-\nabla T_k(v_j^k)_\mu\chi_s^j]\ h_m(u_n)\,dx\,dt\\ \leq C\int_{Q\setminus Q^s} l_{m+1}\ |\nabla T_k(u)|\,dx\,dt+\varepsilon(n,j,s,\mu).\end{aligned}$$

Similarly, by taking $v=u_n+\exp(-G(u_n))(T_k(u_n)-T_k(v_j^k)_\mu)^-h_m(u_n)$ as a test function in the approximate problem $(\mathcal{P}_n)$, we get

$$\begin{aligned}(3.36)\qquad \exp(G(-\infty))\int_{\{T_k(u_n)-T_k(v_j^k)_\mu\leq 0\}}&[a(x,t,T_k(u_n),\nabla T_k(u_n))\\ &-a(x,t,T_k(u_n),\nabla T_k(v_j^k)_\mu\chi_s^j)]\\ &\cdot[\nabla T_k(u_n)-\nabla T_k(v_j^k)_\mu\chi_s^j]h_m(u_n)\,dx\,dt\\ \leq C\int_{Q\setminus Q^s} l_{m+1}|\nabla T_k(u)|\,dx\,dt+\varepsilon(n,j,s,\mu).\end{aligned}$$

Consequently, from (3.35) and (3.36), we get

$$\begin{aligned}(3.37)\quad \exp(G(-\infty))\int_Q&[a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(v_j^k)_\mu\chi_s^j)]\\ &\cdot[\nabla T_k(u_n)-\nabla T_k(v_j^k)_\mu\chi_s^j]h_m(u_n)\,dx\,dt\\ \leq C\int_{Q\setminus Q^s}& l_{m+1}|\nabla T_k(u)|\,dx\,dt+\varepsilon(n,j,s,\mu).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(3.38)\quad &\int_Q[a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)]\\ &\cdot[\nabla T_k(u_n)-\nabla T_k(u)_\mu\chi_s]\,h_m(u_n)\,dx\,dt\\ &-\int_Q[a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(v_j^k)_\mu\chi_s^j)]\\ &\cdot[\nabla T_k(u_n)-\nabla T_k(v_j^k)_\mu\chi_s^j]\,h_m(u_n)\,dx\,dt\\ =&\int_Q a(x,t,T_k(u_n),\nabla T_k(v_j^k)_\mu\chi_s^j)[\nabla T_k(u_n)-\nabla T_k(v_j^k)_\mu\chi_s^j]\,h_m(u_n)\,dx\,dt\\ &-\int_Q a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)[\nabla T_k(u_n)-\nabla T_k(u)_\mu\chi_s]\,h_m(u_n)\,dx\,dt\\ &+\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\\ &\cdot[\nabla T_k(v_j^k)_\mu\chi_s^j-\nabla T_k(u)_\mu\chi_s]\,h_m(u_n)\,dx\,dt.\end{aligned}$$

It is not difficult to show that each integral on the right-hand side of (3.38) has the form $\varepsilon(n,j,\mu)$ or $\varepsilon(n,j,s,\mu)$, which gives

$$\begin{aligned}(3.39)\quad &\int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)]\\ &\qquad\qquad \cdot [\nabla T_k(u_n) - \nabla T_k(u)_\mu \chi_s]\, h_m(u_n)\, dx\, dt\\ &- \int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(v_j^k)_\mu \chi_s^j)]\\ &\qquad\qquad \cdot [\nabla T_k(u_n) - \nabla T_k(v_j^k)_\mu \chi_s^j]\, h_m(u_n)\, dx\, dt \le \varepsilon(n,j,s,\mu).\end{aligned}$$

Also, using (3.37) and (3.39), we have

$$\begin{aligned}(3.40)\quad &\int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)]\\ &\qquad\qquad \cdot [\nabla T_k(u_n) - \nabla T_k(u)_\mu \chi_s]\, h_m(u_n)\, dx\ , dt\\ &\qquad\qquad \le C \int_{Q\setminus Q^s} l_k |\nabla T_k(u)|\, dx\, dt + \varepsilon(n,j,s,\mu).\end{aligned}$$

