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CRITICAL BREZIS–NIRENBERG PROBLEM FOR NONLOCAL SYSTEMS

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Abstract. We deal with the existence of solutions to a critical elliptic system involving the fractional Laplacian operator. We consider the primitive of the nonlinearity interacting with the spectrum of the operator. The one side resonant case is also considered. Variational methods are used to obtain the existence, and our result improves earlier results of the authors.

1. Introduction

Let $s \in (0,1)$, $N > 2s$ and let $\Omega \subset \mathbb{R}^N$ be a smooth and bounded domain. In this paper, we study the existence of solutions to the following fractional system:

$$(1.1)\qquad \begin{cases} (-\Delta)^s u = au + bv + \dfrac{2p}{p+q}\,|u|^{p-2}u|v|^q + 2\xi_1|u|^{p+q-2}u & \text{in } \Omega,\\ (-\Delta)^s v = bu + cv + \dfrac{2q}{p+q}\,|u|^{p}|v|^{q-2}v + 2\xi_2|v|^{p+q-2}v & \text{in } \Omega,\\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

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where $(-\Delta)^s$ is the fractional Laplacian operator defined by

$$(-\Delta)^s u(x) := C(N,s) \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \quad x \in \mathbb{R}^N,$$

where $C(N,s)$ is a suitable positive normalization constant, $\xi_1, \xi_2 > 0$ and $p, q > 1$ are constants such that $p+q = 2_s^* := 2N/(N-2s)$ denotes the fractional critical Sobolev exponent. By a solution (u,v) to (1.1) we shall always mean a weak solution. Under suitable assumptions, one can also obtain a solution in the viscosity and in the strong sense, as described in [17].

It is convenient to rewrite system (1.1) in the vector and matrix forms such as

$$\begin{cases} (-\overrightarrow{\Delta})^s U = AU + \nabla F(U) & \text{in } \Omega, \\ U = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.2}$$

where

$$U^t = \begin{pmatrix} u \\ v \end{pmatrix} \in M_{2\times 1}(\mathbb{R}), \qquad (-\overrightarrow{\Delta})^s U^t = \begin{pmatrix} (-\Delta)^s u \\ (-\Delta)^s v \end{pmatrix},$$

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in M_{2\times 2}(\mathbb{R}),$$

$$F(U) = \frac{2}{p+q} \big(|u|^p |v|^q + \xi_1 |u|^{p+q} + \xi_2 |v|^{p+q} \big),$$

and ∇ is the gradient operator.

We shall denote by $0 < \lambda_{1,s} < \lambda_{2,s} \le \lambda_{3,s} \le \ldots$ the sequence of eigenvalues of the operator $(-\Delta)^s$ with homogeneous Dirichlet boundary datum (that is, $((-\Delta)^s, X(\Omega))$, where $X(\Omega) := \{u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$), and by μ_1 and μ_2 the eigenvalues of the symmetric matrix A given above. Without loss of generality, we may assume $\mu_1 \le \mu_2$.

When $\mu_2 < \lambda_{1,s}$, system (1.1) is related to the seminal paper [2], where the authors showed that the critical growth semi-linear problem

$$\begin{cases} -\Delta u = \lambda u + u^{2^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

admits a solution provided that $\lambda \in (0, \lambda_1)$ and $N \ge 4$, λ_1 being the first eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary condition and $2^* = 2N/(N-2)$. Furthermore, in dimension $N = 3$, the same existence result holds provided that $\mu < \lambda < \lambda_1$, for a suitable $\mu > 0$. After that, considerable attention has been paid to (1.3) throughout the years. Later on, in 1984, Cerami, Fortunato and Struwe obtained in [4] multiplicity results for the nontrivial solutions to

$$
\begin{cases} -\Delta u = \lambda u + u^{2^*-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}
$$

when λ belongs to a left neighbourhood of an eigenvalue of $-\Delta$. In 1985, Capozzi, Fortunato and Palmieri proved in [3] the existence of a nontrivial solution to (1.4) for all $\lambda > 0$ and $N \geq 5$ or for $N \geq 4$ and $\lambda > 0$ different from the eigenvalues of $-\Delta$. We would like to cite [11], [13], [15] for scalar nonlocal case, and [1] for local system case. For critical fractional equation in the resonant case, we would like to cite [12] and references therein. For fractional equation with critical exponent in $\mathbb{R}^N$, we would like to cite [7]. For a survey in the critical system case involving nonlocal operators, see [8].

The aim of this paper is to prove the existence of a nontrivial solution to (1.1) considering the eigenvalues $\mu_1 \leq \mu_2$ of the symmetric matrix A, interacting with the spectrum of the fractional Laplacian operator $(-\Delta)^s$. In this paper, we complement the results achieved in [8], proving that system (1.1) (or (1.2)) has at least a solution via the Linking Theorem when $\lambda_{k,s} \leq \mu_1 \leq \mu_2 < \lambda_{k+1,s}$, for some $k \in \mathbb{N}$. In this case, some complications arise due to the presence of the term

$$
F(u,v) = \frac{2}{\alpha+\beta}\left[|u|^p|v|^q + \xi_1|u|^{p+q} + \xi_2|v|^{p+q}\right]
$$

that includes either an uncoupled or a coupled nonlinearity. Therefore, it is necessary to require that the constants ξ_1, ξ_2 are assumed to be strictly positive. The resonant case ($\lambda_{k,s} = \mu_1$) is also treated here, except for $N = 4s$. As it happens in the Laplacian case when $n = 4$, also in the nonlocal framework there is a dimension ($n = 4s$) where resonance creates a problem.

It is important to point out that, with the aid of [6], our results are still valid for the general case $\nabla F(u,v)$ when F is a $(p+q)$-homogeneous nonlinearity, which includes a larger class of functions.

The following is the main result of the paper.

THEOREM 1.1. *Let $s \in (0,1)$, $N > 2s$, $p+q = 2_s^*$, $\xi_1, \xi_2 > 0$, and let $\Omega \subset \mathbb{R}^N$ be a smooth and bounded domain. Suppose $\lambda_{k,s} \leq \mu_1 \leq \mu_2 < \lambda_{k+1,s}$, for some $k \in \mathbb{N}$. Then* (1.1) *admits a nontrivial solution provided that either*

(a) *$N > 4s$, or*
(b) *$N = 4s$ and $\mu_1 \neq \lambda_{j,s}$ for all $j \in \mathbb{N}$, or*
(c) *$N < 4s$ and μ_1 is large enough.*

2. Notations and preliminary stuff

For any measurable function $u\colon \mathbb{R}^N \to \mathbb{R}$ the Gagliardo seminorm is defined as

$$[u]_s := \bigg(C(N,s)\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\bigg)^{1/2} = \bigg(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2\,dx\bigg)^{1/2}.$$

The second equality follows by [9, Proposition 3.6] when the above integrals are finite. The fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined as follows:

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : [u]_s < \infty\},$$

equipped with the norm

$$\|u\|_{H^s} = (\|u\|^2_{L^2(\mathbb{R}^{\mathbb{N}})} + [u]^2_s)^{1/2},$$

it is a Hilbert space. We shall consider the closed linear subspace

$$X(\Omega) := \{u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}. \tag{2.1}$$

By Theorems 6.5 and 7.1 in [9], the imbedding $X(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for $r \in [1, 2^*_s]$ and compact for $r \in [1, 2^*_s)$. Due to the fractional Sobolev inequality, $X(\Omega)$ is a Hilbert space with inner product

$$\langle u, v\rangle_X := C(N,s)\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy, \tag{2.2}$$

which induces the norm $\|\cdot\|_X = [\,\cdot\,]_s$. Observe that by Proposition 3.6 in [9], we have the following identity:

$$\|u\|^2_X = \frac{2}{C(N,s)}\,\|(-\Delta)^{s/2}u\|^2_{L^2(\mathbb{R}^{\mathbb{N}})}, \quad u \in X(\Omega).$$

Then it is proved that, for $u, v \in X(\Omega)$,

$$\frac{2}{C(N,s)}\int_{\mathbb{R}^N} u(x)(-\Delta)^s v(x)\,dx = \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy, \tag{2.3}$$

in particular, $(-\Delta)^s$ is self-adjoint in $X(\Omega)$.

