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MULTIPLICITY OF POSITIVE SOLUTIONS FOR KIRCHHOFF TYPE PROBLEMS IN $\mathbb{R}^3$

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Abstract. We are concerned with the multiplicity of positive solutions for the following Kirchhoff type problem:

$$\begin{cases} -\left(\varepsilon^2 a + \varepsilon b \displaystyle\int_{\mathbb{R}^3} |\nabla u|^2\,dx\right)\Delta u + u = Q(x)|u|^{p-2}u, & x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \quad u > 0, & x \in \mathbb{R}^3, \end{cases}$$

where $\varepsilon > 0$ is a parameter, $a, b > 0$ are constants, $p \in (2,6)$, and $Q \in C(\mathbb{R}^3)$ is a nonnegative function. We show how the profile of Q affects the number of positive solutions when ε is sufficiently small.

1. Introduction

In this paper, we study the existence and multiplicity of positive solutions to the following nonlinear problem of Kirchhoff type:

$$\text{(P)} \qquad \begin{cases} -\left(\varepsilon^2 a + \varepsilon b \displaystyle\int_{\mathbb{R}^3} |\nabla u|^2\,dx\right)\Delta u + u = Q(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases}$$

where $\varepsilon > 0$ is a parameter, $a, b > 0$ are constants, $p \in (2,6)$, and Q is a nonnegative continuous function satisfying

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(Q) $\lim_{|x|\to\infty} Q(x) = Q_\infty > 0$ and there exist some points $x^1, \dots, x^k$ in $\mathbb{R}^3$ such that $Q(x^i)$ are the strict maxima and satisfy

$$Q(x^i) = Q_m := \max_{x\in\mathbb{R}^3} Q(x) > Q_\infty \quad \text{for all } i = 1, \dots, k.$$

Problem (P) is a particular case of the following Dirichlet problem of Kirchhoff type:

$$\begin{cases} -\Big(a + b\displaystyle\int_\Omega |\nabla u|^2\,dx\Big)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth domain. Such problems are often referred to be nonlocal because of the presence of the Kirchhoff term $(\int_\Omega |\nabla u|^2)\Delta u$ which implies that (1.1) is no longer a pointwise identity. This phenomenon provokes some mathematical difficulties that make the study of such problems particularly interesting. Various results on the existence of positive solutions, multiple solutions, sign-changing solutions, ground states have been obtained, see for example [4], [6], [7], [11], [13], [16]–[18], [20], [25], [27] and the references therein.

Recently, there has been increasing interest in studying the following perturbed Kirchhoff type equation (see [12, 10, 9, 23] and the references therein):

$$\begin{cases} -\Big(\varepsilon^2 a + \varepsilon b\displaystyle\int_{\mathbb{R}^3} |\nabla u|^2\,dx\Big)\Delta u + V(x)u = f(x,u) & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{1.2}$$

where $V \in C(\mathbb{R}^3, \mathbb{R})$, $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, $a, b > 0$ are constants, and ε is a positive parameter. First, it is important to consider the following autonomous problem with $\varepsilon = 1$ and $V(x) \equiv \nu \in \mathbb{R}$ in (1.2):

$$-\Big(a + b\int_{\mathbb{R}^3} |\nabla u|^2\,dx\Big)\Delta u + \nu u = f(u) \quad \text{in } \mathbb{R}^3. \tag{1.3}$$

He and Zou in [12] established the existence of ground state solution to (1.2) under the condition that $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies the Ambrosetti–Rabinowitz condition ((AR) in short)

$$\text{there exists} \quad \mu > 4 \quad \text{such that} \quad 0 < \mu\int_0^u f(s)\,ds \le f(u)u,$$

$f(u) = o(u^3)$ as $u \to 0$, $f(u)/|u|^q \to 0$ as $|u| \to \infty$ for some $3 < q < 5$ and $f(u)/u^3$ is increasing for $u > 0$. Ye in [26] extended the above result to the case without the (AR) condition under the conditions that f is superlinear, subcritical and $f(u)/u$ is increasing for $u > 0$. Wang et al. [23], He et al. [10] proved the existence of ground state solution to (1.3) with critical nonlinearity, i.e. $f(u) \sim \lambda|u|^{p-2}u + |u|^4u$ for some $4 < p < 6$, $\lambda > 0$. Latter, He and Li [9] filled the gap when $f(u) = \lambda|u|^{p-2}u + |u|^4u$ where $2 < p \le 4$. Besides, for

$\varepsilon > 0$ sufficiently small, by using Lusternik–Schnirelmann theory, multiplicity of positive solutions to (1.2) has been obtained by employing the topology of the set where the potential V attains its global or local minimum, see [10], [12], [23] and the references therein.

Making the change of variable $u_\varepsilon := u(\varepsilon x)$, we can rewrite (P) as the following equivalent equation:

$$(\mathrm{P}_\varepsilon) \qquad \begin{cases} -\bigg(a + b\displaystyle\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2\,dx\bigg)\Delta u_\varepsilon + u_\varepsilon = Q_\varepsilon(x)|u_\varepsilon|^{p-2}u_\varepsilon & \text{in } \mathbb{R}^3, \\ u_\varepsilon \in H^1(\mathbb{R}^3), \end{cases}$$

where $Q_\varepsilon(x) = Q(\varepsilon x)$. Throughout this paper, we denote by $H := H^1(\mathbb{R}^3)$ the usual Sobolev space equipped with the inner product and norm

$$(u,v) = \int_{\mathbb{R}^3} a\nabla u\nabla v + uv\,dx, \quad \|u\| = (u,u)^{1/2}.$$

Define the energy functional $I_\varepsilon\colon H \to \mathbb{R}$ by

$$I_\varepsilon(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\bigg(\int_{\mathbb{R}^3} |\nabla u|^2\,dx\bigg)^2 - \frac{1}{p}\int_{\mathbb{R}^3} Q_\varepsilon(x)|u|^p\,dx.$$

The functional I_ε is well defined for every $u \in H$ and belongs to $C^1(H,\mathbb{R})$. Clearly, weak solutions to (P_ε) correspond to the critical points of I_ε and for any $\varphi \in H$, we have

$$\langle I'_\varepsilon(u),\varphi\rangle = (u,\varphi) + b\int_{\mathbb{R}^3} |\nabla u|^2\,dx \int_{\mathbb{R}^3} \nabla u\nabla\varphi\,dx - \int_{\mathbb{R}^3} Q_\varepsilon(x)|u|^{p-2}u\varphi\,dx.$$

Let

$$G_\varepsilon(u) := \langle I'_\varepsilon(u),u\rangle = \|u\|^2 + b\bigg(\int_{\mathbb{R}^3} |\nabla u|^2\,dx\bigg)^2 - \int_{\mathbb{R}^3} Q_\varepsilon|u|^p\,dx. \tag{1.4}$$

Obviously, all critical points of I_ε belong to the Nehari manifold

$$M_\varepsilon := \{u \in H \setminus \{0\} : G_\varepsilon(u) = 0\}. \tag{1.5}$$

In the following, we mainly show how the profile of Q affects the number of positive solutions to (P_ε). To the authors' knowledge, there is no result on this topic for Kirchhoff type problems.

To make use of the profile of Q, we need the barycenter map. The barycenter map is a continuous map $\beta\colon H \setminus \{0\} \to \mathbb{R}^3$ which is equivalent to the action of the group of Euclidean motions in $\mathbb{R}^3$. That is, for any given $z \in \mathbb{R}^3$, $t \neq 0$ and $u \in H \setminus \{0\}$, we have

$$\beta(u) = \beta(|u|), \quad \beta(u(x-z)) = \beta(u) + z, \quad \beta(tu) = \beta(u),$$

and if u is radially symmetric function, $\beta(u) = 0$. Such a map was constructed and used by Cerami and Passaseo in [3], etc.

When $a = 1$, $b = 0$, (P_ε) reduces to the following semilinear equation:

$$-\Delta u + u = Q_\varepsilon(x)|u|^{p-2}u, \quad u \in H^1(\mathbb{R}^N). \tag{1.6}$$

For $N \geq 3$, $2 < p < 2N/(N-2)$, and $Q \in C(\mathbb{R}^N, \mathbb{R})$ satisfying condition (Q), Cao and Noussair in [2] established the existence of both positive and nodal solutions to (1.6) for sufficiently small $\varepsilon > 0$, which is affected by the shape of the graph of Q. Motivated by [2], we can first choose k open balls $\{B(x^i, r^i)\}_{i=1}^k$ with center x^i and radius r^i such that

$$Q(x^i) > Q(x) \quad \text{for all } x \in \overline{B(x^i, r^i)} \setminus \{x^i\},\ i = 1, \ldots, k,$$
$$B(x^i, r^i) \cap B(x^j, r^j) = \emptyset \quad \text{for } i \neq j.$$

We then seek a minimizer u^i of the energy functional I_ε over a suitable subset of the Nehari manifold M_ε with the constraint $\beta(u^i) \in B_\varepsilon^i$ as $\varepsilon > 0$ is sufficiently small, where $B_\varepsilon^i := B(x^i/\varepsilon, r^i/\varepsilon)$ for $i = 1, \ldots, k$. Since $\{B_\varepsilon^i\}_{i=1}^k$ are disjoint from each other, we can assert that k minimizers $\{u^i\}_{i=1}^k$ are distinct solutions to (P_ε).