Now, remark that

$$\begin{aligned}(3.41)\quad &\int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)]\\ &\qquad\qquad \cdot [\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\, dx\, dt\\ &= \int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)]\\ &\qquad\qquad \cdot [\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\, h_m(u_n)\, dx\ dt\\ &+ \int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)]\\ &\qquad\qquad \cdot [\nabla T_k(u_n) - \nabla T_k(u)\chi_s](1 - h_m(u_n))\, dx\, dt.\end{aligned}$$

Taking into account that $1-h_m(u_n) = 0$ in $\{|u_n| \le m\}$ and $\{|u_n| \le k\} \subset \{|u_n| \le m\}$. For m large enough the second term on the right-hand side of (3.41) can be written as

$$\begin{aligned}(3.42)\quad &\int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)]\\ &\qquad \cdot [\nabla T_k(u_n) - \nabla T_k(u)\chi_s](1 - h_m(u_n))\, dx\, dt\\ &= -\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)\chi_s(1 - h_m(u_n))\, dx\, dt\\ &\quad - \int_Q a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)\nabla T_k(u_n)(1 - h_m(u_n))\, dx\, dt.\end{aligned}$$

Since $(a(x,t,T_k(u_n),\nabla T_k(u_n)))_n$ is bounded in $L_{\overline{\varphi}}(Q)^N$ and $\nabla T_k(u)\chi_s(1 - h_m(u_n))$ converges strongly to zero in $E_\varphi(Q)^N$, then, the first term on the right-hand side of (3.42) converges to zero as $n \to \infty$.

The second term converges also to zero as $n \to \infty$, because

$$a(x,t,T_k(u_n),\nabla T_k(u)\chi_s) \to a(x,t,T_k(u),\nabla T_k(u)\chi_s) \quad \text{strongly in } L_{\overline{\varphi}}(\mathrm{Q})^N,$$
$$\nabla T_k(u_n)(1-h_m(u_n)) \rightharpoonup \nabla T_k(u)(1-h_m(u)) \qquad \text{weakly in } E_{\varphi}(\mathrm{Q})^N.$$

Consequently,

$$\lim_{n\to\infty}\int_{\mathrm{Q}}[a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)] \cdot [\nabla T_k(u_n)-\nabla T_k(u)\chi_s](1-h_m(u_n))\,dx\,dt = 0. \tag{3.43}$$

Combining (3.40), (3.41) and (3.43), we get

$$\int_{\mathrm{Q}}[a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)] \cdot [\nabla T_k(u_n)-\nabla T_k(u)\chi_s](1-h_m(u_n))\,dx\,dt \le C\int_{\mathrm{Q}\setminus \mathrm{Q}^s} l_{m+1}|\nabla T_k(u)|\,dx\,dt + \varepsilon(n,j,m,s,\mu).$$

Letting n, j, m, s and μ tend to infinity, we conclude

$$\int_{\mathrm{Q}}[a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)] \cdot [\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\,dx\,dt \to 0 \quad \text{as } n\to\infty, \tag{3.44}$$

and thus, as in the elliptic case (see [8]), there exists a subsequence also denoted by u_n such that

$$\nabla u_n \to \nabla u \quad \text{a.e. in Q}. \tag{3.45}$$

Moreover, by virtue of (3.3), Lemma 2.5 and Vitali's Theorem, one can deduce that

$$\varphi(x,|\nabla T_k(u_n)|) \to \varphi(x,|\nabla T_k(u)|) \quad \text{strongly in } L^1(\mathrm{Q}).$$

Now, we will show that $u \ge \psi$ almost everywhere in Q.

Turning to inequality (3.9), we have $\int_{\mathrm{Q}} T_n(u_n-\psi)^-\,dx\,dt \le C/n$ and by using Fatou's Lemma, we deduce that $\int_{\mathrm{Q}}(u-\psi)^-\,dx\,dt$ converges to zero as $n\to\infty$, then $(u-\psi)^- = 0$ almost everywhere in Q. Consequently $u\ge\psi$ almost everywhere in Q.