We shall work in the Hilbert space given by the product space

$$Y(\Omega) := X(\Omega) \times X(\Omega),$$

equipped with the inner product

$$\langle (u,v), (\varphi,\psi)\rangle_Y := \langle u,\varphi\rangle_X + \langle v,\psi\rangle_X$$

and the norm

$$\|(u,v)\|_Y := (\|u\|^2_X + \|v\|^2_X)^{1/2}.$$

The space $L^r(\Omega) \times L^r(\Omega)$ $(r > 1)$ is considered with the standard norm

$$\|(u,v)\|_{L^r(\mathbb{R}^{\mathbb{N}})\times L^r(\mathbb{R}^{\mathbb{N}})} := \big(\|u\|^r_{L^r(\mathbb{R}^{\mathbb{N}})} + \|v\|^r_{L^r(\mathbb{R}^{\mathbb{N}})}\big)^{1/r}.$$

Besides, we recall that

$$\mu_1|U|^2 \le (AU,U)_{\mathbb{R}^2} \le \mu_2|U|^2, \quad \text{for all } U := (u,v) \in \mathbb{R}^2. \tag{2.4}$$

In this paper, we consider the following notation for the product space $\mathcal{S}\times\mathcal{S} := \mathcal{S}^2$.

2.1. The eigenvalue problem. For $\lambda \in \mathbb{R}$, we consider the problem with homogeneous Dirichlet boundary datum

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.5}$$

If (2.5) admits a weak solution $u \in X(\Omega) \setminus \{0\}$, then λ is called an eigenvalue and u a λ-eigenfunction. The set of all eigenvalues is referred as the spectrum of $(-\Delta)^s$ in $X(\Omega)$ and denoted by $\sigma((-\Delta)^s)$. Since $K = [(-\Delta)^s]^{-1}$ is a compact operator, problem (2.5) can be written as $u = \lambda K u$ with $u \in L^2(\Omega)$, hence the following results are true (see [14], [16]):

(i) problem (2.5) admits an eigenvalue $\lambda_{1,s} = \min \sigma((-\Delta)^s) > 0$ that can be characterized as follows:

$$\lambda_{1,s} = \min_{u \in X\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u(x)|^2\,dx}{\displaystyle\int_{\mathbb{R}^N} |u(x)|^2\,dx}; \tag{2.6}$$

(ii) there exists a non-negative function $\varphi_{1,s} \in X(\Omega)$, which is an eigenfunction corresponding to $\lambda_{1,s}$, attaining the minimum in (2.6);

(iii) all $\lambda_{1,s}$-eigenfunctions are proportional, and if u is a $\lambda_{1,s}$-eigenfunction, then either $u(x) > 0$ almost everywhere in Ω or $u(x) < 0$ almost everywhere in Ω;

(iv) the set of the eigenvalues of problem (2.5) consists of a sequence $\{\lambda_{k,s}\}$ satisfying

$$0 < \lambda_{1,s} < \lambda_{2,s} \le \lambda_{3,s} \le \ldots \le \lambda_{j,s} \le \lambda_{j+1,s} \le \ldots, \ \lambda_{k,s} \to \infty, \quad \text{as } k \to \infty,$$

which is characterized by

$$\lambda_{k+1,s} = \min_{u \in \mathbb{P}_{k+1}\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u(x)|^2\,dx}{\displaystyle\int_{\mathbb{R}^N} |u(x)|^2\,dx}, \tag{2.7}$$

where

$$\mathbb{P}_{k+1} = \{u \in X(\Omega) : \langle u, \varphi_{j,s}\rangle_X = 0,\ j = 1,\ldots,k\}; \tag{2.8}$$

(v) if $\lambda \in \sigma((-\Delta)^s) \setminus \{\lambda_{1,s}\}$ and u is a λ-eigenfunction, then u changes sign in Ω;

(vi) for each $k \in \mathbb{N}$, let $\varphi_{k,s}$ be an eigenfunction associated to the eigenvalue $\lambda_{k,s}$, then the sequence $\{\varphi_{k,s}\}$ is an orthonormal basis either of $L^2(\Omega)$ or of $X(\Omega)$.

Remark 2.1. Every eigenfunction of $(-\Delta)^s$ belongs to $C^{0,\sigma}(\overline{\Omega})$ for some $\sigma \in (0,1)$ (see Theorem 1 of [14] or Proposition 2.4 of [11]).

Remark 2.2. For each $k \in \mathbb{N}$ we can assume $\lambda_{k,s} < \lambda_{k+1,s}$. Otherwise, we can suppose that $\lambda_{k,s}$ has multiplicity $p \in \mathbb{N}$, that is

$$\lambda_{k-1,s} < \lambda_{k,s} = \lambda_{k+1,s} = \ldots = \lambda_{k+p-1,s} < \lambda_{k+p,s}.$$

In this case, we denote $\lambda_{k+p,s} = \lambda_{k+1,s}$.

Observe that the weak solutions to (1.2) are the critical points of the functional $I_s \colon Y(\Omega) \to \mathbb{R}$ given by

$$\begin{aligned} I_s(U) = {} & \frac{C(N,s)}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2 + |v(x)-v(y)|^2}{|x-y|^{N+2s}}\, dx\, dy \\ & - \frac{1}{2} \int_\Omega (AU,U)_{\mathbb{R}^2}\, dx - \int_\Omega F(U)\, dx, \end{aligned}$$

where

$$F(U) := \frac{2}{p+q} \big[|u|^p |v|^q + \xi_1 |u|^{p+q} + \xi_2 |v|^{p+q} \big], \quad \text{for every } U = (u,v) \in \mathbb{R}^2.$$

Remark 2.3 (Properties of homogeneous functions). If G is a C^1-function and α-homogeneous with $\alpha \geq 1$, then:

(a) there exists $K_G > 0$ such that

$$|G(s,t)| \leq K_G(|s|^\alpha + |t|^\alpha), \quad \text{for } s,t \in \mathbb{R},$$

where $K_G = \max\{G(s,t) : s,t \in \mathbb{R},\ |s|^\alpha + |t|^\alpha = 1\}$ is attained in some $(s_o,t_o) \in \mathbb{R}^2$;

(b) $(\nabla G(s,t),(s,t))_{\mathbb{R}^2} = sG_s(s,t) + tG_t(s,t) = \alpha G(s,t)$, for all $(s,t) \in \mathbb{R}^2$;

(c) G_s and G_t are $(\alpha-1)$-homogeneous.

Remark 2.4. The nonlinearity F is $(p+q)$-homogeneous, i.e.

$$F(\lambda U) = \lambda^{p+q} F(U), \quad \text{for all } U \in \mathbb{R}^2, \text{ for all } \lambda \geq 0.$$

In this paper, we apply the following generalized Mountain Pass Theorem [10, Theorem 5.3, Remark 5.5 (iii)]. In what follows, B_r denotes a ball centered at the origin with radius r.

Theorem 2.5. *Let Y be a real Banach space with $Y = V \oplus W$, where V is finite dimensional. Suppose $I \in C^1(Y,\mathbb{R})$ and that*

(a) *there are constants $\rho, \alpha > 0$ such that $I_{|\partial B_\rho \cap W} \geq \alpha$, and*

(b) *there are constants $R_1, R_2 > \rho$ and $e \in \partial B_1 \cap W$ such that $I_{|\partial Q} \leq 0$, where $Q = (\overline{B}_{R_1} \cap V) \oplus \{re,\ 0 < r < R_2\}$.*

Then I possesses a $(\mathrm{PS})_c$ sequence, where $c \geq \alpha$ can be characterized as

$$c = \inf_{h \in \Gamma} \max_{u \in Q} I(h(u)) \quad \text{and} \quad \Gamma = \{h \in C(\overline{Q},Y) : h = \mathrm{id} \text{ on } \partial Q\}.$$

Remark 2.6. Here, ∂Q denotes the boundary of Q relatively to the space $V \oplus \operatorname{span}\{e\}$. When $V = \{0\}$, this theorem refers to the usual Mountain Pass Theorem. We recall that if $I_{|V} \leq 0$ and $I(u) \leq 0$, for all $u \in V \oplus \operatorname{span}\{e\}$ with $\|u\| \geq R$, then I verifies (b) in Theorem 2.5 for R large.

To conclude this section, define the subspaces

$$V_k = \operatorname{span}\{(0,\varphi_{1,s}),(\varphi_{1,s},0),\ldots,(0,\varphi_{k,s}),(\varphi_{k,s},0)\}$$

and $W_k = V_k^{\perp} = (\mathbb{P}_{k+1})^2$, for $k \in \mathbb{N}$.