For $4 \leq p < 6$, we shall seek the minimizer of I_ε on $\overline{M_\varepsilon^i} := M_\varepsilon^i \cup \partial M_\varepsilon^i$ directly as $\varepsilon > 0$ is sufficiently small, where

$$M_\varepsilon^i := \{u \in M_\varepsilon : \beta(u) \in B_\varepsilon^i\}, \quad \partial M_\varepsilon^i := \{u \in M_\varepsilon : \beta(u) \in \partial B_\varepsilon^i\}.$$

We obtain the following result:

Theorem 1.1. *Suppose that $4 \leq p < 6$ and condition* (Q) *is satisfied. Then there exists $\varepsilon_0 > 0$ such that (P_ε) has at least k positive solutions for every $\varepsilon \in (0, \varepsilon_0)$.*

When $2 < p < 4$, due to the Kirchhoff term $(\int_{\mathbb{R}^3} |\nabla u|^2\,dx)^2$ being homogeneous of degree 4, the energy functional I_ε is unbounded from below in M_ε. We shall overcome this obstacle by adding the following constraint:

$$(4-p)b\bigg(\int_{\mathbb{R}^3} |\nabla u|^2\,dx\bigg)^2 < (p-2)\|u\|^2. \tag{1.7}$$

Obviously, for any $u \in M_\varepsilon$ satisfies (1.7),

$$I_\varepsilon(u) = J(u) := \frac{p-2}{2p}\,\|u\|^2 + \frac{(p-4)b}{4p}\bigg(\int_{\mathbb{R}^3} |\nabla u|^2\,dx\bigg)^2 > \frac{p-2}{4p}\,\|u\|^2 \tag{1.8}$$

which implies that I_ε is bounded from below and coercive. Therefore, we can establish the boundedness of the minimizing sequence of I_ε.

The idea of adding constraint (1.7) derives from Sun et al. [21], where they studied the multiplicity of positive solutions for the following Schrödinger–Poisson system:

$$\begin{cases} -\Delta u + \lambda u + K(x)\phi u = Q(x)|u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.9}$$

where $2 < p < 6$, $\lambda > 0$, Q satisfies condition (Q), and $K \in C(\mathbb{R}^3, \mathbb{R})$ is assumed to satisfy certain conditions. They proved the existence of at least k positive solutions to (1.9) for $\lambda > 0$ sufficiently large. Their results extended the earlier one in Chen et al. [5] in which a related problem for (1.9) with $4 \le p < 6$ and $K \equiv 1$ was considered. To deal with the case $2 < p < 4$, Sun et al. in [21] developed a variational method to minimize the energy functional of (1.9) on a certain set which can be regarded as a filtration of the Nehari manifold with some nice properties. However, the constraint condition in [21] does not work for Kirchhoff type problems. In addition, the limiting equation in [21] is the following semilinear equation:

$$-\Delta u + u = Q_{\max}|u|^{p-2}u \quad \text{in } \mathbb{R}^3,$$

while the associated limiting equation of our problem is (1.3), and the Kirchhoff term $b\int_{\mathbb{R}^3}|\nabla u|^2\,dx\Delta u$ may cause some technical difficulties.

Let

$$N_\varepsilon^i := \{u \in M_\varepsilon : K(u) < 0,\ \beta(u) \in B_\varepsilon^i\}, \tag{1.10}$$

for $j = 1, \ldots, k$, where

$$K(u) := (4-p)b\bigg(\int_{\mathbb{R}^3}|\nabla u|^2\,dx\bigg)^2 - (p-2)\|u\|^2. \tag{1.11}$$

Denote by $\overline{N_\varepsilon^i}$ the closure of N_ε^i in H, then $\overline{N_\varepsilon^i} = N_\varepsilon^i \cup \partial\overline{N_\varepsilon^i} \cup \partial\widehat{N}_\varepsilon^i$, where

$$\partial\overline{N}_\varepsilon^i := \{u \in M_\varepsilon : K(u) < 0,\ \beta(u) \in \partial B_\varepsilon^i\}, \tag{1.12}$$

$$\partial\widehat{N}_\varepsilon^i := \{u \in M_\varepsilon : K(u) = 0,\ \beta(u) \in \overline{B_\varepsilon^i}\}, \tag{1.13}$$

for $i = 1, \ldots, k$.

In the following, we shall seek a minimizer of I_ε on $\overline{N_\varepsilon^i}$ for $2 < p < 4$ as $\varepsilon > 0$ is sufficiently small. To rule out the case that the minimizer of I_ε on $\overline{N_\varepsilon^i}$ is located on the boundary of N_ε^i, we require $b > 0$ to be suitably small. We mention that the limitation of b is also crucial in dealing with the autonomous problem. Our main results in this direction can be stated as follows.

THEOREM 1.2. *Let $2 < p < 4$, suppose that condition* (Q) *is satisfied, and $0 < b < b_0(b)$, $b_0(b)$ is defined in* (2.5). *Then there exists $\varepsilon_0 > 0$ such that* (P_ε) *has at least k positive solutions for every $\varepsilon \in (0, \varepsilon_0)$.*

The paper is organized as follows. After proving several lemmas about the autonomous problem in Section 2, we establish some energy estimates in Section 3, and demonstrate proofs of Theorems 1.1 and 1.2 by variational method in Section 4.

2. The autonomous problem (P_ν)

We start this section by consider following autonomous equation:

$$(\mathrm{P}_\nu) \qquad -\bigg(a + b\int_{\mathbb{R}^3} |\nabla u|^2\,dx\bigg)\Delta u + u = Q_\nu |u|^{p-2}u, \quad u \in H,$$

where $a, b > 0$, $2 < p < 6$, $\nu \in \{m, \infty\}$, Q_m and Q_∞ are defined in condition (Q). Since (P_ν) is variational, its solutions are critical points of the energy functional given by

$$I_\nu^b(u) = \frac{1}{2}\,\|u\|^2 + \frac{b}{4}\bigg(\int_{\mathbb{R}^3} |\nabla u|^2\,dx\bigg)^2 - \frac{1}{p}\int_{\mathbb{R}^3} Q_\nu |u|^p\,dx.$$

The Nehari manifold $M_\nu^b := \{u \in H : G_\nu^b(u) = 0\}$ contains all solutions to (P_ν), where

$$G_\nu^b(u) := \|u\|^2 + b|\nabla u|_2^4 - Q_\nu |u|_p^p.$$

As mentioned in Section 1, the existence of the ground states to (P_ν) has been obtained in [26].

Lemma 2.1 (Theorem 1.1 in [26]). *(P_ν) has a radially symmetric and positive ground state solution $w_{b,\nu} \in H$ with the least energy*

$$c_\nu(b) := \inf_{\gamma\in\Gamma_\nu}\max_{t\in[0,1]} I_\nu^b(\gamma(t)) > 0, \tag{2.1}$$

where $\Gamma_\nu^b = \{\gamma \in C([0,1], H) : \gamma(0) = 0,\ I_\nu^b(\gamma(1)) < 0\} \neq \emptyset$.

Remark 2.2. By definition of $c_\nu(b)$, it is easy to see that $c_m(b) < c_\infty(b)$.

Let U be the unique positive radial solution of the well-known scalar field equation (see [1], [8], [15])

$$-a\Delta U + U = Q_m U^{p-1} \quad \text{in } \mathbb{R}^3.$$

We can obtain the following uniqueness result.

Lemma 2.3. *Let $u \in H \setminus \{0\}$ be any given positive solution to (P_m), then there exists $z \in \mathbb{R}^3$, such that*

$$u(x) = U(\mu x - z), \quad \text{where } \mu = \frac{-b|\nabla U|_2^2 + \sqrt{b^2|\nabla U|_2^4 + 4a^2}}{2a}.$$

Proof. Let $1/\mu^2 = 1 + b|\nabla u|_2^2/a$. Obviously, $v(x) = u(x/\mu)$ solves the equation

$$-a\Delta v + v = Q_m v^{p-1} \quad \text{in } \mathbb{R}^3. \tag{2.2}$$

By the uniqueness results of (2.2), there exists $z \in \mathbb{R}^3$ such that $v(x) = U(x-z)$. Therefore, we obtain that $u(x) = U(\mu x - z)$, then

$$|\nabla u|_2^2 = \frac{1}{\mu}\int_{\mathbb{R}^3} |\nabla U|^2\,dx.$$

It yields a quadratic equation of μ

$$a\mu^2 + b|\nabla U|_2^2\mu - a = 0,$$

which implies that

$$\mu = \frac{-b|\nabla U|_2^2 + \sqrt{b^2|\nabla U|_2^4 + 4a^2}}{2a}. \qquad \square$$

REMARK 2.4. It follows from Lemmas 2.1 and 2.3 that $w_b := w_{b,m}$ in Lemma 2.1 is unique, and $w_b(x) = U(\mu x)$.

