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COINCIDENCE DEGREE METHODS IN ALMOST PERIODIC DIFFERENTIAL EQUATIONS

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Abstract. We consider the existence of almost periodic solutions to differential equations by using coincidence degree theory. A new equivalent spectral condition for the compactness of integral operators on almost periodic function spaces is established. It is shown that semigroup conditions are crucial in applications.

1. Introduction

The theory of almost periodic functions was mainly created by the Danish mathematician H. Bohr in 1920s. Almost periodic functions are intended to be a generalization of periodic functions in some sense. It is well known that almost periodic theory is interesting and at the same time difficult. In celestial mechanics, almost periodic solutions and stable solutions are intimately related. In the same way, stable electronic circuits exhibit almost periodic behavior. The methods to study the existence of almost periodic solutions can be found, e.g. in [13], [16], [27], [35]–[37].

The coincidence degree theory was established by Mawhin [25]. This theory, based on the Leray–Schauder degree theory, has a successful application in the study of the existence of periodic solutions and some boundary value problems

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of differential equations $x' = \psi(x,t)$, which is written in an abstract operator form as

$$\mathcal{L}x = \mathcal{N}x,$$

where $\mathcal{L}$ is a Fredholm linear operator of index zero. In the Continuation Theorem (Theorem 2.13), there is a condition: $\mathcal{N}$ is $\mathcal{L}$-compact, which is closely related to the compactness of the integral operator in the applications to differential equations. The $\mathcal{L}$-compactness is usually shown by using the Arzela–Ascoli theorem. The underlying reason is the compactness of the space on which the functions are defined.

A natural generalization of the study of periodic solutions could be application of the degree theory in almost periodic world. Once this is achieved, a new method will be available for almost periodic differential equations. However, this problem is very difficult. The underlying reason for this is the non-compactness of the space on which the functions are defined. There are no general theorems of analysis which yield uniform convergence on $\mathbb{R}$.

As it is commented in [5], the compactness is very difficult to exhibit, because the analog of the Arzela–Ascoli theorem for almost periodic functions, the so-called Lusternik theorem (Theorem 2.3), contains a condition of equi-almost periodicity that is practically unverifiable. In [28] the author provides several good examples to which the degree theory is not applicable. There exist both first and second order differential equations for which the associated operators on almost periodic function spaces have no fixed points but they map the closed unit ball into its interior ([28, Theorems 2.1 and 3.2]). It is also mentioned in [29] that it seems that the standard techniques (variational methods, continuation and degree theory, upper and lower solutions) are not applicable and that new phenomena appear. So, it is of significant interest to work out this problem and show these new phenomena. Indeed, we find that coincidence degree theory is applicable to complex almost periodic differential equations.

To our knowledge, there are a few papers that investigate the existence of almost periodic solutions by using the coincidence degree methods, see e.g. [2], [19]–[24], [32]–[34]. However, there exists a gap in these papers. The authors in these papers assume that the Arzela–Ascoli theorem and the module containment could imply $\mathcal{L}$-compactness, which means that the uniform convergence on any compact subsets of $\mathbb{R}$ could imply uniform convergence on $\mathbb{R}$, but this is not the case as pointed out by Zhou and Shao [38].

In the present paper, we continue to investigate such problems. We find that the mentioned above gap appears because the compactness of integral operators is not discussed in these papers. So, it is of great interest to study the compactness of integral operators and find almost periodic solutions to differential equations by involving coincidence degree theory.

We solve the problem of applying coincidence degree theory to almost periodic differential equations by answering four basic questions, that is, given an almost periodic function space, when an integral operator maps the space to itself, when an integral operator is compact on the space, when a nonlinear operator maps the space to itself, and what class of differential equations admits a priori estimate structure.

Our main theorems are formulated as follows. Notations and terminologies will be explained later.

Theorem 1.1. *Let $H = \{\lambda_k\}_{k=1}^{\infty} \subset \mathbb{R}$ be a set of different numbers such that $0 \notin \overline{H}$, the closure of H, and $AP_H(\mathbb{R},\mathbb{C})$ be the space of almost periodic functions f with $\Lambda_f \subset H$. Then the integral operator*

$$\begin{gathered}\mathcal{I}\colon AP_H(\mathbb{R},\mathbb{C}) \to AP_H(\mathbb{R},\mathbb{C}),\\ f(t) \mapsto \int_0^t f(s)\,ds - \mathfrak{M}\Big\{\int_0^t f(s)\,ds\Big\}\end{gathered} \tag{1.1}$$

is compact if and only if H has no limit point.