3. The geometry of the functional

Associating with problem (1.2) we define the functional $I_s \colon Y(\Omega) \to \mathbb{R}$ given by

$$\begin{aligned} I_s(u,v) = {} & \frac{1}{2}\int_{\mathbb{R}^N} (|(-\Delta)^{s/2}u|^2 + |(-\Delta)^{s/2}v|^2)\,dx \\ & - \frac{1}{2}\int_\Omega (A(u,v),(u,v))_{\mathbb{R}^2}\,dx - \int_\Omega F(u(x),v(x))\,dx, \end{aligned}$$

whose Fréchet derivative is given by

$$\begin{aligned} (3.1)\quad & I_s'(u,v)(\phi,\psi) \\ & = C(N,s)\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\phi(x)-\phi(y)) + (v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dx\,dy \\ & \qquad - \int_\Omega (A(u,v),(\phi,\psi))_{\mathbb{R}^2}\,dx - \int_\Omega (\nabla F(u,v),(\phi,\psi))_{\mathbb{R}^2}\,dx, \end{aligned}$$

for every $(\phi,\psi) \in Y(\Omega)$.

We shall observe that the weak solutions to problem (1.2) correspond to the critical points of the functional I_s.

Under the hypothesis $\lambda_{k,s} \leq \mu_1 \leq \mu_2 < \lambda_{k+1,s}$, for some $k \in \mathbb{N}$, we will show that the functional I_s has the geometric structure required by the Linking Theorem.

Proposition 3.1. *Suppose Ω is a smooth bounded domain of $\mathbb{R}^N$, $p+q = 2_s^*$ and $\lambda_{k,s} \leq \mu_1 \leq \mu_2 < \lambda_{k+1,s}$, for some $k \in \mathbb{N}$. Then the functional I_s has the following properties*:

(a) *there exist $\alpha,\rho > 0$ such that $I_s(u,v) \geq \alpha$ for all $(u,v) \in W_k$ with $\|(u,v)\|_Y = \rho$;*
(b) *let $\mathbb{F}$ be a finite dimensional subspace of $Y(\Omega)$, then there exists $R > \rho$ such that $I_s(u,v) \leq 0$, for all $(u,v) \in \mathbb{F}$ with $||(u,v)||_Y \geq R$.*

Proof. Let $(u,v) \in W_k$. Since

$$|u(x)|^p|v(x)|^q \leq \frac{p}{p+q}\,|u(x)|^{p+q} + \frac{q}{p+q}\,|v(x)|^{p+q},$$

by (2.4) we have

$$I_s(u,v) \geq \frac{1}{2}\bigg(1 - \frac{\mu_2}{\lambda_{k+1,s}}\bigg)\|(u,v)\|_Y^2 - C\|(u,v)\|_Y^{2_s^*},$$

where $C > 0$ is a constant. This proves (a).

To prove (b), notice that for all $(u,v) \in \mathbb{F}$ we have

$$\begin{aligned} I_s(u,v) &\leq \frac{1}{2}\,\|(u,v)\|_Y^2 - \frac{\mu_1}{2}\,\|(u,v)\|_{(L^2(\mathbb{R}^N))^2}^2 \\ &\quad - \frac{2}{2_s^*}\int_\Omega \big(|u(x)|^p|v(x)|^q + \xi_1|u(x)|^{p+q} + \xi_2|v(x)|^{p+q}\big)\,dx \\ &\leq \frac{1}{2}\,\|(u,v)\|_Y^2 - \frac{2}{2_s^*}\min\{\xi_1,\xi_2\}\|(u,v)\|_{(L^{2_s^*}(\mathbb{R}^N))^2}^{2_s^*} \\ &\leq \frac{1}{2}\,\|(u,v)\|_Y^2 - K\|(u,v)\|_Y^{2_s^*}, \end{aligned}$$

for some positive constant K, due to the fact that in any finite dimensional space all the norms are equivalent. Since $2_s^* > 2$, we have that $I_s(u,v) \leq 0$, for all $(u,v) \in \mathbb{F}$ with $||(u,v)||_Y \geq R$. □

Remark 3.2. By using [16, Proposition 9], for all $(u,v) \in V_k$, we have

$$(u,v) = \bigg(\sum_{i=1}^k u_i e_{i,s}, \sum_{i=1}^k v_i e_{i,s}\bigg)$$

and

$$\int_{\mathbb{R}^N} |u|^2\,dx = \sum_{i=1}^k u_i^2, \qquad \int_{\mathbb{R}^N} |v|^2\,dx = \sum_{i=1}^k v_i^2.$$

Also

$$\begin{aligned} &\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2 + |v(x)-v(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \\ &\qquad = \sum_{i=1}^k (u_i^2+v_i^2)||\varphi_{i,s}||_X^2 = \sum_{i=1}^k (u_i^2+v_i^2)\lambda_{i,s}. \end{aligned}$$

Since $\mu_1 \geq \lambda_{i,s}$, for all $i = 1,\ldots,k$, by using (2.4), we get

$$\begin{aligned} I_s(u,v) &\leq \frac{1}{2}\sum_{i=1}^k (u_i^2+v_i^2)\lambda_{i,s} - \frac{\mu_1}{2}\sum_{i=1}^k (u_i^2+v_i^2) \\ &\quad - \frac{2}{2_s^*}\int_{\mathbb{R}^N} \big(|u(x)|^p|v(x)|^q + \xi_1|u(x)|^{p+q} + \xi_2|v(x)|^{p+q}\big)\,dx \\ &\leq \frac{1}{2}\sum_{i=1}^k (u_i^2+v_i^2)(\lambda_{i,s}-\mu_1) \leq 0. \end{aligned}$$

In order to prove Theorem 1.1, we shall make use of the following definitions:

$$S^s_{p+q}(\Omega) = \inf_{u\in X(\Omega)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy}{\displaystyle\bigg(\int_{\mathbb{R}^N} |u(x)|^{p+q}\,dx\bigg)^{2/(p+q)}}, \tag{3.2}$$

$$\widetilde{S}^s_{p,q}(\Omega) = \inf_{u,v\in X(\Omega)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2+|v(x)-v(y)|^2}{|x-y|^{N+2s}}\,dx\,dy}{\displaystyle\bigg(\int_{\mathbb{R}^N} \big(|u(x)|^p|v(x)|^q+\xi_1|u(x)|^{p+q}+\xi_2|v(x)|^{p+q}\big)\,dx\bigg)^{2/(p+q)}}. \tag{3.3}$$

We denote $S_s = S^s_{p+q}(\Omega)$, $\widetilde{S}_s = \widetilde{S}^s_{p,q}(\Omega)$, if $p+q=2^*_s$. The following result can be proved along the same lines as in [1], where the local case is considered. For completeness we present its proof.

Lemma 3.3. *Let Ω be a domain (not necessarily bounded) and $p+q=2^*_s$. Then there exists a constant m such that*

$$\widetilde{S}_s = mS_s. \tag{3.4}$$

Moreover, if w_o realizes S_s then (s_ow_o,t_ow_o) realizes $\widetilde{S}_s$, for some $s_o,t_o>0$.

Proof. Let $\{w_n\}\subset X(\Omega)\setminus\{0\}$ be a minimizing sequence for $S^s_{p+q}(\Omega)$ and consider the sequence $(\widetilde{u}_n,\widetilde{v}_n)=(s_ow_n,t_ow_n)$, with $s_o,t_o>0$ to be chosen later. Substituting $(\widetilde{u}_n,\widetilde{v}_n)$ in quotient (3.3), we get

$$\frac{\displaystyle(s_o^2+t_o^2)\int_{\mathbb{R}^{2N}} \frac{|w_n(x)-w_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy}{\displaystyle(s_o^pt_o^q+\xi_1s_o^{p+q}+\xi_2t_o^{p+q})^{2/p+q}\bigg(\int_{\mathbb{R}^N}|w_n(x)|^{p+q}\,dx\bigg)^{2/(p+q)}} \geq \widetilde{S}^s_{p,q}(\Omega). \tag{3.5}$$

Define the function

$$H(u,v) := \frac{p+q}{2}\,F(u,v) = |u|^p|v|^q+\xi_1|u|^{p+q}+\xi_2|v|^{p+q}.$$

Since $H(u,v)^{2/(p+q)}$ is 2-homogeneous, there exists a constant $M>0$ satisfying

$$H(u,v)^{2/(p+q)} \leq M(|u|^2+|v|^2), \text{ for all } u,v\in\mathbb{R}, \tag{3.6}$$

where M is the maximum of the function $H^{2/(p+q)}$ attained in some (s_o,t_o) (with $s_o,t_o\geq 0$) of the compact set $\{(s,t): s,t\in\mathbb{R},\ |s|^2+|t|^2=1\}$.