REMARK 2.5. Note that the uniqueness of positive solution to a Kirchhoff type equation in $\mathbb{R}^3$ has also been established recently by Xie et al. in [25] and Huang et al. in [14].

If $4 < p < 6$, He and Zou in [12] (Proposition 2.4, Lemma 4.2 therein) proved that the mountain pass solution w_b minimizes the energy functional I_m^b on M_m^b. For the reader's convenience we give an alternative approach to this result here for $4 \le p < 6$ by considering the following minimization problem:

$$\widehat{c}(b) = \inf_{u \in M_m^b} I_m^b(u). \tag{2.3}$$

PROPOSITION 2.6. *Let $4 \le p < 6$, then all the minimizing sequences of* (2.3) *are relatively compact and $\widehat{c}(b)$ is achieved. Moreover, $\widehat{c}(b) = c_m(b)$.*

However, when $2 < p < 4$, I_m^b is unbounded from below on M_m^b. By considering the following minimization problem:

$$\widetilde{c}(b) = \inf_{u \in N_m^b} I_m^b(u), \tag{2.4}$$

we shall prove that, under the condition

$$b < b_0(a,p,b) := \frac{(p-2)^2a^2}{8p(4-p)\widetilde{c}(b)}, \tag{2.5}$$

w_b minimizes the energy functional I_m^b on $N_m^b := \{u \in M_m^b \mid K(u) < 0\}$, where K is defined in (1.11).

PROPOSITION 2.7. *Suppose that $2 < p < 4$ and $b > 0$ satisfies condition* (2.5) *then all the minimizing sequences of* (2.4) *are relatively compact, and $\widetilde{c}(b)$ is achieved. Moreover, $\widetilde{c}(b) = c_m(b)$.*

REMARK 2.8. If $b > 0$ is sufficiently small, then (2.5) holds. Indeed, let $g(t,b) = (t^2 - t^p)\|U\|^2 + t^4 b|\nabla U|_2^4$, we have

$$g(1,0) = 0, \qquad \frac{dg}{dt}(1,0) = (2-p)\|U\|^2 \neq 0.$$

Applying the implicit function theorem, there exist a small $\delta > 0$ and a continuous differentiable function $t(b)\colon (0,\delta) \to (1, 1+\varepsilon)$ such that $g(t(b), b) = 0$ and $K(t(b)U) < 0$, for some sufficiently small $\varepsilon > 0$. Thus

$$\widetilde{c}(b) \le I_m^b(t(b)U) = \frac{p-2}{2p}\,\|t(b)U\|^2 - \frac{p-4}{4p}\,b|t(b)\nabla U|_2^4 < \frac{p-2}{2p}\,\|t(b)U\|^2 \le \|U\|^2,$$

which implies $b_0(a,p,b)$ is bounded away from zero as $b > 0$ is sufficiently small.

Before proving Proposition 2.6 and 2.7, we first establish several lemmas.

Lemma 2.9. *Suppose that $2 < p < 4$, $A, B, C > 0$ and*

$$f(t) = A + Bt^2 - Ct^{p-2}, \qquad h(t) = \frac{p-2}{2p}\,At^2 - \frac{4-p}{4p}\,Bt^4,$$

then

(a) *if $A + B < C$, there exist $t_\pm > 0$ such that $0 < t_- < 1 < t_+$, $f(t_\pm) = 0$, $f'(t_-) < 0$, $f'(t_+) > 0$;*
(b) *if $(A/((4-p)/2))^{(4-p)/2}(B/((p-2)/2))^{(p-2)/2} < C$, there exist $t^\pm > 0$ such that $f(t^\pm) = 0$, $f'(t^-) < 0$, $f'(t^+) > 0$;*
(c) *if $A + B = C$ and $B < ((p-2)/(4-p)) \cdot A$, there exist $t^\pm > 0$ such that $f(t^\pm) = 0$, $f'(t^-) < 0$, $f'(t^+) > 0$;*
(d) *h is increasing in the interval $[0, t_2)$, $t_2 = \sqrt{(p-2)A/((4-p)B)}$.*

Proof. (a) Note that $f(0) = A > 0$, $f(1) = A + B - C < 0$ and $f(t) \to +\infty$ as $t \to +\infty$, so we can find the desired $t_\pm > 0$.

(b) Since

$$f'(t) = t\biggl(2B - \frac{(p-2)C}{t^{4-p}}\biggr) \quad \text{for } t > 0,$$

then f has only one critical point (minimum point)

$$t_1 = \biggl(\frac{p-2}{2} \cdot \frac{C}{B}\biggr)^{1/(4-p)},$$

and

$$\begin{aligned} f(t_1) &= A - \frac{4-p}{2}\biggl((p-2)/2\biggr)^{(p-2)/(4-p)} \frac{C^{2/(4-p)}}{B^{(p-2)/(4-p)}} \\ &= \frac{4-p}{2} \cdot \frac{\biggl(\dfrac{p-2}{2}\biggr)^{(p-2)/(4-p)}}{B^{(p-2)/(4-p)}} \\ &\quad \cdot \biggl\{\biggl[\biggl(\frac{A}{(4-p)/2}\biggr)^{(4-p)/2}\biggl(\frac{B}{(p-2)/2}\biggr)^{(p-2)/2}\biggr]^{2/(4-p)} - C^{2/(4-p)}\biggr\} < 0. \end{aligned}$$

Since $f(0) = A > 0$, $f(t) \to +\infty$ as $t \to +\infty$, then there exist $0 < t^- < t_1 < t^+$ such that $f(t^\pm) = 0$, $f'(t^-) < 0$ and $f'(t^+) > 0$.

(c) By Young's inequality,

$$\left(\frac{A}{(4-p)/2}\right)^{(4-p)/2}\left(\frac{B}{(p-2)/2}\right)^{(p-2)/2} \leq A+B,$$

and the equality holds if and only if $(4-p)B=(p-2)A$.

Since $B<(p-2)A/(4-p)$, we have

$$\left(\frac{A}{(4-p)/2}\right)^{(4-p)/2}\left(\frac{B}{(p-2)/2}\right)^{(p-2)/2} < A+B=C.$$

By the results of (b), we can also find the desired $t^{\pm}>0$.

(d) Since

$$h'(t)=\frac{t}{p}\left((p-2)A-(4-p)Bt^2\right) \quad \text{for } t>0,$$

then h has only one (maximum) critical point $t_2=\sqrt{(p-2)A/((4-p)B)}$ hence $h'(t)>0$ for $t\in[0,t_2)$. □

The next lemma contains the statement of the main properties of M_m^b and N_m^b.

Lemma 2.10.

(a) *Suppose that* $2<p<6$, *then there exist constants* $\eta_0>0$ *and* $\eta_1>0$ *such that*

$$\|u\|>\eta_0, \quad |u|_p^p>\eta_1 \quad \text{for all } u\in M_m^b. \tag{2.6}$$

(b) *Suppose that* $4\leq p<6$, *then* M_m^b *is a* C^1*-manifold. If* u *is a critical point of* I_m^b *constrained on* M_m^b, *then* u *is a critical point of* I_m^b *on* H.

(c) *Suppose that* $2<p<4$, *then* N_m^b *is a nonempty* C^1*-manifold. If* u *is a critical point of* I_m^b *constrained on* N_m^b, *then* u *is a critical point of* I_m^b *on* H.

Proof. (a) Let $u\in M_m^b$, then $G_m^b(u)=0$ and

$$\frac{1}{C}|u|_p^2\leq\|u\|^2+b|\nabla u|_2^4=Q_m|u|_p^p\leq CQ_m\|u\|^p$$

for some $C>0$. Thus, we can find $\eta_0>0$ and $\eta_1>0$ satisfying (2.6).

(b) Obviously, $G_m^b(u)$ and $K(u)$ are C^1-functionals in H. Note that for all $u\in M_m^b$, $4\leq p<6$,

$$\langle (G_m^b)'(u),u\rangle = 2\|u\|^2+4b|\nabla u|_2^4-pQ_m|u|_p^p \tag{2.7}$$

$$= (2-p)\|u\|^2+(4-p)b|\nabla u|_2^4<0. \tag{2.8}$$

Let u be a critical point of I_m^b constrained on M_m^b, then there exists $\lambda\in\mathbb{R}$ such that $(I_m^b)'(u)=\lambda(G_m^b)'(u)$. Hence

$$0=G_m^b(u)=\langle (I_m^b)'(u),u\rangle=\lambda\langle (G_m^b)'(u),u\rangle.$$

It follows from (2.7) that $\lambda=0$, then $(I_m^b)'(u)=0$.