Theorem 1.2. *Consider complex differential equations of the form*

$$z' = \alpha z + \psi(z,t) + \varphi(t). \tag{1.2}$$

Let the following conditions hold:

(A1) $\alpha \in \mathbb{C}$, $\alpha \neq 0$.

(A2) $\psi(z,t)\colon \mathbb{C}\times\mathbb{R} \to \mathbb{C}$ *is almost periodic in t uniformly for compact subsets of $\mathbb{C}$. $\Lambda_\psi \subset [0,\infty)$ has no limit point.*

(A3) $\varphi \in AP(\mathbb{R},\mathbb{C})$ *and $\Lambda_\varphi \subset [0,\infty)$ has no limit point.*

Let $H = \{\lambda_k\}_{k=1}^{\infty} \subset (0,\infty)$ be the semigroup generated by $(\Lambda_\psi \cup \Lambda_\varphi)\setminus\{0\}$, and $\delta > 0$ be a number such that $\lambda_k \geq \delta$ for each $k \in \mathbb{Z}_+$. Assume further that there exists $R > 0$ with $\psi(z,t)$ being analytic in z for $|z| \leq R$ and

$$\sup_{|z|\leq R,\, t\in\mathbb{R}} |\psi(z,t)| + \|\varphi\| < \frac{1 - \dfrac{\beta}{\delta}|\alpha|}{\dfrac{1}{|\alpha|} + \dfrac{\beta}{\delta}} \cdot R,$$

where $\beta > 0$ is an absolute constant given by Theorem 2.7. *Then equation* (1.2) *has at least one solution in $AP^1_{H_0}(\mathbb{R},\mathbb{C})$, where $H_0 = H \cup \{0\}$.*

Theorem 1.1 remains true for integral operators on both the space $AP_1(\mathbb{R},\mathbb{C})$ of almost periodic functions with absolutely convergent Fourier series and the space $B^2(\mathbb{R},\mathbb{C})$ of almost periodic functions in the sense of Besicovitch, but with different proofs (Theorems 3.7 and 3.9). The semigroup condition for H in Theorem 1.2 is crucial for both the compactness of an integral operator and the definition of a nonlinear operator. Compared with fixed point methods for almost periodic differential equations of [6], [8], [12], [14], the applicability of coincidence degree theory depends much upon the obtention of a priori bound for the solutions to the equations. If $\operatorname{Re}\alpha = 0$, equation (1.2) does not possess an exponential dichotomy and the non-resonance condition in [6], [8], [12], [14] may fail, so those fixed point methods will not work. Moreover, in view of Lemma 2.9 ([28, Proposition 3.4]) the constant α in equation (1.2) may not be easily replaced by the almost periodic function $\alpha(t)$. This reveals somewhat the difficulties that one may find when working on the existence theorems for almost periodic solutions.

We organize this paper as follows. Section 2 introduces some basic notations, terminologies and known results. In Section 3 we prove Theorem 1.1. In Section 4 we study semigroups and the composition of almost periodic functions to define nonlinear operators. In Section 5 we construct suitable real function spaces for applying coincidence degree theory to complex equations. In Section 6 we prove Theorem 1.2.

2. Preliminaries

In this section, we recall some basic knowledge that will be used in this paper. For more details, see e.g. [3], [9], [13], [15], [18].

Definition 2.1 ([18, p. 1]). A continuous function $f\colon \mathbb{R} \to \mathbb{C}$ is called almost periodic (in the sense of Bohr) if for each $\varepsilon > 0$, the ε-translation set (or ε-almost periodic set) of f,

$$T(f,\varepsilon) := \{\tau \in \mathbb{R} : |f(t+\tau) - f(t)| < \varepsilon \text{ for all } t \in \mathbb{R}\}$$

is relatively dense, that is, there is a number $l = l(\varepsilon) > 0$ such that $[a, a+l] \cap T(f,\varepsilon) \neq \emptyset$ for every $a \in \mathbb{R}$. In this case, l is called the inclusion length for $T(f,\varepsilon)$. Members of $T(f,\varepsilon)$ are called ε-translation numbers (ε-almost periods) of f.