Let $m=M^{-1}$, so we have

$$H(s_o,t_o)^{2/(p+q)} = m^{-1}(s_o^2+t_o^2), \tag{3.7}$$

and consequently, by (3.5), it follows that

$$\widetilde{S}_s \le m \frac{\displaystyle\int_{\mathbb{R}^{2N}} \frac{|w_n(x)-w_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy}{\bigg(\displaystyle\int_{\mathbb{R}^N} |w_n(x)|^{p+q}\,dx\bigg)^{2/(p+q)}}. \tag{3.8}$$

Taking the limit in (3.8), we obtain $\widetilde{S}_s \le mS_s$.

In order to prove the reversed inequality, let $\{(u_n,v_n)\}$ be a minimizing sequence for $\widetilde{S}_s$, i.e.

$$\frac{\displaystyle\int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2+|v_n(x)-v_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy}{\bigg(\displaystyle\frac{p+q}{2}\int_{\mathbb{R}^N} F(u_n(x),v_n(x))\,dx\bigg)^{2/(p+q)}} \to \widetilde{S}_s, \quad \text{as } n\to\infty.$$

By using the Hölder inequality, we get

$$\int_{\mathbb{R}^N} F(u_n(x),v_n(x))\,dx \le F\big(\|u_n\|_{L^{p+q}(\mathbb{R}^{\mathbb{N}})}, \|v_n\|_{L^{p+q}(\mathbb{R}^{\mathbb{N}})}\big),$$

for each $u_n, v_n \in L^{p+q}(\mathbb{R}^{\mathbb{N}})$. Therefore, the above estimate guarantees that

$$\begin{aligned} &\frac{\displaystyle\int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2+|v_n(x)-v_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy}{\bigg(\displaystyle\frac{p+q}{2}\int_{\mathbb{R}^N} F(u_n(x),v_n(x))\,dx\bigg)^{2/(p+q)}} \\ &\qquad\ge \frac{S_s\big(\|u_n\|^2_{L^{p+q}(\mathbb{R}^{\mathbb{N}})}+\|v_n\|^2_{L^{p+q}(\mathbb{R}^{\mathbb{N}})}\big)}{\bigg(\displaystyle\frac{p+q}{2}F\big(\|u_n\|_{L^{p+q}(\mathbb{R}^{\mathbb{N}})}, \|v_n\|_{L^{p+q}(\mathbb{R}^{\mathbb{N}})}\big)\bigg)^{2/(p+q)}}. \end{aligned} \tag{3.9}$$

Now, by inequality (3.6),

$$\begin{aligned} &m\bigg(\frac{p+q}{2}F\big(\|u_n\|_{L^{p+q}(\mathbb{R}^{\mathbb{N}})}, \|v_n\|_{L^{p+q}(\mathbb{R}^{\mathbb{N}})}\big)\bigg)^{2/(p+q)} \\ &\qquad\le \|u_n\|^2_{L^{p+q}(\mathbb{R}^{\mathbb{N}})}+\|v_n\|^2_{L^{p+q}(\mathbb{R}^{\mathbb{N}})}. \end{aligned} \tag{3.10}$$

Hence, by (3.9) and (3.10), we have

$$\frac{\displaystyle\int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2+|v_n(x)-v_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy}{\bigg(\displaystyle\frac{p+q}{2}\int_{\mathbb{R}^N} F(u_n(x),v_n(x))\,dx\bigg)^{2/(p+q)}} \ge mS_s.$$

Therefore, passing to the limit in the above inequality, we have the desired reversed inequality. □

From [5, Theorem 1.1], S_s is attained, namely, $S_s = S_s(\widetilde{u})$, where

$$\widetilde{u}(x) = k(\mu^2 + |x - x_0|^2)^{-(N-2s)/2}, \tag{3.11}$$

for $x \in \mathbb{R}^N$, $k \in \mathbb{R} \setminus \{0\}$, $\mu > 0$, fixed $x_0 \in \mathbb{R}^N$. Equivalently,

$$S_s = \inf_{\substack{u \in X(\Omega) \setminus \{0\} \\ ||u||_{L^{2_s^*}(\mathbb{R}^N)} = 1}} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}}\, dx\, dy = \int_{\mathbb{R}^{2N}} \frac{|\overline{u}(x) - \overline{u}(y)|^2}{|x - y|^{N+2s}}\, dx\, dy,$$

where $\overline{u}(x) = \widetilde{u}(x)/||\widetilde{u}||_{L^{2_s^*}(\mathbb{R}^N)}$. In what follows, we suppose that, up to a translation, $x_0 = 0$ in (3.11).

The function

$$u^*(x) = \overline{u}\left(\frac{x}{S_s^{1/(2s)}}\right), \quad \text{for } x \in \mathbb{R}^N,$$

is a solution to the problem

$$(-\Delta)^s u = |u|^{2_s^*-2} u \quad \text{in } \mathbb{R}^N, \tag{3.12}$$

verifying the property

$$||u^*||^{2_s^*}_{L^{2_s^*}(\mathbb{R}^N)} = S_s^{N/2s}. \tag{3.13}$$

Notice that the family of functions

$$U_\varepsilon(x) = \varepsilon^{-(N-2s)/2} u^*\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^N,$$

solves (3.12) and verifies, for all $\varepsilon > 0$,

$$\int_{\mathbb{R}^{2N}} \frac{|U_\varepsilon(x) - U_\varepsilon(y)|^2}{|x - y|^{N+2s}}\, dx\, dy = \int_{\mathbb{R}^N} |U_\varepsilon(x)|^{2_s^*}\, dx = S_s^{N/2s}. \tag{3.14}$$

Fix $\delta > 0$, such that $B_{4\delta} \subset \Omega$, and $\eta \in C^\infty(\mathbb{R}^N)$ a cut-off function such that $0 \le \eta \le 1$ in $\mathbb{R}^N$, $\eta = 1$ in B_δ and $\eta = 0$ in $B^c_{2\delta} = \mathbb{R}^N \setminus B_{2\delta}$.

Now define the family of nonnegative truncated functions

$$u_\varepsilon(x) = \eta(x) U_\varepsilon(x), \quad x \in \mathbb{R}^N, \tag{3.15}$$

and note that $u_\varepsilon \in X$.

Now, we recall some well-known results for the local case. For the nonlocal case, its proof can be found in [15].

PROPOSITION 3.4. *Let $\rho > 0$ and $\mu > 0$ be as in* (3.11)*. If $x \in B^c_\rho$, then*

(a) $|u_\varepsilon(x)| \le |U_\varepsilon(x)| \le C\varepsilon^{(N-2s)/2}$, *for all* $\varepsilon > 0$,

(b) $|\nabla u_\varepsilon(x)| \le C\varepsilon^{(N-2s)/2}$, *for all* $\varepsilon > 0$,

(c) *for any* $x \in \mathbb{R}^N$ *and* $y \in B^c_\delta$ $(B_{4\delta} \subset \Omega)$ *with* $|x - y| \le \delta/2$, *we have*

$$|u_\varepsilon(x) - u_\varepsilon(y)| \le C\varepsilon^{(N-2s)/2}|x - y|, \quad \textit{for all } \varepsilon > 0,$$

(d) *for any* $x, y \in B^c_\delta$, *we have*

$$|u_\varepsilon(x) - u_\varepsilon(y)| \le C\varepsilon^{(N-2s)/2} \min\{1, |x - y|\}, \quad \textit{for all } \varepsilon > 0,$$

where C is a positive constant which possibly can depend on μ, ρ, s and N.

Proposition 3.5. *For $s \in (0,1)$ and $N > 2s$, we have*:

(a) $\displaystyle\int_{\mathbb{R}^{2N}} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \le S_s^{N/2s} + O(\varepsilon^{N-2s})$, *as $\varepsilon \to 0$.*

(b) $$\int_{\mathbb{R}^N} |u_\varepsilon(x)|^2\,dx \ge \begin{cases} C_s\varepsilon^{2s} + O(\varepsilon^{N-2s}) & \text{if } N > 4s, \\ C_s\varepsilon^{2s}|\log\varepsilon| + O(\varepsilon^{2s}) & \text{if } N = 4s, \\ C_s\varepsilon^{N-2s} + O(\varepsilon^{2s}) & \text{if } 2s < N \le 4s, \end{cases}$$

as $\varepsilon \to 0$. Here C_s is a positive constant depending only on s.

(c) $\displaystyle\int_{\mathbb{R}^N} |u_\varepsilon(x)|^{2_s^*}\,dx = S_s^{N/2s} + O(\varepsilon^N)$, *as $\varepsilon \to 0$.*

Now consider the following minimization problem:

$$S_{s,\lambda} = \inf_{v \in X(\Omega)\setminus\{0\}} S_{s,\lambda}(v),$$

where

$$S_{s,\lambda}(v) = \frac{\displaystyle\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \lambda\int_{\mathbb{R}^N} |v(x)|^2\,dx}{\bigg(\displaystyle\int_{\mathbb{R}^N} |v(x)|^{2_s^*}\,dx\bigg)^{2/2_s^*}}.$$

Arguing as in [2], the following Brezis–Nirenberg estimates for nonlocal setting were proved in [15, Section 4.2] the first item, while in [13, Corollary 8] the second.