(c) Let $w_{b,\infty}$ be the unique positive radially symmetric ground state solution to problem (P_∞), which satisfies $\|w_{b,\infty}\|^2 + b|\nabla w_{b,\infty}|_2^4 = Q_\infty |w_{b,\infty}|_p^p < Q_m|w_{b,\infty}|_p^p$. Set $A = \|w_{b,\infty}\|^2$, $B = b|\nabla w_{b,\infty}|_2^4$, $C = Q_m|w_{b,\infty}|_p^p$, then $A+B < C$. Consider the function $f(t) = A + Bt^2 - Ct^{p-2}$, by Lemma 2.9 (a) there exists $t \in (0,1)$ such that $f(t) = 0$ and $f'(t) < 0$. Since

$$\begin{aligned} G_m^b(tw_{b,\infty}) &= t^2 f(t) = 0, \\ K(tw_{b,\infty}) &= (2-p)t^2 f(t) + t^3 f'(t) < 0, \end{aligned}$$

it yields that $tw_{b,\infty} \in N_m^b$, which implies that N_m^b is not an empty set.

Since N_m^b is relatively open in M_m^b, if u is a critical point of I_m^b constrained on N_m^b, there also exists $\lambda \in \mathbb{R}$ such that $(I_m^b)'(u) = \lambda (G_m^b)'(u)$. Note that

$$\langle (G_m^b)'(u), u\rangle = K(u) < 0 \quad \text{for } u \in N_m^b,$$

we can obtain the statement for N_m^b by a similar argument as in (b). □

Remark 2.11. Let $u \in M_m^b$, then $K(u) = (4-p)b|\nabla u|_2^4 - (p-2)\|u\|^2 < 0$ for $4 \le p < 6$. Therefore, M_m^b and N_m^b coincide when $4 \le p < 6$. Thus, we can treat (2.3) and (2.4) in a unified framework by considering the minimization problem

$$c(b) = \inf_{u \in N_m^b} I_m^b(u) \quad \text{for } 2 < p < 6. \tag{2.9}$$

Proof of Propositions 2.6 and 2.7. 1. Let $\{u_n\} \subset N_m^b$ be a minimizing sequence of (2.9), namely, $G_m^b(u_n) = 0$, $K(u_n) < 0$ and $I_m^b(u_n) \to c(b)$ as $n \to \infty$. Since

$$I_m^b(u_n) = \frac{p-2}{4p}\|u_n\|^2 - \frac{1}{4p}K(u_n) \ge \frac{p-2}{4p}\|u_n\|^2,$$

we have

$$\|u_n\|^2 \le \frac{4pc(b)}{p-2} + o(1). \tag{2.10}$$

Let $u_n \rightharpoonup u \in H$, $u_n \to u$ almost everywhere in $\mathbb{R}^3$ and $u_n \to u$ in $L^q_{\mathrm{loc}}(\mathbb{R}^3)$ for $2 \le q < 6$. By Lemma 2.10 (a), it yields that $|u_n|_p^p > \eta_1 > 0$. From the Lions Concentration–Compactness Lemma (e.g. [24]), we further assume that $u \ne 0$.

Denote $v_n = u_n - u$. By using the Brezis–Lieb Lemma [24], we have

$$G_m^b(u_n) = G_m^b(u) + G_m^b(v_n) + 2b|\nabla u|_2^2|\nabla v_n|_2^2 + o(1) = 0. \tag{2.11}$$

On the other hand,

$$c(b) + o(1) = I_m^b(u_n) = \frac{p-2}{2p}\|u\|^2 + \frac{(p-4)b}{4p}|\nabla u|_2^4 + J_0(u, v_n), \tag{2.12}$$

where

$$J_0(u, v_n) = \frac{p-2}{2p}\,\|v_n\|^2 + \frac{(p-4)b}{4p}\,|\nabla v_n|_2^4 + \frac{(p-4)b}{2p}\,|\nabla u|_2^2|\nabla v_n|_2^2 \tag{2.13}$$

$$\geq \begin{cases} 0, & 4 \leq p < 6, \\ |\nabla v_n|_2^2\Big(\dfrac{(p-2)a}{2p} - \dfrac{b(4-p)}{2ap}\,\|u_n\|^2\Big), & 2 < p < 4. \end{cases} \tag{2.14}$$

Since b satisfies condition (2.5) when $2 < p < 4$, one has

$$\frac{(p-2)a}{2p} - \frac{b(4-p)}{2ap}\,\|u_n\|^2 \geq \frac{(p-2)a}{2p} - \frac{b(4-p)}{2ap}\cdot\frac{4pc(b)}{p-2} + o(1) > 0, \tag{2.15}$$

where (2.10) is used. Combining (2.12), (2.13) and (2.15) together, we have

$$c(b) \geq \frac{p-2}{2p}\,\|u\|^2 + \frac{p-4}{4p}\,b|\nabla u|_2^4 = I_m^b(u). \tag{2.16}$$

2. We next prove that $G_m^b(u) \geq 0$. Arguing by contradiction, we assume that $G_m^b(u) < 0$. In the following, we shall prove that there exists a positive number $t > 0$ such that $tu \in N_m^b$, and

$$\begin{aligned} I_m^b(u) &= \frac{p-2}{2p}\,\|u\|^2 + \frac{(p-4)b}{4p}\,|\nabla u|_2^4 \\ &> \frac{p-2}{2p}\,\|tu\|^2 + \frac{(p-4)b}{4p}\,|t\nabla u|_2^4 = I_m^b(tu). \end{aligned} \tag{2.17}$$

Once such $t > 0$ is found, by the definition of $c(b)$, we have $c(b) \leq I_m^b(tu) < I_m^b(u)$, which contradicts (2.16).

If $4 \leq p < 6$,

$$G_m^b(tu) = \begin{cases} t^2\|u\|^2 + t^4 b|\nabla u|_2^4 - t^p Q_m|u|_p^p, & 4 < p < 6, \\ t^2\|u\|^2 + t^4(b|\nabla u|_2^4 - Q_m|u|_p^p), & p = 4. \end{cases} \tag{2.18}$$

Since $b|\nabla u|_2^4 - Q_m|u|_p^p < -\|u\|^2 < 0$, there exists a $0 < t < 1$ such that $G_m^b(tu) = 0$. Moreover,

$$I_m^b(tu) = \frac{(p-2)t^2}{2p}\,\|u\|^2 + \frac{(p-4)t^4}{4p}\,b|\nabla u|_2^4 < \frac{p-2}{2p}\,\|u\|^2 + \frac{p-4}{4p}\,b|\nabla u|_2^4 = I_m^b(u).$$

On the other hand, if $2 < p < 4$, let

$$\begin{aligned} f(t) &= \|u\|^2 + t^2 b|\nabla u|_2^4 - t^{p-2}Q_m|u|_p^p, \\ h(t) &= \frac{(p-2)t^2}{2p}\,\|u\|^2 - \frac{(4-p)t^4}{4p}\,b|\nabla u|_2^4. \end{aligned}$$

Since $G_m^b(u) < 0$, applying Lemma 2.9 (a) with $A = \|u\|^2$, $B = b|\nabla u|_2^4$ and $C = Q_m|u|_p^p$, we can find $0 < t_- < 1$ such that

$$\begin{aligned} G_m^b(t_- u) &= t_-^2 f(t_-) = 0, \\ K(t_- u) &= (2-p)t_-^2 f(t_-) + t_-^3 f'(t_-) < 0. \end{aligned} \tag{2.19}$$

Moreover, h increases in the interval $[0,t_2]$,

$$t_2=\bigg(\frac{p-2}{4-p}\cdot\frac{\|u\|^2}{b|\nabla u|_2^4}\bigg)^{1/2}.$$

By (2.5) and (2.10),

$$t_2=\bigg(\frac{a^2}{b}\cdot\frac{p-2}{4-p}\cdot\frac{\|u\|^2}{a^2|\nabla u|_2^4}\bigg)^{1/2}\geq\frac{a(p-2)}{2\sqrt{bp(4-p)c(b)}}>1>t_-, \tag{2.20}$$

hence $h(t_-)<h(1)$ and (2.17) holds.

3. A similar argument as in part 2. shows that $\liminf\limits_{n\to\infty}G_m^b(v_n)\geq 0$. Due to (2.11), we have $G_m^b(u)=0$, $G_m^b(v_n)=o(1)$ and $|\nabla u|_2^2|\nabla v_n|_2^2=o(1)$. Since $u\neq 0$, we obtain that $|\nabla v_n|_2^2=o(1)$ and $|v_n|_2^2=Q_m|v_n|_p^p+o(1)$. Using the Gagliardo–Nirenberg inequality

$$|v_n|_2^2=Q_m|v_n|_p^p+o(1)\leq CQ_m|\nabla v_n|_2^{(3p-6)/2}|v_n|_2^{(6-p)/2}+o(1),$$

we deduce that $\|v_n\|^2\to 0$ as $n\to\infty$, which implies that $\{u_n\}$ converges to u in H.