Denote by $AP(\mathbb{R},\mathbb{C})$ the Banach space [13, p. 5] of complex almost periodic functions with uniform convergence norm $\|f\| = \sup\limits_{t\in\mathbb{R}} |f(t)|$. For every $f \in AP(\mathbb{R},\mathbb{C})$, the mean value

$$\mathfrak{M}\{f\} := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\,dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T+s}^{T+s} f(t)\,dt$$

exists uniformly with respect to $s \in \mathbb{R}$. Denote by $\Lambda_f = \{\lambda_k\}$ the set of all real numbers such that

$$a(f,\lambda) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t) e^{-i\lambda t}\,dt \neq 0.$$

Then Λ_f is countable, which is called the spectrum of f. If $a_k = a(f, \lambda_k)$, we associate the Fourier series

$$f(t) \sim \sum_k a_k e^{i\lambda_k t}. \tag{2.1}$$

The elements $a_k \in \mathbb{C}$ are called the Fourier coefficients and the numbers λ_k the Fourier exponents of f.

Theorem 2.2 (Approximation theorem [18, p. 17]). *For every $f \in AP(\mathbb{R}, \mathbb{C})$ and every $\varepsilon > 0$ there is a trigonometric polynomial*

$$P_\varepsilon(t) = \sum_{k=1}^{N_\varepsilon} b_{k,\varepsilon} e^{i\lambda_{k,\varepsilon} t}, \quad b_{k,\varepsilon} \in \mathbb{C},\ \lambda_{k,\varepsilon} \in \Lambda_f,$$

such that $\|P_\varepsilon - f\| < \varepsilon$.

The module of f, denoted by $\operatorname{mod}(f)$, is defined to be the additive group

$$\operatorname{mod}(f) := \bigg\{ \sum_{k=1}^{n} m_k \lambda_k : \lambda_k \in \Lambda_f,\ m_k \in \mathbb{Z},\ n \in \mathbb{Z}_+ \bigg\}.$$

An additive semigroup [17, p. 24] $G \subset \mathbb{R}$ is a set of real numbers such that $a + b \in G$ for all $a, b \in G$. The semi-module of f, denoted by $\operatorname{smod}(f)$, is defined to be the additive semigroup

$$\operatorname{smod}(f) := \bigg\{ \sum_{k=1}^{n} m_k \lambda_k : \lambda_k \in \Lambda_f,\, m_k \in \mathbb{N},\, \sum_{k=1}^{n} m_k \neq 0,\, n \in \mathbb{Z}_+ \bigg\},$$

such that its members do not necessarily have an inverse.

The next property gives a condition for the compactness of a set in $AP(\mathbb{R}, \mathbb{C})$.

Theorem 2.3 (Lusternik [13, p. 21], [18, p. 7]). *A set $E \subset AP(\mathbb{R}, \mathbb{C})$ is compact if and only if the following conditions are satisfied*:

(a) *For every fixed $t_0 \in \mathbb{R}$ the set $E(t_0) = \{f(t_0) \in \mathbb{C} : f \in E\}$ is compact.*
(b) *The set E is equicontinuous, that is, for every $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that $|f(t') - f(t'')| < \varepsilon$ whenever $|t' - t''| < \delta$ for all $f \in E$.*
(c) *The set E is equi-almost periodic, that is, for every $\varepsilon > 0$ the set*

$$T(E, \varepsilon) = \bigcap_{f \in E} T(f, \varepsilon)$$

is relatively dense.

Remark 2.4. A family $\mathfrak{F}$ of almost periodic functions is a uniformly almost periodic family (u.a.p. family [13, p. 17]) if it is uniformly bounded, and if given $\varepsilon > 0$, then $T(\mathfrak{F}, \varepsilon) = \bigcap_{f \in \mathfrak{F}} T(f, \varepsilon)$ is relatively dense and includes an interval about 0. It is easy to check that the family $\mathfrak{F}$ is equicontinuous if and only if $T(\mathfrak{F}, \varepsilon)$ includes an interval about 0 for each $\varepsilon > 0$. Consequently, $\mathfrak{F}$ is a u.a.p.

family if and only if $\mathfrak{F} \subset AP(\mathbb{R}, \mathbb{C})$ is relatively compact. This remark will also be referenced when considering issues related to relative compactness.