Proposition 3.6. *By considering the above definitions one can deduce that*:

(a) *For $N \ge 4s$, $s \in (0,1)$, we have $S_{s,\lambda}(u_\varepsilon) < S_s$, for all $\lambda > 0$ and provided $\varepsilon > 0$ is sufficiently small.*
(b) *For $2s < N < 4s$, $s \in (0,1)$, there exists $\lambda_s > 0$ such that for all $\lambda > \lambda_s$, we have $S_{s,\lambda}(u_\varepsilon) < S_s$, provided $\varepsilon > 0$ is sufficiently small.*

Proof. For the sake of the completeness, we give a sketch of the proof. Let us distinguish the three different cases $N > 4s$, $N = 4s$ and $2s < N < 4s$. By Proposition 3.5, we infer that

Case $N > 4s$.

$$\begin{aligned} S_{s,\lambda}(u_\varepsilon) &\le \frac{S_s^{N/2s} + O(\varepsilon^{N-2s}) - \lambda C_s\varepsilon^{2s}}{(S_s^{N/2s} + O(\varepsilon^N))^{2/2_s^*}} \\ &\le S_s + O(\varepsilon^{N-2s}) - \lambda\widetilde{C}_s\varepsilon^{2s} \le S_s + \varepsilon^{2s}(O(\varepsilon^{N-4s}) - \lambda\widetilde{C}_s) < S_s, \end{aligned}$$

if $\lambda > 0$, $\varepsilon > 0$ is sufficiently small and $\widetilde{C}_s > 0$ is a constant.

Case $N = 4s$.

$$
\begin{aligned}
S_{s,\lambda}(u_\varepsilon) &\le \frac{S_s^{N/2s} + O(\varepsilon^{N-2s}) - \lambda C_s \varepsilon^{2s}|\log\varepsilon| + O(\varepsilon^{2s})}{\left(S_s^{N/2s} + O(\varepsilon^N)\right)^{2/2_s^*}} \\
&\le S_s + O(\varepsilon^{2s}) - \lambda \widetilde{C}_s \varepsilon^{2s}|\log\varepsilon| \le S_s + \varepsilon^{2s}(O(1) - \lambda\widetilde{C}_s)|\log\varepsilon|) < S_s,
\end{aligned}
$$

for $\lambda > 0$, $\varepsilon > 0$ sufficiently small and $\widetilde{C}_s > 0$ a constant.

Case $2s < N < 4s$.

$$
\begin{aligned}
S_{s,\lambda}(u_\varepsilon) &\le \frac{S_s^{N/2s} + O(\varepsilon^{N-2s}) - \lambda C_s \varepsilon^{N-2s} + O(\varepsilon^{2s})}{\left(S_s^{N/2s} + O(\varepsilon^N)\right)^{2/2_s^*}} \\
&\le S_s + \varepsilon^{N-2s}(O(1) - \lambda\widetilde{C}_s) + O(\varepsilon^{2s}) < S_s,
\end{aligned}
$$

for all $\lambda > 0$ large enough ($\lambda \ge \lambda_s$), $\varepsilon > 0$ sufficiently small and $\widetilde{C}_s > 0$ a constant. □

For our purposes, we need to define the following minimization problem:

$$
\widetilde{S}_{s,A} = \inf_{u,v \in X(\Omega)\setminus\{0\}} S_{s,A}(u,v),
$$

where

$$
\begin{aligned}
S_{s,A}(u,v) = \Bigg(\int_{\mathbb{R}^{2N}} &\frac{|u(x)-u(y)|^2 + |v(x)-v(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \\
&- \int_{\mathbb{R}^N} (A(u(x),v(x)),(u(x),v(x)))_{\mathbb{R}^2}\,dx\Bigg) \Bigg/ \\
&\left(\int_{\mathbb{R}^N} \left(|u(x)|^p|v(x)|^q + \xi_1|u(x)|^{p+q} + \xi_2|v(x)|^{p+q}\right)dx\right)^{2/(p+q)}
\end{aligned}
$$

and $p + q = 2_s^*$.

Proposition 3.7. *Let μ_1 be given in* (2.4).

(a) *If $N \ge 4s$, $s \in (0,1)$ and μ_1 is positive, then $\widetilde{S}_{s,A} < \widetilde{S}_s$.*
(b) *For $2s < N < 4s$, $s \in (0,1)$, there exists a constant $\mu_s > 0$ such that if $\mu_1 > \mu_s$, we have $\widetilde{S}_{s,A} < \widetilde{S}_s$.*

Proof. From Proposition 3.6, we have

(a) For $N \ge 4s$, $s \in (0,1)$, $S_{s,\mu_1}(u_\varepsilon) < S_s$ thanks to the fact that $\mu_1 > 0$, and provided $\varepsilon > 0$ is sufficiently small.

(b) For $2s < N < 4s$, $s \in (0,1)$, there exists $\mu_s > 0$ such that if $\mu_1 > \mu_s$, we have $S_{s,\mu_1}(u_\varepsilon) < S_s$, provided $\varepsilon > 0$ is sufficiently small.

Let $s_o, t_o > 0$ be obtained in Lemma 3.3. From (2.4) and (3.7), combined with the above estimate, we infer that

$$\begin{aligned}\widetilde{S}_{s,A} &\leq S_{s,A}(s_o u_\varepsilon, t_o u_\varepsilon) \\ &\leq \frac{(s_o^2+t_o^2)}{(s_o^p t_o^q + \xi_1 s_o^{p+q} + \xi_2 t_o^{p+q})^{2/2_s^*}} \\ &\quad \cdot \frac{\displaystyle\int_{\mathbb{R}^{2N}} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \mu_1 \int_{\mathbb{R}^N} |u_\varepsilon(x)|^2\,dx}{\displaystyle\bigg(\int_{\mathbb{R}^N} |u_\varepsilon(x)|^{2_s^*}\,dx\bigg)^{2/2_s^*}} \\ &= mS_{s,\mu_1}(u_\varepsilon) < mS_s = \widetilde{S}_s.\end{aligned}$$

This concludes the proof. □

Remark 3.8. Notice that, by Remark 3.2, we can choose the finite dimensional subspace $\mathbb{F}$ of $Y(\Omega)$ as

$$\mathbb{F} \equiv \mathbb{F}_\varepsilon = V_k \oplus \operatorname{span}\{(\widetilde{z}_\varepsilon, 0)\},$$

where $V_k = \operatorname{span}\{(0,\varphi_{1,s}),(\varphi_{1,s},0),(0,\varphi_{2,s}),(\varphi_{2,s},0),\ldots,(0,\varphi_{k,s}),(\varphi_{k,s},0)\}$, $\widetilde{z}_\varepsilon = z_\varepsilon/||z_\varepsilon||_X$, with

$$z_\varepsilon = u_\varepsilon - \sum_{j=1}^{k}\bigg(\int_\Omega u_\varepsilon \varphi_{j,s}\,dx\bigg)\varphi_{j,s},$$

and u_ε defined in (3.15).

From Proposition 3.1, we can apply Theorem 2.5 to the functional I_s with

$$Q = (\overline{B}_R \cap V_k) \oplus \{r(\widetilde{z}_\varepsilon, 0) : 0 < r < R\},$$

which critical level is characterized as

$$c = \inf_{h\in\Gamma}\max_{(u,v)\in Q} I_s(h(u,v)),$$

where $\Gamma = \{h \in C(\overline{Q}, Y) : h = \mathrm{id} \text{ on } \partial Q\}$.

4. Palais–Smale condition for the functional

Lemma 4.1. *Suppose $\lambda_{k,s} \leq \mu_1 \leq \mu_2 < \lambda_{k+1,s}$ and let $c \in \mathbb{R}$ be such that*

$$c < \frac{2s}{N}\bigg(\frac{\widetilde{S}_s}{2}\bigg)^{N/2s}. \tag{4.1}$$

Then, the functional I_s satisfies the $(\mathrm{PS})_c$ *condition.*

Proof. Let $(U_n) = (u_n, v_n)$ in $Y(\Omega)$ be a (PS)-sequence for I_s. In order to prove Lemma 4.1, we proceed by the following steps.

Step 1. *Any* $(\mathrm{PS})_c$*-sequence is bounded in the space* $Y(\Omega)$.