4. It remains to show that $c(b)=c_m(b)$, where $c_m(b)$ is defined in (2.1). If $u\geq 0$, by Lemma 2.3, Remark 2.4, we have $K(w_b)=K(u)<0$ and $c_m(b)=I_m^b(w_b)=I_m^b(u)=c(b)$. While, if $u^\pm\neq 0$, multiplying on both sides of (P_m) by $u^\pm$, we get $G_m^b(u^\pm)=\|u^\pm\|^2+b|\nabla u^\pm|_2^4-Q_m|u^\pm|_p^p=-b|\nabla u^\mp|_2^4<0$. By Lemma 2.9 (a) and a similar argument as (2.19), there exist $t^\pm\in(0,1)$ such that $t^\pm u^\pm\in N_m^b$ and

$$J(u^\pm)>J(t^\pm u^\pm)=I_m^b(t^\pm u^\pm)\geq c(b),$$

where J is defined in (1.8). Note that

$$\begin{aligned}c(b)=I_m^b(u)&=J(u^+)+J(u^-)+\frac{(p-4)b}{2p}|\nabla u^+|_2^2|\nabla u^-|_2^2\\&>2c(b)+\frac{(p-4)b}{2p}|\nabla u^+|_2^2|\nabla u^-|_2^2.\end{aligned}$$

If $4\leq p<6$, we directly have a contradiction. On the other hand, if $2<p<4$, by (2.5) and (2.10),

$$\frac{(4-p)b}{2p}|\nabla u^+|_2^2|\nabla u^-|_2^2\leq\frac{(4-p)b}{8p}|\nabla u|_2^4\leq\frac{c(b)}{4}. \tag{2.21}$$

Therefore, $c(b)=I_m^b(u)>7c(b)/4$ which derives a contradiction. □

3. Energy estimates

In this section, in order to rule out the case that the minimizer of I_ε on $\overline{N_\varepsilon^i}$ is located on the boundary of N_ε^i, we shall establish some energy estimates for perturbation problem (P_ε) by considering the following minimization problems in N^i_ε, $\partial\overline{N^i_\varepsilon}$ and $\partial\widehat{N}^i_\varepsilon$ for I_ε:

$$\gamma^i_\varepsilon = \inf_{u\in N^i_\varepsilon} I_\varepsilon(u), \quad \overline{\gamma}^i_\varepsilon = \inf_{u\in\partial\bar{N}^i_\varepsilon} I_\varepsilon(u) \quad \text{and} \quad \widehat{\gamma}^i_\varepsilon = \inf_{u\in\partial\hat{N}^i_\varepsilon} I_\varepsilon(u)$$

for $i = 1,\ldots,k$, where N^i_ε, $\partial\overline{N^i_\varepsilon}$ and $\partial\widehat{N}^i_\varepsilon$ are defined in (1.10)–(1.13). In fact, N^i_ε is not an empty set. Let $w^i_{b,\infty}(x) = w_{b,\infty}(x - x^i/\varepsilon)$, then

$$\begin{aligned}\|w^i_{b,\infty}\|^2 + b|\nabla w^i_{b,\infty}|^4_2 &= Q_\infty \int_{\mathbb{R}^3} |w^i_{b,\infty}|^p \\ &< \int_{B(0,1/\sqrt{\varepsilon})} Q(\varepsilon x + x^i)|w_{b,\infty}|^p\,dx < \int_{\mathbb{R}^3} Q_\varepsilon |w^i_{b,\infty}|^p\,dx,\end{aligned}$$

for sufficiently small $\varepsilon > 0$. Similarly as the proof of the non-emptiness of N^b_m, we can find $t^i_\varepsilon > 0$ such that $t^i_\varepsilon w^i_{b,\infty} \in N^i_\varepsilon$.

For $\varepsilon > 0$ sufficiently small, we define a cutoff function $\psi_\varepsilon(x) \in C^\infty(\mathbb{R}^3, [0,1])$ such that

$$\begin{aligned}\psi_\varepsilon(x) &= 1 \quad \text{in } B(0,\varepsilon^{-1/2}),\\ \psi_\varepsilon(x) &= 0 \quad \text{in } \mathbb{R}^3 \setminus B(0,\varepsilon^{-1/2}+1),\\ |\Delta\psi_\varepsilon(x)| &\le 3 \quad \text{in } \mathbb{R}^3.\end{aligned}$$

Let $w_{\varepsilon,i}(x) = w_b(x - x^i/\varepsilon)\psi_\varepsilon(x - x^i/\varepsilon)$ where $w_b = w_{b,m}$ denotes the radially symmetric and positive ground state in Lemma 2.1. We start with the following upper bound for γ^i_ε.

LEMMA 3.1. *Let $2 < p < 6$ and suppose that* (2.5) *holds if $2 < p < 4$. Then, for any $\delta > 0$,*

(3.1) $$\gamma^i_\varepsilon \le c_m(b) + \delta \quad \textit{for } \varepsilon > 0 \textit{ is sufficiently small.}$$

PROOF. 1. We first compute

$$\begin{aligned}|w_{\varepsilon,i}|^2_2 &= |w_b\psi_\varepsilon|^2_2 = |w_b|^2_2 + o_\varepsilon(1),\\ |\nabla w_{\varepsilon,i}|^2_2 &= |\nabla w_b\psi_\varepsilon + w_b\nabla\psi_\varepsilon|^2_2\\ &= |\nabla w_b\psi_\varepsilon|^2_2 - \int_{\mathbb{R}^3} w_b^2\psi_\varepsilon\Delta\psi_\varepsilon\,dx = |\nabla w_b|^2_2 + o_\varepsilon(1),\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^3} Q_\varepsilon(x) w^p_{\varepsilon,i}\,dx &= \int_{\mathbb{R}^3} Q(\varepsilon x + x^i) w^p_b\psi^p_\varepsilon\,dx\\ &= \int_{B(0,1/\sqrt{\varepsilon})} Q(\varepsilon x + x^i) w^p_b\,dx\\ &\quad + \int_{\mathbb{R}^3\setminus B(0,1/\sqrt{\varepsilon})} Q(\varepsilon x + x^i) w^p_b\psi^p_\varepsilon\,dx = Q_m|w_b|^p_p + o_\varepsilon(1).\end{aligned}$$

It yields that $\|w_{\varepsilon,i}\|^2 = \|w_b\|^2 + o_\varepsilon(1)$, $b|\nabla w_{\varepsilon,i}|_2^4 = b|\nabla w_b|_2^4 + o_\varepsilon(1)$ and

$$\int_{\mathbb{R}^3} Q_\varepsilon w_{\varepsilon,i}^p\,dx = Q_m|w_b|_p^p + o_\varepsilon(1).$$

2. We claim that there exists $t_\varepsilon > 0$ such that $t_\varepsilon w_{\varepsilon,i} \in N_\varepsilon^i$ and $t_\varepsilon \to 1$ as $\varepsilon \to 0$.

If $4 \le p < 6$, similarly to (2.18), there exists $0 < t_\varepsilon < 1$ such that

$$\begin{aligned}0 &= G_\varepsilon(t_\varepsilon w_{\varepsilon,i})\\ &= \begin{cases} t_\varepsilon^2\|w_{\varepsilon,i}\|^2 + t_\varepsilon^4 b|\nabla w_{\varepsilon,i}|_2^4 - t_\varepsilon^p \displaystyle\int_{\mathbb{R}^3} Q_\varepsilon w_{\varepsilon,i}^p\,dx, & 4 < p < 6,\\ t_\varepsilon^2\|w_{\varepsilon,i}\|^2 + t_\varepsilon^4\Big(b|\nabla w_{\varepsilon,i}|_2^4 - \int_{\mathbb{R}^3} Q_\varepsilon w_{\varepsilon,i}^p\,dx\Big), & p = 4.\end{cases}\end{aligned} \tag{3.2}$$

By Lemma 2.10 (a), there exists $\eta_2 > 0$ such that $\|t_\varepsilon w_{\varepsilon,i}\| > \eta_2$. Since $\{\|w_{\varepsilon,i}\|\}$ is bounded, we obtain that t_ε must converge to $t \in (0,+\infty)$ as $\varepsilon \to 0$. Let $\varepsilon \to 0$ in (3.2), we have

$$t^2\|w_b\|^2 + t^4 b|\nabla w_b|_2^4 - t^p Q_m|w_b|_p^p = 0, \quad 4 \le p < 6.$$

Combining with $\|w_b\|^2 + b|\nabla w_b|_2^4 - Q_m|w_b|_p^p = 0$, it yields that $t = 1$.

If $2 < p < 4$, from Proposition 2.7, we can deduce that

$$K(w_b) = (4-p)b|\nabla w_b|_2^4 - (p-2)\|w_b\|^2 < 0.$$

By Young's inequality, we have

$$\left(\frac{\|w_b\|^2}{(4-p)/2}\right)^{(4-p)/2}\left(\frac{b|\nabla w_b|_2^4}{(p-2)/2}\right)^{(p-2)/2} < \|w_b\|^2 + b|\nabla w_b|_2^4 = Q_m|w_b|_p^p.$$

Then

$$\left(\frac{\|w_{\varepsilon,i}\|^2}{(4-p)/2}\right)^{(4-p)/2}\left(\frac{b|\nabla w_{\varepsilon,i}|_2^4}{(p-2)/2}\right)^{(p-2)/2} < \int_{\mathbb{R}^3} Q_\varepsilon w_{\varepsilon,i}^p\,dx \tag{3.3}$$

holds for $\varepsilon > 0$ sufficiently small. Let $A = \|w_{\varepsilon,i}\|^2$, $B = b|\nabla w_{\varepsilon,i}|_2^4$, $C = \int_{\mathbb{R}^3} Q_\varepsilon w_{\varepsilon,i}^p dx$ and

$$f(t) = A + Bt^2 + Ct^{p-2},$$

applying Lemma 2.9 (b), there exist $0 < t_\varepsilon^- < t_1 < t_\varepsilon^+$ such that $f(t_\varepsilon^\pm) = 0$, $f'(t_\varepsilon^-) < 0$ and $f'(t_\varepsilon^+) > 0$. Similarly to (2.19), we deduce that $G_\varepsilon(t_\varepsilon^\pm w_{\varepsilon,i}) = 0$, $K(t_\varepsilon^- w_{\varepsilon,i}) < 0$ and $K(t_\varepsilon^+ w_{\varepsilon,i}) > 0$. From $G_\varepsilon(t_\varepsilon^\pm w_{\varepsilon,i}) = 0$, it is easy to see that $t_\varepsilon^\pm$ must converge to $t^\pm \in (0,+\infty)$, which implies that $K(t^- w_b) \le 0$, $K(t^+ w_b) \ge 0$. It follows from $K(w_b) < 0$ that $t_\varepsilon^- \to t^- = 1$ as $\varepsilon \to 0$.