The following property of u.a.p. families is useful for determining the compactness of integral operators.

THEOREM 2.5 ([12], [13, p. 70]). *Let $\mathfrak{F}$ be a family of almost periodic functions, whose exponents all lie in a given countable set H with no limit point. If there is K so that $|f(t) - f(s)| \le K|t - s|$ for all $t, s \in \mathbb{R}$ and $f \in \mathfrak{F}$, and if there is M such that $\|f\| \le M$ for all $f \in \mathfrak{F}$, then $\mathfrak{F}$ is a u.a.p. family. Conversely, if $\mathfrak{F}$ is the family of all almost periodic functions such that $\|f\| \le M$ and $|f(t) - f(s)| \le K|t - s|$ with $\Lambda_f \subset H$, then $\mathfrak{F}$ is u.a.p. only if H has no limit point.*

We are interested in the case when the primitive $\int_0^t f(s)\,ds$ of an almost periodic f is also almost periodic. There indeed exist almost periodic functions whose primitives are unbounded on $\mathbb{R}$.

LEMMA 2.6 ([28]). *Assume that $G \subset \mathbb{R}$ is a group which is not cyclic, then there exists $f \in AP(\mathbb{R}, \mathbb{C})$ with $\operatorname{mod}(f) \subset G$ such that its primitives F satisfy*

$$F(t) \to \infty \quad \text{as } |t| \to \infty.$$

The following theorem gives a simple condition under which the integral operator $\mathcal{I}$ defined by (1.1) is linear and bounded on an almost periodic function space.

THEOREM 2.7 ([13, p. 74]). *Suppose that $f \in AP(\mathbb{R}, \mathbb{C})$, $f(t) \sim \sum_k a_k e^{i\lambda_k t}$, where $|\lambda_k| \ge \delta > 0$. Then $\int_0^t f(s)\,ds$ is in $AP(\mathbb{R}, \mathbb{C})$ and if g is the integral of f with $a(g, 0) = 0$, then $\|g\| \le \beta \|f\|/\delta$, where $\beta > 0$ is an absolute constant which depends only on δ.*

To define a nonlinear operator on an almost periodic function space, we need the following semi-module containment theorem.

THEOREM 2.8 ([8]). *Suppose that $f(z, t)$ is almost periodic in t uniformly for $z \in \mathbb{C}$, $|z| \le r$, and analytic in z for $|z| \le r$. Then, for every $\varphi \in AP(\mathbb{R}, \mathbb{C})$ with $\Lambda_\varphi \subset \operatorname{smod}(f)$ and $\|\varphi\| \le r$, one has $f(\varphi(\cdot), \cdot) \in AP(\mathbb{R}, \mathbb{C})$ and $\Lambda_{f(\varphi(\cdot), \cdot)} \subset \operatorname{smod}(f)$.*

The difficulties in establishing the existence theorems for almost periodic solutions can be seen from the following result, which shows the problem in a wider perspective.

LEMMA 2.9 ([28]). *Assume that $G \subset \mathbb{R}$ is a group which is not cyclic, then there exist $a, b \in AP(\mathbb{R}, \mathbb{C})$ with $[\mathrm{mod}(a) \cup \mathrm{mod}(b)] \subset G$ such that for the linear equation*

$$x' = a(t)x + b(t)$$

all the solutions are bounded but none of them is almost periodic.

Real almost periodic functions can be defined in the same way and have the same properties as the complex ones. We introduce another two types of almost periodic functions. They have the same Fourier series theory as Bohr almost periodic functions.