Step 3. Equi-integrability of the nonlinearities. As a consequence of (3.22) and (3.45), one has $g_n(x,t,u_n,\nabla u_n) \to g(x,t,u,\nabla u)$ almost everywhere in Q, so it suffices to show that $g_n(x,t,u_n,\nabla u_n)$ are uniformly equi-integrable in Q. Let

$$\widetilde{B_h}(r) = \int_0^r \exp(G(s))\int_0^s b(\tau)\chi_{\{\tau>h\}}\,d\tau\,ds.$$

Taking

$$v = u_n - \exp(G(u_n))\int_0^{u_n} b(s)\chi_{\{s>h\}}\,ds$$

as a test function in the approximate problem $(\mathcal{P}_n)$, we get

$$\begin{aligned}\int_\Omega \widetilde{B_h}(u(T))\,dx &+ \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n b(u_n)\chi_{\{s>h\}}\exp(G(u_n))\,dx\,dt\\ &+ \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n \frac{b(u_n)}{\alpha}\int_0^{u_n} b(s)\chi_{\{s>h\}}\,ds\,dx\,dt\\ &+ \int_Q g_n(x,t,u_n,\nabla u_n)\exp(G(u_n))\int_0^{u_n} b(s)\chi_{\{s>h\}}\,ds\,dx\,dt\\ &- n\int_Q T_n(u_n-\psi)^-\exp(G(u_n))\int_0^{u_n} b(s)\chi_{\{s>h\}}\,ds\,dx\,dt\\ \le \int_Q |f_n|\exp(G(u_n))&\int_0^{u_n} b(s)\chi_{\{s>h\}}\,ds\,dx\,dt + \int_\Omega \widetilde{B_h}(u_{0n})\,dx,\end{aligned}$$

then

$$\begin{aligned}&\int_Q a(x,t,u_n,\nabla u_n)\nabla u_n b(u_n)\chi_{\{s>h\}}\,dx\,dt\\ &\le \bigg(\int_h^\infty b(s)\,ds\bigg)\exp\bigg(\frac{\|b\|_{L^1(\mathbb{R})}}{\alpha}\bigg)(\|c'\|_{L^1(Q)}+\|f_n\|_{L^1(Q)}+\|u_{0n}\|_{L^1(\Omega)}+C).\end{aligned}$$

Since

$$\int_0^{u_n} b(s)\chi_{\{s>h\}}\,ds \le \int_h^\infty b(s)\,ds$$

and by the coercivity condition (2.1), we get

$$\int_{\{u_n>h\}} \varphi(x,|\nabla u_n|)b(u_n)\,dx\,dt \le C\int_h^\infty b(s)\,ds,$$

which gives, since $b\in L^1(\mathbb{R})$,

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n>h\}} \varphi(x,|\nabla u_n|)b(u_n)\,dx\,dt = 0. \tag{3.46}$$

As above, using

$$v = u_n - \exp(-G(u_n))\int_{u_n}^0 b(s)\chi_{\{s<-h\}}\,ds$$

as a test function in the approximate problem $(\mathcal{P}_n)$, we obtain

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n<-h\}} \varphi(x,|\nabla u_n|)b(u_n)\,dx\,dt = 0. \tag{3.47}$$

Combining (3.46) and (3.47), we conclude that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>h\}} \varphi(x,|\nabla u_n|)b(u_n)\,dx\,dt = 0.$$

It follows that, for h large enough and for a subset E of Q,

$$\lim_{|E|\to 0}\int_E \varphi(x,|\nabla u_n|)b(u_n)\,dx\,dt$$

$$\leq \max_{|s|<h}(b(s)) \lim_{|E|\to 0} \int_E \varphi(x, |\nabla T_h(u_n)|)\, dx\, dt$$
$$+ \lim_{|E|\to 0} \int_{E\cap\{|u_n|>h\}} \varphi(x, |\nabla u_n|) b(u_n)\, dx\, dt.$$

Then $\varphi(x, |\nabla u_n|)b(u_n)$ is equi-integrable and so

$$\varphi(x, |\nabla u_n|)b(u_n) \to \varphi(x, |\nabla u|)b(u) \quad \text{in } L^1(\mathrm{Q}).$$

Finally, by (3.4) and Vitali's Theorem, we conclude the equi-integrability of the nonlinearities.