3.

$$\begin{aligned}\gamma_\varepsilon^i \le I_\varepsilon(t_\varepsilon w_{\varepsilon,i}) &= \frac{1}{2}\,t_\varepsilon^2\|w_{\varepsilon,i}\|^2 + \frac{b}{4}\,t_\varepsilon^4|\nabla w_{\varepsilon,i}|_2^4 - \frac{1}{p}\,t_\varepsilon^p\int_{\mathbb{R}^3} Q_\varepsilon(x)w_{\varepsilon,i}^p\,dx\\ &= I_m^b(w_b) + o_\varepsilon(1) = c_m(b) + o_\varepsilon(1) < c_m(b) + \delta,\end{aligned}$$

for $\varepsilon > 0$ sufficiently small. □

LEMMA 3.2. *Let $2 < p < 6$ and suppose that* (2.5) *holds if* $2 < p < 4$. *Then, there exists* $\eta_3 > 0$, *independent of* ε *such that*

$$\bar{\gamma}^i_\varepsilon > c_m(b) + \eta_3 \quad \textit{for } \varepsilon > 0 \textit{ sufficiently small.}$$

PROOF. Assume to the contrary that there exists a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ as $n \to \infty$ such that $\liminf_{n\to\infty} \overline{\gamma}^i_{\varepsilon_n} = c_m(b)$. Then there exists a sequence $\{u_n\} \subset \overline{\partial N^i_{\varepsilon_n}}$ such that

$$I_{\varepsilon_n}(u_n) \to c \le c_m(b) \quad \text{as } n \to \infty. \tag{3.4}$$

We first claim that there exists a sequence $\{t_n\}$ such that $0 < t_n < 1$, $t_n u_n \in N^b_m$ and $I^b_m(t_n u_n) < I^b_m(u_n)$.

If $4 \le p < 6$, since $b|\nabla u_n|_2^4 - Q_m|u_n|_p^p = -\|u_n\|^2 < 0$, similarly to (2.18) the above claim holds.

If $2 < p < 4$, define

$$f_n(t) := \|u_n\|^2 + t^2 b|\nabla u_n|_2^4 - t^{p-2} Q_m |u_n|_p^p,$$
$$h_n(t) := \frac{(p-2)t^2}{2p}\,\|u_n\|^2 + \frac{(p-4)t^4}{4p}\,b|\nabla u_n|_2^4.$$

Since $G(u_n) = 0$ and $K(u_n) < 0$, Lemma 2.9 (c) yields that there exists a sequence $\{t_n\}$ such that $0 < t_n < 1$, $f_n(t_n) = 0$ and $f_n'(t_n) < 0$. Similarly to (2.19), we have $t_n u_n \in N^b_m$. By (3.4), we can deduce that

$$\|u_n\|^2 \le \frac{4pc_m(b)}{p-2} + o(1).$$

By Proposition 2.7, $c_m(b) = \widetilde{c}(b)$. Similarly to (2.20), we have $h_n(t) < h_n(1)$, i.e. $I^b_m(t_n u_n) < I^b_m(u_n)$. Thus,

$$\begin{aligned} c_m(b) \le I^b_m(t_n u_n) &= \frac{1}{2}\,\|t_n u_n\|^2 + \frac{b}{4}\,|t_n \nabla u_n|_2^4 - \frac{1}{p}\,Q_m |t_n u_n|_p^p \\ &< \frac{1}{2}\,\|u_n\|^2 + \frac{b}{4}\,|\nabla u_n|_2^4 - \frac{1}{p}\int_{\mathbb{R}^3} Q_{\varepsilon_n} |u_n|^p\,dx = I_\varepsilon(u_n) \le c_m(b) + o(1), \end{aligned}$$

which implies that $\{t_n u_n\}$ are minimizing sequences of (2.9). Moreover $t_n \to 1$ as $n \to \infty$ and

$$Q_m |u_n|_p^p = \int_{\mathbb{R}^3} Q_{\varepsilon_n} |u_n|^p\,dx + o(1). \tag{3.5}$$

By Propositions 2.6, 2.7, there exists a sequence $\{z_n\} \subset \mathbb{R}^3$ such that $u_n(x + z_n) \to w_b$ in H as $n \to \infty$.

Note that $\beta(u_n) \in \partial B^i_{\varepsilon_n}$, it follows from the properties of barycenter map that $\{\varepsilon_n z_n\}$ converges to a point $z \in \partial B(x^i, r^i)$. Therefore

$$\begin{aligned} \lim_{n\to\infty} \int_{\mathbb{R}^3} Q_{\varepsilon_n}(x) u_n^p(x)\,dx &= \lim_{n\to\infty} \int_{\mathbb{R}^3} Q(\varepsilon_n x + \varepsilon_n z_n) u_n^p(x + z_n)\,dx = Q(z)|w_b|_p^p, \\ \lim_{n\to\infty} Q_m |u_n|_p^p &= Q_m |w_b|_p^p > Q(z)|w_b|_p^p, \end{aligned}$$

which contradicts (3.5). This completes the proof of Lemma 3.2. $\square$

Lemma 3.3. *Let $2 < p < 4$ and suppose that* (2.5) *holds. Then*

$$\widehat{\gamma}_\varepsilon^i \geq 2c_m(b) \quad \textit{for all } \varepsilon > 0.$$

Proof. For any $u \in \partial \widehat{N}_\varepsilon^i$,

$$I_\varepsilon(u) = \frac{p-2}{4p}\,\|u\|^2,$$
$$(p-2)a|\nabla u|_2^2 \leq (p-2)\|u\|^2 = (4-p)b|\nabla u|_2^4.$$

By Lemma 2.10 (a), $\|u\| > \eta_2 > 0$, which implies that

$$|\nabla u|_2^2 \geq \frac{(p-2)a}{(4-p)b}.$$

It follows from (2.5) that

$$I_\varepsilon(u) \geq \frac{(p-2)^2a^2}{4bp(4-p)} > 2c_m(b). \qquad \square$$

Corollary 3.4. *Let $2 < p < 6$ and suppose that* (2.5) *holds if $2 < p < 4$. Then*

$$\gamma_\varepsilon^i < \min\{c_\infty(b), \overline{\gamma}_\varepsilon^i, \widehat{\gamma}_\varepsilon^i\} \quad \textit{for } i = 1, \ldots, k,$$

for $\varepsilon > 0$ sufficiently small.

4. Proof of Theorems 1.1 and 1.2

Motivated by Tarantello [22] (also see Ni and Takagi [19]), we have the following result.

Lemma 4.1. *For $\varepsilon > 0$ sufficiently small, $u \in N_\varepsilon^i$, there exist $\sigma > 0$ and a function $t^*\colon B(0,\sigma) \subset H \to \mathbb{R}_+$ such that $t^*(0) = 1$ and $t^*(v)(u-v) \in N_\varepsilon^i$ for all $v \in B(0,\sigma)$ and, for all $\phi \in H$,*

$$\begin{aligned}\langle (t^*)'(0), \phi\rangle &= \frac{dt^*(s\phi)}{ds}\bigg|_{s=0} \\ &= \frac{2(u,\phi) + 4b|\nabla u|_2^2 \int_{\mathbb{R}^3} \nabla u \nabla \phi\,dx - p\int_{\mathbb{R}^3} Q_\varepsilon |u|^{p-2}u\phi\,dx}{(4-p)b|\nabla u|_2^4 - (p-2)\|u\|^2}.\end{aligned}$$

Proof. For $u \in N_\varepsilon^i$, define $F_u\colon \mathbb{R}\times H \to \mathbb{R}$ by

$$F_u(t,v) = G_\varepsilon(t(u-v)) = t^2\|u-v\|^2 + t^4 b|\nabla u - \nabla v|_2^4 - t^p\int_{\mathbb{R}^3} Q_\varepsilon |u-v|^p\,dx.$$