Let $AP_1(\mathbb{R}, \mathbb{C}) \subset AP(\mathbb{R}, \mathbb{C})$ be the Banach space [9, p. 31] of Bohr almost periodic functions with absolutely convergent Fourier series

$$AP_1(\mathbb{R}, \mathbb{C}) := \Big\{ f(t) = \sum_k a_k e^{i\lambda_k t} : \mathbb{R} \to \mathbb{C} : a_k \in \mathbb{C},\ \lambda_k \in \mathbb{R} \text{ and } \sum_k |a_k| < \infty \Big\}$$

equipped with the norm

$$\|f\|_1 := \sum_k |a_k|.$$

The space $AP_1(\mathbb{R}, \mathbb{C})$ is a Banach algebra with the operation of multiplication being the usual point-wise multiplication.

Denote by $TP(\mathbb{R}, \mathbb{C})$ the set of all trigonometric polynomials,

$$TP(\mathbb{R}, \mathbb{C}) = \Big\{ P(t) = \sum_{k=1}^{n} a_k e^{i\lambda_k t} : \mathbb{R} \to \mathbb{C} : a_k \in \mathbb{C},\ \lambda_k \in \mathbb{R} \\ \text{for } k = 1, \ldots, n \text{ and } n \in \mathbb{Z}_+ \Big\}. \tag{2.2}$$

A metric on $TP(\mathbb{R}, \mathbb{C})$ is given by

$$d_1(P, Q) = \sum_{\lambda \in \Lambda_P \cup \Lambda_Q} |a(P, \lambda) - a(Q, \lambda)|,$$

where $P, Q \in TP(\mathbb{R}, \mathbb{C})$. The completion of the space $(TP(\mathbb{R}, \mathbb{C}), d_1)$ is exactly the space $AP_1(\mathbb{R}, \mathbb{C})$ with the metric induced by $\| \cdot \|_1$ [9, p. 18]. Let

$$\mathcal{M} = \Big\{ f \in L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}) : \limsup_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 dt < \infty \Big\}$$

be a linear space with the semi-norm

$$\|f\|_{\mathcal{M}} = \Big[\limsup_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 dt \Big]^{1/2}. \tag{2.3}$$

If $K = \{ f \in \mathcal{M} : \|f\|_{\mathcal{M}} = 0 \}$, then the quotient space $\mathcal{M}/K =: \mathcal{M}_2(\mathbb{R}, \mathbb{C})$, called a Marcinkiewicz function space, is complete with respect to norm (2.3) (see [9, p. 45]).

Definition 2.10 ([9, p. 46]). The space $B^2(\mathbb{R},\mathbb{C})$ of almost periodic functions in the sense of Besicovitch is the closure of the linear manifold $TP(\mathbb{R},\mathbb{C})+K$ in the space $(\mathcal{M}_2(\mathbb{R},\mathbb{C}),\|\cdot\|_{\mathcal{M}})$, where $TP(\mathbb{R},\mathbb{C})$ stands for the set of trigonometric polynomials given by (2.2).

Two functions from the same equivalence class of $B^2(\mathbb{R},\mathbb{C})$ may differ at a set of points even of infinite measure [3, p. 74]. Among various almost periodic function spaces, e.g. $AP_1(\mathbb{R},\mathbb{C})$, $AP(\mathbb{R},\mathbb{C})$, $B^2(\mathbb{R},\mathbb{C})$, etc., $B^2(\mathbb{R},\mathbb{C})$ is the largest for which the Parseval equality holds [10], i.e.

$$\mathfrak{M}\{|f|^2\}=\sum_k |a_k|^2, \tag{2.4}$$

where f takes the form of (2.1). Moreover, there holds the following important result.

Theorem 2.11 (Riesz–Fisher–Besicovitch [3, p. 110]). *To any series* $\sum_k a_k e^{i\lambda_k t}$ *for which* $\sum_k |a_k|^2$ *converges, there corresponds a function from* $B^2(\mathbb{R},\mathbb{C})$ *having this series as its Fourier series.*

Following Vo–Khac (see [4]), given $f\in B^2(\mathbb{R},\mathbb{C})$, denote by ∇f the limit (if it exists) in $B^2(\mathbb{R},\mathbb{C})$ of the quotients $(f(\,\cdot\,+r)-f(\,\cdot\,))/r$ when $r\to 0, r\neq 0$. In this case, the simple relation $a(\nabla f,\lambda)=i\lambda a(f,\lambda)$ holds for all $\lambda\in\mathbb{R}$.