Let $(U_n) = (u_n, v_n)$ in $Y(\Omega)$ be a (PS)-sequence for I_s, then

$$
\begin{aligned}
(4.2)\quad & 2I_s(U_n) - I_s'(U_n)(U_n) \\
&= 2\left(1 - \frac{2}{2_s^*}\right) \int_\Omega \big(|u_n|^p |v_n|^q + \xi_1 |u_n|^{p+q} + \xi_2 |v_n|^{p+q}\big)\, dx \\
&\qquad\qquad \le c + o(1)\|U_n\|_Y.
\end{aligned}
$$

Using the Young inequality, we obtain

$$
\|(u_n, v_n)\|^2_{(L^2(\Omega))^2} \le k_1 + k_2 \|(u_n, v_n)\|^{2_s^*}_{(L^{2_s^*}(\Omega))^2}. \tag{4.3}
$$

Combining (4.2) and (4.3), we conclude

$$
\|U_n\|_Y^2 \le 2I_s(U_n) + \frac{4}{2_s^*} \int_\Omega \big(|u_n|^p |v_n|^q + \xi_1 |u_n|^{p+q} + \xi_2 |v_n|^{p+q}\big)\, dx \tag{4.4}
$$

$$
+ \|(u_n, v_n)\|^2_{(L^2(\Omega))^2} \le c + o(1)\|U_n\|_Y. \tag{4.5}
$$

Therefore, we conclude that the sequence (U_n) is bounded.

Step 2. *Problem* (1.1) *admits a solution* $U \in Y(\Omega)$.

Since U_n is bounded in $Y(\Omega)$, up to a subsequence, still denoted by U_n, there exists $U \in Y(\Omega)$ such that $U_n \rightharpoonup U$ in $Y(\Omega)$.

Since $Y(\Omega) \hookrightarrow L^{2_s^*}(\Omega) \times L^{2_s^*}(\Omega)$, we have that U_n is bounded in $L^{2_s^*}(\Omega) \times L^{2_s^*}(\Omega)$, and so, up to a subsequence,

$$
U_n \rightharpoonup U \quad \text{in } L^{2_s^*}(\Omega) \times L^{2_s^*}(\Omega), \tag{4.6}
$$

$$
U_n \to U \quad \text{a.e. } x \text{ in } \Omega, \tag{4.7}
$$

$$
U_n \to U \quad \text{in } L^r(\Omega) \times L^r(\Omega), \text{ for all } r \in [1, 2_s^*). \tag{4.8}
$$

Moreover, by Remark 2.3 (c), there exists a constant $K > 0$ such that

$$
|\nabla F(U_n)| \le K[|u_n|^{2_s^*-1} + |v_n|^{2_s^*-1}]. \tag{4.9}
$$

We have that $|u_n|^{2_s^*-1}$ and $|v_n|^{2_s^*-1}$ are bounded in $L^{2_s^*/(2_s^*-1)}(\Omega)$ and consequently $|\nabla F(U_n)|$ is bounded in $L^{2_s^*/(2_s^*-1)}(\Omega)$. Therefore, by (4.6), it follows that

$$
\nabla F(U_n) \rightharpoonup \nabla F(U) \quad \text{in } L^{2_s^*/(2_s^*-1)}(\Omega) \times L^{2_s^*/(2_s^*-1)}(\Omega). \tag{4.10}
$$

Since $(2_s^*/2_s^* - 1)' = 2_s^*$, it is easily seen that, for all $\Theta \in L^{2_s^*}(\Omega) \times L^{2_s^*}(\Omega)$,

$$
\int_\Omega (\nabla F(U_n), \Theta)_{\mathbb{R}^2}\, dx \to \int_\Omega (\nabla F(U), \Theta)_{\mathbb{R}^2}\, dx.
$$

In particular

$$
\int_\Omega (\nabla F(U_n), \Theta)_{\mathbb{R}^2}\, dx \to \int_\Omega (\nabla F(U), \Theta)_{\mathbb{R}^2}\, dx, \quad \text{for all } \Theta \in Y(\Omega), \tag{4.11}
$$

as $n \to \infty$. On the other hand, for any $\Theta \in Y(\Omega)$, we have the convergence to zero of $I'_s(U_n)(\Theta)$, i.e.

$$\langle U_n, \Theta\rangle_Y - \int_\Omega (AU_n, \Theta)_{\mathbb{R}^2}\, dx - \int_\Omega (\nabla F(U_n), \Theta)_{\mathbb{R}^2}\, dx \to 0, \tag{4.12}$$

so that, passing to the limit in this expression as $n \to \infty$ and taking into account the convergences (4.6), (4.8) and (4.11), we get

$$\langle U, \Theta\rangle_Y - \int_\Omega (AU, \Theta)_{\mathbb{R}^2}\, dx - \int_\Omega (\nabla F(U), \Theta)_{\mathbb{R}^2}\, dx = 0,$$

for all $\Theta \in Y(\Omega)$, and consequently the Step 2 follows.

Step 3. *The following relations hold true*:

(a) $I_s(U) = \left(\dfrac{2^*_s}{2} - 1\right) \displaystyle\int_\Omega F(U)\, dx \geq 0.$

(b) $I_s(U_n) = I_s(U) + \dfrac{1}{2}\, \|U_n - U\|^2_Y - \displaystyle\int_\Omega F(U_n - U)\, dx + o(1).$

(c) $\|U_n - U\|^2_Y = 2^*_s \displaystyle\int_\Omega F(U_n - U)\, dx + o(1).$

Proof of (a). Taking $\Theta = U \in Y(\Omega)$ as a test function in (3.1), we get

$$0 = I'_s(U)U = \|U\|^2_Y - \int_\Omega (AU, U)_{\mathbb{R}^2}\, dx - \int_\Omega (\nabla F(U), U)_{\mathbb{R}^2}\, dx.$$

Therefore,

$$\begin{aligned} I_s(U) &= \frac{1}{2}\left(\|U\|^2_Y - \int_\Omega (AU, U)_{\mathbb{R}^2}\, dx\right) - \int_\Omega F(U)\, dx \\ &= \frac{1}{2}\int_\Omega (\nabla F(U), U)_{\mathbb{R}^2}\, dx - \int_\Omega F(U)\, dx \\ &= \frac{2^*_s}{2}\int_\Omega F(U)\, dx - \int_\Omega F(U)\, dx = \left(\frac{2^*_s}{2} - 1\right)\int_\Omega F(U)\, dx. \end{aligned}$$

Proof of (b). By Step 1, the sequence U_n is bounded in $Y(\Omega) \hookrightarrow L^{2^*_s}(\Omega) \times L^{2^*_s}(\Omega)$, hence U_n is bounded in $L^{2^*_s}(\Omega) \times L^{2^*_s}(\Omega)$. Since $U_n \to U$ almost everywhere in Ω, by the Brezis–Lieb Lemma (see [7, Theorem 1]), we have

$$\|U_n\|^2_Y = \|U_n - U\|^2_Y + \|U\|^2_Y + o(1), \tag{4.13}$$

$$\|U_n\|^{2^*_s}_{L^{2^*_s}} = \|U_n - U\|^{2^*_s}_{L^{2^*_s}} + \|U\|^{2^*_s}_{L^{2^*_s}} + o(1). \tag{4.14}$$

Otherwise, by the Brezis–Lieb Lemma for homogeneous functions (Lemma 5 in [6]),

$$\int_\Omega F(U_n)\, dx = \int_\Omega F(U)\, dx + \int_\Omega F(U_n - U)\, dx + o(1), \quad \text{as } n \to \infty. \tag{4.15}$$

Therefore, using that $U_n \to U$ in $L^r(\Omega) \times L^r(\Omega)$, for all $r \in [1, 2_s^*)$, by the definition of I_s, (4.13)–(4.15), we deduce that

$$\begin{aligned} I_s(U_n) &= \frac{1}{2}\,\|U_n\|_Y^2 - \frac{1}{2}\int_\Omega (AU_n, U_n)_{\mathbb{R}^2}\,dx - \int_\Omega F(U_n)\,dx \\ &= \frac{1}{2}\,\|U_n - U\|_Y^2 + \frac{1}{2}\,\|U\|_Y^2 - \frac{1}{2}\int_\Omega (AU, U)_{\mathbb{R}^2}\,dx \\ &\quad - \int_\Omega F(U)\,dx - \int_\Omega F(U_n - U)\,dx + o(1) \\ &= I_s(U) + \frac{1}{2}\,\|U_n - U\|_Y^2 - \int_\Omega F(U_n - U)\,dx + o(1). \end{aligned}$$