Then, $F_u(1,0) = G_\varepsilon(u) = 0$ and

$$\frac{\partial F_u}{\partial t}(1,0) = 2\|u\|^2 + 4b|\nabla u|_2^4 - p\int_{\mathbb{R}^3} Q_\varepsilon(x)|u|^p\,dx = K(u) < 0.$$

According to the implicit function theorem, there exist $\sigma > 0$ and a differentiable function $t^*\colon B(0,\sigma) \subset H \to \mathbb{R}_+$ such that $t^*(0) = 1$, $F_u(t^*(v), v) = G(t^*(v)(u - v)) = 0$ for all $v \in B(0,\sigma)$ and

$$\begin{aligned}\langle (t^*)'(0), \phi\rangle &= \frac{dt^*(s\phi)}{ds}\bigg|_{s=0} \\ &= \frac{2(u,\phi) + 4b|\nabla u|_2^2 \displaystyle\int_{\mathbb{R}^3} \nabla u \nabla \phi\, dx - p \int_{\mathbb{R}^3} Q_\varepsilon |u|^{p-2} u\phi\, dx}{(4-p)b|\nabla u|_2^4 - (p-2)\|u\|^2}\end{aligned}$$

for all $\phi \in H$. Furthermore, due to continuity of the functional K, β and $t^*(u)$, we have $K(t^*(v)(u - v)) < 0$ and $\beta(t^*(v)(u - v)) \in B_\varepsilon^i$ for all $v \in B(0,\sigma) \subset H$. Thus, $t^*(v)(u - v) \in N_\varepsilon^i$ for all $v \in B(0,\sigma) \subset H$. □

Lemma 4.2. *Let $2 < p < 6$ and condition* (2.5) *hold if $2 < p < 4$. Then, for $\varepsilon > 0$ sufficiently small, there exists a sequence $\{u_n\} \subset N_\varepsilon^i$ such that*

$$I_\varepsilon(u_n) = \gamma_\varepsilon^i + o(1) \quad \textit{and} \quad I'_\varepsilon(u_n) = o(1) \quad \textit{in } H^{-1}.$$

Proof. Let $\overline{N_\varepsilon^i}$ be the closure of N_ε^i. By Corollary 3.4, we have

$$\gamma_\varepsilon^i = \inf_{u \in \overline{N_\varepsilon^i}} I_\varepsilon(u), \quad \text{for } 2 < p < 6.$$

Applying the Ekeland variational principle, there exists a minimizing sequence $\{u_n\} \subset \overline{N_\varepsilon^i}$ such that $I_\varepsilon(u_n) < \gamma_\varepsilon^i + 1/n$, and

$$I_\varepsilon(u_n) \le I_\varepsilon(w) + \frac{1}{n}\,\|w - u_n\| \quad \text{for all } w \in \overline{N_\varepsilon^i}. \tag{4.1}$$

By Corollary 3.4, we assume that $u_n \in N_\varepsilon^i$ for n sufficiently large. Applying Lemma 4.1 with $u = u_n$, we obtain the function $t_n^*\colon B(0,\sigma_n) \to \mathbb{R}_+$ for some $\sigma_n > 0$ such that $t_n^*(v)(u_n - v) \in N_\varepsilon^i$ for all $v \in B(0,\sigma_n)$. Let $s < \sigma_n$, $v \in H$ and $\|v\| = 1$, we have $t_n^*(sv)(u_n - sv) \in N_\varepsilon^i$. It follows from (4.1) that

$$I_\varepsilon(u_n) - I_\varepsilon(t_n^*(sv)(u_n - sv)) < \frac{1}{n}\,\|t_n^*(sv)(u_n - sv) - u_n\|.$$

Since $t_n^*(sv) \to t_n^*(0) = 1$ as $s \to 0$, we have $I_\varepsilon(u_n) - I_\varepsilon(t_n^*(sv)(u_n - sv)) \to 0$ as $s \to 0$. Therefore

$$\lim_{s\to 0} \frac{I_\varepsilon(u_n) - I_\varepsilon(t_n^*(sv)(u_n - sv))}{s} = \big\langle I'_\varepsilon(u_n), -\langle (t_n^*)'(0), v\rangle u_n + v\big\rangle = \langle I'_\varepsilon(u_n), v\rangle.$$

On the other hand,

$$\begin{aligned}\lim_{s\to 0} \frac{I_\varepsilon(u_n) - I_\varepsilon(t_n^*(sv)(u_n - sv))}{s} &\le \lim_{s\to 0} \frac{\|t_n^*(sv)(u_n - sv) - u_n\|}{sn} \\ \le \lim_{s\to 0} \bigg(\frac{t_n^*(sv) - 1}{s}\bigg)\frac{\|u_n\|}{n} &+ \lim_{s\to 0} t_n^*(sv)\,\frac{\|v\|}{n} = \langle (t_n^*)'(0), v\rangle\,\frac{\|u_n\|}{n} + \frac{1}{n}.\end{aligned}$$

Since $\{\|u_n\|\}$ is bounded, we have

$$\langle I'_\varepsilon(u_n), v\rangle = o(1)\langle (t_n^*)'(0), v\rangle + o(1).$$

By Lemma 4.1, $\langle (t_n^*)'(0), v\rangle \le \|u_n\|/|K(u_n)|$. Once we prove that $\{|K(u_n)|\}$ is bounded away form zero, we get

$$\langle I'_\varepsilon(u_n), v\rangle = o(1) \quad \text{for all } v \in H \text{ and } \|v\| = 1.$$

Note that, if $4 \le p < 6$, $|K(u_n)| = (p-2)\|u_n\|^2 + (p-4)b|\nabla u_n|_2^4 > C\eta_2 > 0$. On the other hand, if $2 < p < 4$, suppose that $\{K(u_n)\}$ converges to zero,

$$(p-2)a|\nabla u_n|_2^2 \le (p-2)\|u_n\|^2 = (4-p)b|\nabla u_n|_2^4 + o(1),$$

which implies $|\nabla u_n|_2^2 \ge (p-2)a/((4-p)b) + o(1)$. By Lemma 3.1, let $\delta = c_m(b)/4$ in (3.1),

$$\begin{aligned} \frac{5}{4} c_m(b) + \frac{1}{n} > \gamma_\varepsilon^i + \frac{1}{n} > I_\varepsilon(u_n) &= \frac{p-2}{4p} \|u_n\|^2 + o(1) \\ &\ge \frac{(p-2)^2 a^2}{4bp(4-p)} + o(1) > 2c_m(b) + o(1), \end{aligned}$$

which derives a contradiction. This completes the proof of Lemma 4.2. □

Lemma 4.3. *Fix $i_0 \in \{1,\ldots,k\}$, let $\{u_n\} \subset N_\varepsilon^{i_0}$ be a $(\mathrm{PS})_c$ sequence, namely $I_\varepsilon(u_n) \to c$, $I'(u_n) \to 0$ in H^{-1} as $n \to \infty$. Then, up to a subsequence, either u_n is strongly convergent in H, or there exist a function $u \in H$, an integer $l > 0$, l functions $\{W_j\}$, $W_j \ne 0$ for $1 \le j \le l$ and l sequences $\{y_n^j\} \subset \mathbb{R}^3$ for $1 \le j \le l$, such that*

$$\begin{cases} -\Big(a + b|\nabla u|_2^2 + \displaystyle\sum_{j=1}^{l} b|\nabla W_j|_2^2\Big)\Delta u + u = Q_\varepsilon(x)|u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Big(a + b|\nabla u|_2^2 + \displaystyle\sum_{i=1}^{l} b|\nabla W_i|_2^2\Big)\Delta W_j + W_j = Q_\infty|W_j|^{p-2}W_j & \text{in } \mathbb{R}^3, \\ & j = 1,\ldots,l, \end{cases} \tag{4.2}$$

and

$$|y_n^j| \to +\infty, \qquad |y_n^j - y_n^i| \to +\infty,\ i \ne j,$$

$$u_n(x) - \sum_{j=1}^{l} W_j(x - y_n^j) \to u(x) \quad \text{in } H. \tag{4.3}$$

Moreover,

$$\begin{aligned} \lim_{n\to\infty} I_\varepsilon(u_n) = J(u) + \sum_{j=1}^{l} J(W_j) &+ \frac{p-4}{2p}\Big(\sum_{j=1}^{l} b|\nabla u|_2^2|\nabla W_j|_2^2\Big) \\ &+ \frac{p-4}{2p}\Big(\sum_{1\le i<j\le l} b|\nabla W_i|_2^2|\nabla W_j|_2^2\Big), \end{aligned} \tag{4.4}$$

where J is defined in (1.8).