The following result which is closely related to the compactness of integral operators on $AP_1(\mathbb{R},\mathbb{C})$ and $B^2(\mathbb{R},\mathbb{C})$ is a special case of [11, Proposition 7.4].

Lemma 2.12. *A set $E\subset l^p(\mathbb{C})$, $p\in[1,\infty)$, is relatively compact if and only if E is bounded and equi-convergent, i.e. for each $\varepsilon>0$ there is an $N(\varepsilon)\in\mathbb{Z}_+$ such that*

$$\sum_{k=N(\varepsilon)+1}^{\infty}|x_k|^p<\varepsilon \quad \textit{for all } x=(x_1,x_2,\ldots)\in E.$$

At last, we would like to recall the basics of coincidence degree theory. Let Y and Z be real Banach spaces, $\mathcal{L}\colon \operatorname{dom}\mathcal{L}\subset Y\to Z$ be a linear operator and $\mathcal{N}\colon Y\to Z$ be a continuous operator. $\mathcal{L}$ is called a Fredholm operator of index zero if $\dim\ker\mathcal{L}=\operatorname{codim}\operatorname{ran}\mathcal{L}<\infty$ and $\operatorname{ran}\mathcal{L}$ is closed in Z. In this case, there exist continuous projectors $\mathcal{P}\colon Y\to Y$ and $\mathcal{Q}\colon Z\to Z$ such that $\operatorname{ran}\mathcal{P}=\ker\mathcal{L}$ and $\operatorname{ran}\mathcal{L}=\ker\mathcal{Q}=\operatorname{ran}(\mathrm{id}_Z-\mathcal{Q})$. The operator $\mathcal{L}|_{\operatorname{dom}\mathcal{L}\cap\ker\mathcal{P}}\colon(\mathrm{id}_Y-\mathcal{P})\operatorname{dom}\mathcal{L}\to\operatorname{ran}\mathcal{L}$ is invertible. Denote by $\mathcal{K}_{\mathcal{P}}$ the inverse of $\mathcal{L}|_{\operatorname{dom}\mathcal{L}\cap\ker\mathcal{P}}$. If Ω is an open bounded subset of Y, then $\mathcal{N}$ is called $\mathcal{L}$-compact on $\overline{\Omega}$ if $\mathcal{Q}\mathcal{N}(\overline{\Omega})$ is bounded and $\mathcal{K}_{\mathcal{P}}(\mathrm{id}_Z-\mathcal{Q})\mathcal{N}\colon\overline{\Omega}\to Y$ is compact. Denote by $\mathcal{J}$ an isomorphism from $\operatorname{ran}\mathcal{Q}$ to $\ker\mathcal{L}$.

Theorem 2.13 (Continuation Theorem [15, p. 40]). *Suppose that $\Omega\subset Y$ is an open bounded set, $\mathcal{L}$ is a Fredholm operator of index zero and $\mathcal{N}$ is $\mathcal{L}$-compact on $\overline{\Omega}$. Let the following conditions hold*:

(C1) $\mathcal{L}y \neq \mu\mathcal{N}y$ *for each* $y \in \partial\Omega \cap \operatorname{dom}\mathcal{L}$ *and* $\mu \in (0,1)$.
(C2) $\mathcal{Q}\mathcal{N}y \neq 0$ *for each* $y \in \partial\Omega \cap \ker\mathcal{L}$.
(C3) *The Brouwer degree* $\deg(\mathcal{J}\mathcal{Q}\mathcal{N}, \Omega \cap \ker\mathcal{L}, 0) \neq 0$.

Then the equation $\mathcal{L}y = \mathcal{N}y$ *has at least one solution in* $\overline{\Omega} \cap \operatorname{dom}\mathcal{L}$.