Step 5. Passage to the limit. Let $\phi \in \mathfrak{D}(\overline{\mathrm{Q}})$ such that $\phi \geq \psi$ and taking $v = u_n - T_k(u_n - \phi)\chi_{(0,\tau)}$ as a test function in $(\mathcal{P}_n)$, we obtain

$$\int_0^T \left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \phi)\chi_{(0,\tau)} \right\rangle dt + \int_{\mathrm{Q}_\tau} a(x, t, u_n, \nabla u_n) \cdot \nabla T_k(u_n - \phi)\, dx\, dt$$
$$+ \int_{\mathrm{Q}_\tau} g_n(x, t, u_n, \nabla u_n) T_k(u_n - \phi)\, dx\, dt - n \int_{\mathrm{Q}_\tau} T_n(u_n - \psi)^- T_k(u_n - \phi)\, dx\, dt$$
$$= \int_{\mathrm{Q}_\tau} f_n T_k(u_n - \phi)\, dx\, dt.$$

Using the fact that $-n \int_{\mathrm{Q}_\tau} T_n(u_n - \psi)^- T_k(u_n - \phi)\, dx\, dt \geq 0$, gives

$$\int_{\mathrm{Q}_\tau} a(x, t, u_n, \nabla u_n).\nabla T_k(u_n - \phi)\, dx\ dt$$
$$= \int_{\mathrm{Q}_\tau} a(x, t, u_n, \nabla u_n).(\nabla T_{k+\|\phi\|_\infty}(u_n) - \nabla\phi)\chi_{\{|u_n-\phi|<k\}}\, dx\, dt$$
$$= \int_{\mathrm{Q}_\tau} a(x, t, u, \nabla u) \cdot (\nabla T_{k+\|\phi\|_\infty}(u) - \nabla\phi)\chi_{\{|u-\phi|<k\}}\, dx\, dt + \varepsilon(n)$$
$$= \int_{\mathrm{Q}_\tau} a(x, t, u, \nabla u).\nabla T_k(u - \phi)\, dx\, dt + \varepsilon(n).$$

Then

$$\int_\Omega S_k(u_n(\tau) - \phi(\tau))\, dx + \int_0^\tau \left\langle \frac{\partial\phi}{\partial t}, T_k(u_n - \phi) \right\rangle dt$$
$$+ \int_{\mathrm{Q}_\tau} a(x, t, u_n, \nabla u_n) \cdot (\nabla T_{k+\|\phi\|_\infty}(u_n) - \nabla\phi)\chi_{\{|u_n-\phi|<k\}}\, dx\, dt$$
$$+ \int_{\mathrm{Q}_\tau} g_n(x, t, u_n, \nabla u_n) T_k(u_n - \phi)\, dx\, dt$$
$$\leq \int_{\mathrm{Q}_\tau} f_n\ T_k(u_n - \phi)\, dx\ dt + \int_\Omega S_k(u_0 - \phi(0))\, dx,$$

which gives, by passing to the limit,

$$\int_\Omega S_k(u(\tau) - \phi(\tau))\, dx + \int_0^\tau \left\langle \frac{\partial\phi}{\partial t}, T_k(u - \phi) \right\rangle dt$$

$$+\int_{Q_\tau} a(x,t,u,\nabla u)\nabla T_k(u-\phi)\,dx\,dt+\int_{Q_\tau} g(x,t,u,\nabla u)T_k(u-\phi)\,dx\,dt \\ \leq \int_{Q_\tau} f\ T_k(u-\phi)\,dx\ dt+\int_\Omega S_k(u_0-\phi(0))\,dx.$$

Finally, for every $v \in W_0^{1,x}L_\varphi(Q)\cap L^\infty(Q)$ with $v \geq \psi$ almost everywhere in Q, there exists $v_j \in W_0^{1,x}L_\varphi(Q)\cap\mathfrak{D}(Q)$ with $v_j \geq \psi$ such that v_j converges to v for the modular convergence in $W_0^{1,x}L_\varphi(Q)$ and $\partial v_j/\partial t$ converges to $\partial v/\partial t$ for the modular convergence in $W_0^{-1,x}L_{\overline{\varphi}}(Q)+L^1(Q)$. Then, u satisfies (3.6) and the proof of Theorem 3.1 is complete. □

References

[1] R.A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] A. Ahmed, A. Benkirane, M.S.B. Elemine Vall and A. Touzani, *Parabolic equation in Musielak–Orlicz Sobolev framework without sign condition* (submitted).

[3] M.L. Ahmed Oubeid, A. Benkirane and M. Sidi El vally, *Strongly nonlinear parabolic problems in Musielak–Orlicz–Sobolev spaces*, Bol. Soc. Parana. Mat. **33** (2015), 191–223.

[4] M. Ait Khellou and A. Benkirane, *Elliptic inequalities with L^1 data in Musielak–Orlicz spaces*, Monatsh. Math. **183** (2017), 1–33.