PROOF. Similarly to (2.10),

$$\|u_n\|^2 \le \frac{4pc}{p-2} + o(1).$$

Let $u_n \rightharpoonup u \in H$, $u_n \to u$ almost everywhere in $\mathbb{R}^3$ and $u_n \to u \in L^q_{\rm loc}(\mathbb{R}^3)$ for $q \in (2,6)$. Set $\widetilde{u}_n^1 = u_n - u$, by the Brezis–Lieb Lemma, we have

$$-(a + b|\nabla u|_2^2 + b|\nabla \widetilde{u}_n^1|_2^2)\Delta u + u = Q_\varepsilon(x)|u|^{p-2}u + o(1) \quad \text{in } H^{-1}, \tag{4.5}$$

$$-(a + b|\nabla u_n|_2^2)\Delta \widetilde{u}_n^1 + \widetilde{u}_n^1 = Q_\infty|\widetilde{u}_n^1|^{p-2}\widetilde{u}_n^1 + o(1) \quad \text{in } H^{-1}. \tag{4.6}$$

Define

$$\delta := \lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |\widetilde{u}_n^1|^2\,dx.$$

If $\delta = 0$, by the Lions Lemma we have $\widetilde{u}_n^1 \to 0$ in $L^p(\mathbb{R}^3)$. It follows from (4.6) that $\|\widetilde{u}_n^1\|^2 \le Q_\infty|\widetilde{u}_n^1|_p^p \to 0$ as $n \to \infty$, which implies that $u_n \to u$ in H.

If $\delta > 0$, by the Concentration–Compactness Argument, there exists a sequence $\{y_n^1\} \subset \mathbb{R}^3$ such that $|y_n^1| \to \infty$ and $\widetilde{u}_n^1(x + y_n^1) \rightharpoonup W_1 \ne 0$ in H. Set $\widetilde{u}_n^2 := \widetilde{u}_n^1(x + y_n^1) - W_1$,

$$|\nabla \widetilde{u}_n^1|_2^2 = |\nabla \widetilde{u}_n^2|_2^2 + |\nabla W_1|_2^2 + o(1).$$

Similarly to (4.5) and (4.6), W_1 and $\widetilde{u}_n^2$ satisfy

$$\begin{aligned}
-\big(a+b|\nabla u|_2^2+b|\nabla W_1|_2^2+b|\nabla \widetilde{u}_n^2|_2^2\big)\Delta W_1 + W_1 &= Q_\infty|W_1|^{p-2}W_1 + o(1) \text{ in } H^{-1},\\
-\big(a+b|\nabla u|_2^2+b|\nabla W_1|_2^2+b|\nabla \widetilde{u}_n^2|_2^2\big)\Delta \widetilde{u}_n^2+\widetilde{u}_n^2 &= Q_\infty|\widetilde{u}_n^2|^{p-2}\widetilde{u}_n^2 + o(1) \quad \text{in } H^{-1}.
\end{aligned}$$

By iterating the above procedure, we construct sequences $\{W_j\}$ and $\{y_n^j\}$ such that $G_\infty^b(W_j) < 0$, however, the iteration must terminate at some finite index l. In fact, similarly to the proof of that in Propositions 2.6, 2.7 and Lemma 2.10 (a), we can find $t_j \in (0,1)$ such that $t_jW_j \in N_\infty^b$ and $\|W_j\| > \|t_jW_j\| \ge \widetilde{\eta}_0$, for some $\widetilde{\eta}_0 > 0$. It follows from $K(u_n) < 0$ that

$$\begin{aligned}
c + 1 \ge I_\varepsilon(u_n) &\ge \frac{p-2}{4p}\,\|u_n\|^2\\
&\ge \frac{p-2}{4p}\bigg(\|u\|^2 + \sum_{j=1}^{l}\|W_j\|^2 + \|\widetilde{u}_n^{l+1}\|^2\bigg) + o(1) \ge \frac{(p-2)\,l\widetilde{\eta}_0^2}{4p}.
\end{aligned}$$

Therefore,

$$u_n(x) - \sum_{j=1}^{l} W_j(x - y_n^j) \to u(x) \quad \text{in } H,$$

and

$$
\begin{aligned}
I_\varepsilon(u_n) &= \frac{p-2}{2p}\,\|u_n\|^2 + \frac{(p-4)b}{4p}\,|\nabla u_n|_2^4 \\
&= J(u) + \frac{p-2}{2p}\,\|\widetilde{u}_n^1\|^2 + \frac{(p-4)b}{4p}\,|\nabla \widetilde{u}_n^1|_2^4 + \frac{(p-4)b}{2p}\,|\nabla u|_2^2|\nabla \widetilde{u}_n^1|_2^2 \\
&= J(u) + J(W_1) + \frac{p-2}{2p}\,\|\widetilde{u}_n^2\|^2 \\
&\quad + \frac{(p-4)b}{4p}\,|\nabla \widetilde{u}_n^2|_2^4 + \frac{(p-4)b}{2p}\,|\nabla W_1|_2^2|\nabla \widetilde{u}_n^2|_2^2 \\
&\quad + \frac{(p-4)b}{2p}\,|\nabla u|_2^2|\nabla W_1|_2^2 + \frac{(p-4)b}{2p}\,|\nabla u|_2^2|\nabla \widetilde{u}_n^2|_2^2 \\
&= J(u) + \sum_{j=1}^{l} J(W_j) + \frac{p-4}{2p}\bigg(\sum_{j=1}^{l} b|\nabla u|_2^2|\nabla W_j|_2^2\bigg) \\
&\quad + \frac{p-4}{2p}\bigg(\sum_{1\le i<j\le l} b|\nabla W_i|_2^2|\nabla W_j|_2^2\bigg). \qquad \square
\end{aligned}
$$

Proof of Theorems 1.1 and 1.2. Fix $i_0 \in \{1,\dots,k\}$, let $\{u_n\} \subset N_\varepsilon^{i_0}$ be a sequence satisfying

$$
I_\varepsilon(u_n) = \gamma_\varepsilon^{i_0} + o(1), \quad I'_\varepsilon(u_n) = o(1) \quad \text{in } H^{-1}.
$$

By Lemma 4.3, if $\{u_n\}$ is not compact, there exist u, $\{W_j\}$ and $\{y_n^j\}$, $j = 1,\dots,l$, satisfying (4.2), (4.4) and

$$
u_n = u + \sum_{j=1}^{l} W_j(x - y_n^j) + o(1) \quad \text{in } H.
$$

Case 1. $u \neq 0$. Similarly to (2.10), by Lemma 3.1, we have

$$
\|u_n\|^2 \le \frac{4p\gamma_\varepsilon^{i_0}}{p-2} + o(1) \le \frac{4p(c_m(b)+\delta)}{p-2} + o(1). \tag{4.7}
$$

Let $\widetilde{u}_n^1 = u_n - u$, by (4.5) and (4.6) we have $G_\varepsilon(u) < 0$ and $G_\infty^b(\widetilde{u}_n^1) < 0$. Using Lemma 2.9 (a) and (2.19), there exist $t_0, t_n^1 \in (0,1)$, such that $t_0 u \in N_\varepsilon^{i_0}$, $t_n^1 \widetilde{u}_n^1 \in N_\infty^b$ and

$$
J(u) > J(t_0 u) \ge c_m(b), \quad J(\widetilde{u}_n^1) > J(t_n^1 \widetilde{u}_n^1) \ge c_\infty(b). \tag{4.8}
$$

Note that

$$
I_\varepsilon(u_n) = J(u) + J(\widetilde{u}_n^1) + \frac{(p-4)b}{2p}\,|\nabla u|_2^2|\nabla \widetilde{u}_n^1|_2^2,
$$

if $4 \le p < 6$, we directly have $I_\varepsilon(u_n) > c_m(b) + c_\infty(b)$ which contradicts to $I_\varepsilon(u_n) < c_m(b) + \delta$, for $\varepsilon > 0$ sufficiently small.

On the other hand, if $2 < p < 4$, by (2.5) and (4.7),

$$
\frac{(4-p)b}{2p}\,|\nabla u|_2^2|\nabla \widetilde{u}_n^1|_2^2 \le \frac{(4-p)b}{8p}\,|\nabla u_n|_2^4 \le \frac{(c_m(b)+\delta)^2}{4c_m} + o(1) < \frac{c_m}{2} \tag{4.9}
$$

for a small $\delta > 0$ and $0 < \varepsilon < \varepsilon_0$. Thus, by (4.8),

$$c_m + o(1) = I_\varepsilon(u_n) = J(u) + J(\widetilde{u}_n^1) - \frac{(4-p)b}{2p} |\nabla u|_2^2 |\nabla \widetilde{u}_n^1|_2^2 \geq \frac{3c_m}{2},$$

which derives a contradiction.

Case 2. $u = 0$. If $l \geq 2$, denote $\widetilde{u}_n^2 = u_n(x + y_n^1) - W_1$. We see that $\|\widetilde{u}_n^2\|$ is bounded away form zero. Following the argument as in case 1, we can get the contradiction similarly. At last we analyze the case $l = 1$, i.e. $u_n = W_1(x - y_n^1) + o(1)$, for some $|y_n^1| \to \infty$ as $n \to \infty$. Note that the barycenter of u_n is bounded in $B_\varepsilon^{i_0}$ for a fixed $\varepsilon > 0$, while $|\beta(W_1(x - y_n^1))| \to \infty$ as $n \to \infty$, which leads to a contradiction. Therefore, $u_n \to u$ in H for some $u \neq 0$. By Corollary 3.4, we know that $u \in N_\varepsilon^{i_0}$ is the minimizer of I_ε in $N_\varepsilon^{i_0}$ □

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