3. Compactness of integral operators

3.1. Integral operators on Bohr almost periodic function spaces. Coincidence degree theory obtains its success in studying alternative problems, which can be written as operator equations

$$\mathcal{L}x = \mathcal{N}x$$

with an appropriate priori estimate structure (see [15] for details). For first-order ordinary differential equations of the form

$$x' = \psi(x,t),$$

with ψ being a continuous function, $\mathcal{L}$ is the usual differential operator $x \mapsto x'$, and $\mathcal{K}_{\mathcal{P}}$ is nothing but an integral operator. In many cases, the nonlinear operator $\mathcal{N} \colon x(\,\cdot\,) \mapsto \psi(x(\,\cdot\,), \,\cdot\,)$ is not compact (Example 4.8) and $\mathcal{Q}$ is finite-dimensional. So, it is natural to study $\mathcal{K}_{\mathcal{P}}$ to obtain the $\mathcal{L}$-compactness of $\mathcal{N}$.

Generally speaking, an integral operator is not compact on $AP(\mathbb{R},\mathbb{C})$. However, this may happen on a subspace of $AP(\mathbb{R},\mathbb{C})$. We first study conditions for an integral operator to be well defined on a subspace of $AP(\mathbb{R},\mathbb{C})$.

Fix an infinitely countable set of different numbers $H = \{\lambda_k\}_{k=1}^{\infty} \subset \mathbb{R}$. A frequently used complete subspace of $AP(\mathbb{R},\mathbb{C})$ is the following:

$$AP_H(\mathbb{R},\mathbb{C}) := \{f \in AP(\mathbb{R},\mathbb{C}) : \Lambda_f \subset H\}. \tag{3.1}$$

The example in the proof of Lemma 2.6 requires the denseness of the group G in $\mathbb{R}$. We provide here a different one with less restrictions.

Lemma 3.1. *If* $0 \in \overline{H}$, *the closure of* H, *then there exists* $f \in AP_H(\mathbb{R},\mathbb{C})$ *such that its primitive*

$$\int_0^t f(s)\,ds \notin AP(\mathbb{R},\mathbb{C}).$$

Proof. If 0 is an isolated point of H, it is obvious that $f(t) \equiv 1 \in AP_H(\mathbb{R},\mathbb{C})$ but $\int_0^t f(s)\,ds = t \notin AP(\mathbb{R},\mathbb{C})$.

If 0 is a limit point of H, there is a sequence of different numbers $\{\mu_j\}_{j=1}^{\infty}$ in $H \setminus \{0\}$ which converges to 0. Let $\{a_k\}_{k=1}^{\infty}$ be an absolutely convergent sequence in $\mathbb{C}\setminus\{0\}$, i.e. $\sum\limits_{k=1}^{\infty} |a_k| < \infty$. For a_1 there is μ_{j_1} in $\{\mu_j\}_{j=1}^{\infty}$ such that $|a_1/\mu_{j_1}| > 1$, for a_2 there is $\mu_{j_2} \neq \mu_{j_1}$ in $\{\mu_j\}_{j=1}^{\infty}$ such that $|a_2/\mu_{j_2}| > 1$, … In this way, we obtain a subsequence $\{\mu_{j_k}\}_{k=1}^\infty$ of $\{\mu_j\}_{j=1}^\infty$ such that $|a_k/\mu_{j_k}| > 1$ for all $k \in \mathbb{Z}_+$. It is obvious that the function

$$f(t) = \sum_{k=1}^\infty a_k e^{i\mu_{j_k} t}, \quad t \in \mathbb{R},$$

is in $AP_H(\mathbb{R}, \mathbb{C})$.

Denote by $F(t) = \int_0^t f(s)\,ds$ a primitive of f and suppose that $F \in AP(\mathbb{R}, \mathbb{C})$. Since $a(f, \lambda) = i\lambda a(F, \lambda)$ for all $\lambda \in \mathbb{R}$, it follows that

$$F(t) \sim a(F, 0) + \sum_{k=1}^\infty \frac{a_k}{i\mu_{j_k}}\, e^{i\mu_{j_k} t}.$$

Therefore, the Fourier coefficients of F are not square summable since $|a_k/\mu_{j_k}| > 1$ for all $k \in \mathbb{Z}_+$, which contradicts the Parseval equality (2.4). $\square$

From Theorem 2.7 and Lemma 3.1 it follows that the integral operator $\mathcal{I}$ defined by (1.1) maps the space $AP_H(\mathbb{R}, \mathbb{C})$ to $AP_H(\mathbb{R}, \mathbb{C})$ if and only if $0 \notin \overline{H}$. In that case, $\mathcal{I}$ is linear and bounded.