Proof of (c). By (4.6), (4.10) and Remark 2.3 (a),

$$\begin{aligned} &\int_\Omega (\nabla F(U_n) - \nabla F(U), U_n - U)_{\mathbb{R}^2}\,dx \\ &\qquad = \int_\Omega (\nabla F(U_n), U_n)_{\mathbb{R}^2}\,dx - \int_\Omega (\nabla F(U_n), U)_{\mathbb{R}^2}\,dx \\ &\qquad\quad - \int_\Omega (\nabla F(U), U_n)_{\mathbb{R}^2}\,dx + \int_\Omega (\nabla F(U), U)_{\mathbb{R}^2}\,dx \\ &\qquad = \int_\Omega (\nabla F(U_n), U_n)_{\mathbb{R}^2}\,dx - \int_\Omega (\nabla F(U), U)_{\mathbb{R}^2}\,dx + o(1) \\ &\qquad = 2_s^* \int_\Omega F(U_n)\,dx - 2_s^* \int_\Omega F(U)\,dx + o(1). \end{aligned}$$

Therefore, using (4.15), we get

$$\int_\Omega (\nabla F(U_n) - \nabla F(U), U_n - U)_{\mathbb{R}^2}\,dx = 2_s^* \int_\Omega F(U_n - U)\,dx + o(1). \tag{4.16}$$

On the other hand, by Steps 1 and 2,

$$\begin{aligned} o(1) &= I_s'(U_n)(U_n - U) = I_s'(U_n)(U_n - U) - I_s'(U)(U_n - U) \\ &= \langle U_n, U_n - U\rangle_Y - \int_\Omega (AU_n, U_n - U)_{\mathbb{R}^2}\,dx - \int_\Omega (\nabla F(U_n), U_n - U)_{\mathbb{R}^2}\,dx \\ &\quad - \langle U, U_n - U\rangle_Y + \int_\Omega (AU, U_n - U)_{\mathbb{R}^2}\,dx + \int_\Omega (\nabla F(U), U_n - U)_{\mathbb{R}^2}dx \\ &= \langle U_n - U, U_n - U\rangle_Y - \int_\Omega (A(U_n - U), U_n - U)_{\mathbb{R}^2}\,dx \\ &\quad - \int_\Omega (\nabla F(U_n) - \nabla F(U), U_n - U)_{\mathbb{R}^2}\,dx. \end{aligned}$$

Hence, from (4.8) and (4.16), it follows that

$$\|U_n - U\|_Y^2 = 2_s^* \int_\Omega F(U_n - U)\,dx + o(1), \quad \text{as } n \to \infty.$$

Now, we can conclude the proof of Lemma 4.1. By Step 3 (c), it follows that

$$\begin{aligned}\frac{1}{2}\,\|U_n-U\|_Y^2 &- \int_\Omega F(U_n-U)\,dx \\ &= \left(\frac{1}{2}-\frac{1}{2_s^*}\right)\|U_n-U\|_Y^2+o(1)=\frac{s}{N}\,\|U_n-U\|_Y^2+o(1).\end{aligned}$$

Therefore, using the Step 3 (b) and above equality, notice that

$$\begin{aligned}I_s(U)\,+\,&\frac{s}{N}\,\|U_n-U\|_Y^2 \\ &=I_s(U)+\frac{1}{2}\,\|U_n-U\|_Y^2-\int_\Omega F(U_n-U)\,dx+o(1) \\ &=I_s(U_n)+o(1)=c+o(1),\quad \text{as } n\to\infty.\end{aligned}\tag{4.17}$$

Now, by Step 1, the sequence $\|U_n\|_Y$ is bounded in $\mathbb{R}$. So, up to a subsequence, if necessary, we can assume that

$$\|U_n-U\|_Y^2\to L\quad \text{as } n\to\infty.\tag{4.18}$$

Again, as a consequence of Step 3 (c),

$$2_s^*\int_{\mathbb{R}^N}F(U_n-U)\,dx\to L,\quad \text{as } n\to\infty\tag{4.19}$$

and consequently $L\in[0,\infty)$ and by definition of $\widetilde{S}_{p,q}(\Omega)$ (see 3.3), since $U_n-U\in Y(\Omega)\setminus\{(0,0)\}$, we have

$$\widetilde{S}_s:=\widetilde{S}_{p,q}(\Omega)\le\frac{\|U_n-U\|_Y^2}{\left(\dfrac{2_s^*}{2}\displaystyle\int_{\mathbb{R}^N}F(U_n-U)\,dx\right)^{2/2_s^*}}.$$

Hence, by (4.18) and (4.19), we conclude that

$$L\ge\frac{1}{2^{(N-2s)/N}}\,L^{2/2_s^*}\widetilde{S}_s,$$

and consequently,

$$\text{either}\quad L=0\quad\text{or}\quad L\ge\frac{1}{2^{(N-2s)/2s}}\,(\widetilde{S}_s)^{N/(2s)}.$$

If $L\ge(\widetilde{S}_s)^{N/(2s)}/2^{(N-2s)/(2s)}$, by (4.17), (4.18) and Step 3 (a), we would get

$$c=I(U)+\frac{s}{N}\,L\ge\frac{s}{N}\,L\ge\frac{s}{N}\,\frac{1}{2^{(N-2s)/(2s)}}\,(\widetilde{S}_s)^{N/(2s)}=\frac{2s}{N}\left(\frac{\widetilde{S}_s}{2}\right)^{N/(2s)},$$

which contradicts (4.1). Thus $L=0$ and therefore, by (4.18), we have

$$\|U_n-U\|_Y^2\to 0\quad\text{as } n\to\infty$$

and so the assertion of lemma 4.1 follows. □

The next result can be proved along the same lines as [13, Proposition 12] and [11, Proposition 7.3].

PROPOSITION 4.2. *Let $s \in (0,1)$, $N > 2s$ and* $M_\varepsilon := \max\limits_{\substack{u \in \mathbb{F}_\varepsilon \\ \|u\|_{L^{2^*_s}} = 1}} S_{s,\mu_1}(u)$. *Suppose $\lambda_{k,s} \le \mu_1 \le \mu_2 < \lambda_{k+1,s}$, for some $k \in \mathbb{N}$, then*

(a) *M_ε is achieved by $u_M \in \mathbb{F}_\varepsilon$, characterized by $u_M = \widetilde{v} + tz_\varepsilon$, where $t > 0$, z_ε is given in Remark* 3.8, *and*

$$\widetilde{v} = v + t\sum_{i=1}^{k}\bigg(\int_\Omega u_\varepsilon \varphi_{i,s}\,dx\bigg)\varphi_{i,s},$$

u_ε defined in (3.15) *and $v \in \operatorname{span}\{\varphi_{1,s}, \dots, \varphi_{k,s}\}$.*

(b) *The following estimate holds for $t > 0$:*

$$M_\varepsilon \le (\lambda_{k,s} - \mu_1)\|v\|^2_{L^2} + \mathcal{S}_{s,\mu_1}(u_\varepsilon)(1 + O(\varepsilon^{(N-2s)/2})\|v\|_{L^2}) + O(\varepsilon^{(N-2s)/2})\|v\|_{L^2}, \quad \text{as } \varepsilon \to 0.$$

(c) *$M_\varepsilon < \mathcal{S}_s$, provided*
  - (c1) *$N > 4s$ and $\mu_1 \ne \lambda_{k,s}$, for all $k \in \mathbb{N}$.*
  - (c2) *$N = 4s$ and $\mu_1 \ne \lambda_{k,s}$, for all $k \in \mathbb{N}$.*
  - (c3) *$N < 4s$ and $\mu_1 \ne \lambda_{k,s}$, for all $k \in \mathbb{N}$ and μ_1 is large enough ($\mu_1 \ge \lambda_s > 0$).*

The next result can be proved along the same lines as in [12, Proposition 3.1] and [11, Proposition 7.3].

PROPOSITION 4.3. *Let $s \in (0,1)$ and $N > 2s$. Suppose $\mu_1 = \lambda_{k,s} \le \mu_2 < \lambda_{k+1,s}$, for some $k \in \mathbb{N}$.*

(a) *M_ε is achieved by $u_M \in \mathbb{F}_\varepsilon$, characterized by $u_M = v + P_k\widetilde{v} + t\widetilde{u}_\varepsilon$, where $t > 0$, $\widetilde{u}_\varepsilon = u_\varepsilon - P_k u_\varepsilon$, u_ε defined in* (3.15), *$P_K w$ denotes the projection operator of w on the direction $\varphi_{k,s}$, that is,*

$$P_k w = \bigg(\int_\Omega w\varphi_{k,s}\,dx\bigg)\varphi_{k,s},$$

$$v = \sum_{i=1}^{k-1}\bigg(\int_\Omega (\widetilde{v} - tu_\varepsilon)\varphi_{i,s}\,dx\bigg)\varphi_{i,s} \in \operatorname{span}\{\varphi_{1,s}, \dots, \varphi_{k-1,s}\},$$

and $\widetilde{v} \in \operatorname{span}\{\varphi_{1,s}, \dots, \varphi_{k-1,s}\}$.