[5] E. Azroul, M.B. Benboubker, H. Redwane and C. Yazough, *Renormalized solutions for a class of nonlinear parabolic equations without sign condition involving nonstandard growth*, Annals of the University of Craiova, Mathematics and Computer Science Series **41** (2014), 69–87.

[6] E. Azroul, M. El Lekhlifi, H. Redwane and A. Touzani, *Entropy solutions of nonlinear parabolic equations in Orlicz–Sobolev spaces without sign condition and L^1 data*, J. Nonlinear Evol. Equ. Appl. **2014** (2015), no. 7, 101–130.

[7] M. Bendahmane, P. Wittbold and A. Zimmermann, *Renormalized solutions for a nonlinear parabolic equation with variable exponent and L^1-data*, J. Differential Equations **249** (2010), 1483–1515.

[8] A. Benkirane and A. Elmahi, *Almost everywhere convergence of the gradients of solutions to elliptic equations in Orlicz spaces and application*, Nonlinear Anal. **28** (1997), 1769–1784.

[9] A. Benkirane and M. Sidi El Vally (Ould Mohamedhen Val), *Some approximation properties in Musielak–Orlicz–Sobolev spaces*, Thai J. Math. **10**, (2012), 371–381.

[10] A. Benkirane and M. Sidi El Vally (Ould Mohamedhen Val), *Variational inequalities in Musielak–Orlicz–Sobolev spaces*, Bull. Belg. Math. Soc. Simon Stevin **21** (2014), 787–811.

[11] L. Diening, P. Harjulehto, P. Hästö and M. Råžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics, Vol. 2017, Springer, Heidelberg, 2011.

[12] T. Donaldson, *Inhomogeneous Orlicz–Sobolev spaces and nonlinear parabolic initial-boundaryvalue problems*, J. Differential Equations **16** (1974), 201–256.

[13] A. Elmahi, *Strongly nonlinear parabolic initial-boundary value problems in Orlicz spaces*, Electron. J. Differential Equations Conf. **9** (2002), 203–220.

[14] A. Elmahi and D. Meskine, *Parabolic equations in Orlicz spaces*, J. London Math. Soc. (2) **72** (2005), 410–428.

[15] A. ELMAHI AND D. MESKINE, *Strongly nonlinear parabolic equations with natural growth terms in Orlicz spaces*, Nonlinear Anal. **60** (2005), 1–35.

[16] A. ELMAHI AND D. MESKINE, *Strongly nonlinear parabolic equations with natural growth terms and L^1 data in Orlicz spaces*, Port. Math. **62** (2005), 143–182.

[17] X.L. FAN, *An imbedding theorem for Musielak–Sobolev spaces*, Nonlinear Anal. **75** (2012), 1959–1971.

[18] P. GWIAZDA AND A. ŚWIERCZEWSKA-GWIAZDA, *On non-Newtonian fluids with the property of rapid thickening under different stimulus*, Math. Models Methods Appl. Sci. **18** (2008), 1073–1092.

[19] P. GWIAZDA, P. WITTBOLD, A. WRÓBLEWSKA AND A. ZIMMERMANN, *Renormalized solutions of nonlinear elliptic problems in generalized Orlicz spaces*, J. Differential Equations **253** (2012), 635–666.

[20] P. GWIAZDA, P. WITTBOLD, A. WRÓBLEWSKA AND A. ZIMMERMANN, *Renormalized solutions to nonlinear parabolic problems in generalized Musielak–Orlicz spaces*, Nonlinear Anal. **129** (2015), 1–36.

[21] J. MUSIELAK, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Math., Vol. 1034, Springer, Berlin, 1983.

[22] H. REDWANE, *Existence results for a class of nonlinear parabolic equations in Orlicz spaces*, Electron. J. Qual. Theory Differ. Equ. **2** (2010), 1–19.

[23] H. REDWANE, *Existence of a solution for a class of parabolic equations with three unbounded nonlinearities*, Adv. Dyn. Syst. Appl. **2** (2007), no. 2, 241–264.

[24] J. ROBERT, *Inéquations variationnelles paraboliques fortement non linéaires*, J. Math. Pures Appl. **53** (1974), 299–321.

[25] A. ŚWIERCZEWSKA-GWIAZDA, *Nonlinear parabolic problems in Musielak–Orlicz spaces*, Nonlinear Anal. **98** (2014), 48–65.

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