(b) *The following estimate holds for $t > 0$:*

$$M_\varepsilon \le (\lambda_{k-1,s} - \mu_1 + \sigma)\|v\|^2_{L^2} + \mathcal{S}_{s,\mu_1}(u_\varepsilon)(1 + O(\varepsilon^{(N-2s)/2})\|v\|_{L^2}) + O(\varepsilon^{(N-2s)/2})\|v\|_{L^2},$$

as $\varepsilon \to 0$, some $\sigma < \mu_1 - \lambda_{k-1,s}$.

(c) *$M_\varepsilon < \mathcal{S}_s$, provided*
  - (c1) *$N > 4s$.*
  - (c2) *$N < 4s$ and μ_1 is large enough ($\mu_1 \ge \lambda_s > 0$).*

5. End of the proof of Theorem 1.1

To complete the proof of Theorem 1.1, we have to show that condition (4.1) is satisfied.

Proposition 5.1. *According to our previous notation, we have*

$$c < \frac{2s}{N}\left(\frac{\widetilde{\mathcal{S}}_s}{2}\right)^{N/2s},$$

where c is the critical level $c = \inf_{h\in\Gamma}\max_{(u,v)\in Q} I_s(h(u,v))$ and $\Gamma = \{h \in C(\overline{Q}, Y) : h = \mathrm{id}$ on $\partial Q\}$.

Proof. Notice that, for all $h \in \Gamma$, we have

$$c = \inf_{h\in\Gamma}\max_{(u,v)\in Q} I_s(h(u,v)) \le \max_{(u,v)\in Q} I_s(h(u,v)).$$

Let $\mathbb{F}_\varepsilon$ be as in Remark 3.8 with ε sufficiently small. Since $Q \subset (\mathbb{F}_\varepsilon)^2$, taking $h = \mathrm{id}$ and recalling that $(\mathbb{F}_\varepsilon)^2$ is a linear subspace, we obtain

$$\begin{aligned} c &= \inf_{h\in\Gamma}\max_{(u,v)\in Q} I_s(h(u,v)) \le \max_{(u,v)\in Q} I_s((u,v)) \\ &\le \max_{\substack{(u,v)\,\in\,(\mathbb{F}_\varepsilon)^2\\ (u,v)\,\ne\,(0,0)}} I_s((u,v)) = \max_{\substack{(u,v)\,\in\,(\mathbb{F}_\varepsilon)^2\\ \eta\,\ne\,0}} I_s\left(|\eta|\left(\frac{u}{|\eta|},\frac{v}{|\eta|}\right)\right) \\ &= \max_{\substack{(u,v)\,\in\,(\mathbb{F}_\varepsilon)^2\\ \eta\,>\,0}} I_s(\eta(u,v)) \le \max_{\substack{(u,v)\,\in\,(\mathbb{F}_\varepsilon)^2\\ \eta\,\ge\,0}} I_s(\eta(u,v)). \end{aligned}$$

Claim. We claim that

$$\max_{\substack{(u,v)\,\in\,(\mathbb{F}_\varepsilon)^2\\ \eta\,\ge\,0}} I_s(\eta(u,v)) < \frac{2s}{N}\left(\frac{\widetilde{\mathcal{S}}_s}{2}\right)^{N/2s}.$$

To verify this claim, fix $U = (u,v) \in (\mathbb{F}_\varepsilon)^2$ such that $uv \ne 0$, by (2.4), for all $r \ge 0$, we infer

$$\begin{aligned} I_s(rU) &\le \frac{r^2}{2}\big(\|U\|_Y^2 - \mu_1\|U\|_{(L^2)^2}^2\big) \\ &\quad - \frac{2r^{2_s^*}}{2_s^*}\int_{\mathbb{R}^N}\big(|u(x)|^p|v(x)|^q + \xi_1|u(x)|^{p+q} + \xi_2|v(x)|^{p+q}\big)\,dx \\ &= \frac{Ar^2}{2} - \frac{2Br^{2_s^*}}{2_s^*} := g(r). \end{aligned}$$

Notice that $r_0 = (A/(2B))^{1/(2_s^*-2)}$ is the maximum point of g, which maximum value is given by

$$\frac{2s}{N}\left(\frac{A}{2B^{2/2_s^*}}\right)^{N/2s}.$$

Then

$$
\begin{aligned}
&\max_{r\geq 0} I_s(rU)\\
&\leq \frac{2s}{N}\left\{\frac{\|U\|_Y^2-\mu_1\|U\|_{(L^2)^2}^2}{2\left[\int_{\mathbb{R}^N}\left(|u(x)|^p|v(x)|^q+\xi_1|u(x)|^{p+q}+\xi_2|v(x)|^{p+q}\right)dx\right]^{2/2_s^*}}\right\}^{N/2s}.
\end{aligned}
$$

Therefore, it is sufficient to show that

$$
\begin{aligned}
\widetilde{M_\varepsilon} &:= \max_{(u,v)\in(\mathbb{F}_\varepsilon)^2}\frac{\|U\|_Y^2-\mu_1\|U\|_{(L^2)^2}^2}{2\left[\int_{\mathbb{R}^N}\left(|u(x)|^p|v(x)|^q+\xi_1|u(x)|^{p+q}+\xi_2|v(x)|^{p+q}\right)dx\right]^{2/2_s^*}}\\
&=\frac{1}{2}\max_{\substack{(u,\,v)\,\in\,(\mathbb{F}_\varepsilon)^2\\ \int_{\mathbb{R}^N}(|u|^p|v|^q+\xi_1|u|^{p+q}+\xi_2|v|^{p+q})\,dx=1}}(\|U\|_Y^2-\mu_1\|U\|_{(L^2)^2}^2)<\frac{\widetilde{\mathcal{S}}_s}{2}.
\end{aligned}
$$

Define

$$
M_\varepsilon := \max_{u\in\mathbb{F}_\varepsilon\setminus\{0\}}\frac{\|u\|_X^2-\mu_1\|u\|_{L^2}^2}{\left(\int_{\mathbb{R}^N}|u|^{2_s^*}\,dx\right)^{2/2_s^*}} = \max_{\substack{u\,\in\,\mathbb{F}_\varepsilon\\ \int_{\mathbb{R}^N}|u|^{2_s^*}dx=1}}(\|u\|_X^2-\mu_1\|u\|_{L^2}^2).
$$

Taking $s_o, t_o > 0$ as in Lemma 3.3 and u_M as in Propositions 4.3 and 4.2, $\widetilde{M_\varepsilon}$ is achieved by function $U_M = (s_o u_M, t_o u_M)$. Therefore, from Propositions 4.3 and 4.2, and using (3.7), we can conclude that

$$
\begin{aligned}
\widetilde{M_\varepsilon} &= \frac{1}{2}\,\frac{\|U_M\|_Y^2-\mu_1\|U_M\|_{(L^2)^2}^2}{\left[\int_{\mathbb{R}^N}\left(|s_o u_M|^p|t_o u_M|^q+\xi_1|s_o u_M|^{p+q}+\xi_2|t_o u_M|^{p+q}\right)dx\right]^{2/2_s^*}}\\
&=\frac{1}{2}\,\frac{(s_o^2+t_o^2)}{(s_o^p t_o^q+\xi_1 s_o^{p+q}+\xi_2 t_o^{p+q})^{2/2_s^*}}\,\frac{(\|u_M\|_X^2-\mu_1\|u_M\|_{L^2}^2)}{\left(\int_{\mathbb{R}^N}|u_M|^{2_s^*}\,dx\right)^{2/2_s^*}}\\
&=\frac{1}{2}\,mM_\varepsilon<\frac{1}{2}\,m\mathcal{S}_s=\frac{1}{2}\,\widetilde{\mathcal{S}}_s,
\end{aligned}
$$

if one of the following conditions holds:

(a) $N > 4s$ and $\mu_1 > 0$.
(b) $N = 4s$ and $\mu_1 \neq \lambda_{k,s}$, for all $k \in \mathbb{N}$.
(c) $N < 4s$ and μ_1 is large enough ($\mu_1 \geq \lambda_s > 0$).

This completes the proof. $\square$

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