Lemma 3.2. *Suppose that $\{\mu_k\}_{k=1}^\infty \subset \mathbb{R}$ is a set of different numbers, then the following statements hold for the set of infinitely many pure oscillations $E = \{e^{i\mu_k t}\}_{k=1}^\infty \subset AP(\mathbb{R}, \mathbb{C})$:*

(a) *There exists no Cauchy sequence in the set E.*
(b) *The set E is equicontinuous if and only if the sequence $\{\mu_k\}_{k=1}^\infty$ is bounded.*
(c) *The set E is equi-almost periodic only if the sequence $\{\mu_k\}_{k=1}^\infty$ has no limit point.*

Proof. (a) A direct calculation shows that

$$\sup_{t\in\mathbb{R}} |e^{i\mu_j t} - e^{i\mu_k t}| = \sup_{t\in\mathbb{R}} \big|[e^{i(\mu_j - \mu_k)t} - 1] \cdot e^{i\mu_k t}\big| = 2$$

for $j \neq k$. Consequently, the set E contains no Cauchy sequence.

(b) Sufficiency. Let the sequence $\{\mu_k\}_{k=1}^\infty$ be bounded by a constant $M > 0$. Choose $\varepsilon > 0$ so small that there is a unique $\theta \in (0, \pi)$ satisfying $|e^{i\theta} - 1| = \varepsilon$, and $\delta > 0$ so small that $M\delta < \theta$. It follows that

$$|e^{i\mu_k s} - e^{i\mu_k t}| = \big|[e^{i\mu_k (s-t)} - 1] \cdot e^{i\mu_k t}\big| < |e^{iM\delta} - 1| < \varepsilon$$

for all $s, t \in \mathbb{R}$, $|s - t| < \delta$, and $k \in \mathbb{Z}_+$. Hence the set E is equicontinuous.

Necessity. Let the set E be equicontinuous. Without loss of generality we may assume the contrary that $\lim_{k\to\infty} \mu_k = \infty$. Given a sufficiently small constant $\varepsilon_0 > 0$, there is δ_0, $0 < \delta_0 < \pi$, such that

$$|e^{it} - 1| > 2 - \varepsilon_0 \tag{3.2}$$

whenever $|t-\pi| < \delta_0$. It is obvious that the continuous function $g(t) = \pi/t$ maps $(0, \delta)$ onto $(\pi/\delta, \infty)$, where $0 < \delta < \delta_0$. So, for each $m \in \mathbb{Z}$, $m > \pi/\delta + 1$, there exists $t \in (0, \delta)$ such that $m \le \pi/t < m + 1$. Furthermore, for each sufficiently large $k \in \mathbb{Z}_+$ there exists a unique $m_k \in \mathbb{Z}_+$ such that

$$\frac{\pi}{\delta} + 1 < m_k \le \mu_k < m_k + 1.$$

Let $t_k \in (0, \delta)$ be such that

$$m_k \le \frac{\pi}{t_k} < m_k + 1.$$

Then $|\mu_k t_k - \pi| < t_k < \delta$ and $|e^{i\mu_k t_k} - 1| > 2 - \varepsilon_0$ by (3.2). Consequently, for each $\delta > 0$ and sufficiently large $k \in \mathbb{Z}_+$ there exists $t_k \in (0, \delta)$ such that

$$|e^{i\mu_k(t+t_k)} - e^{i\mu_k t}| > 2 - \varepsilon_0$$

for all $t \in \mathbb{R}$. This contradicts the equicontinuity of the set E.

(c) Let the set E be equi-almost periodic. Without loss of generality we may assume the contrary that the sequence $\{\mu_k\}_{k=1}^{\infty}$ is bounded. By (b) the set E is equicontinuous. Hence for every $\varepsilon > 0$ the set $T(E, \varepsilon)$ is relatively dense and includes an interval about 0. Therefore, the set E is relatively compact by Remark 2.4. This contradicts (a). □

Now we are in the position to prove Theorem 1.1.

Proof of Theorem 1.1. Sufficiency. Let $\delta > 0$ be a number such that $|\lambda_k| \ge \delta$ for all $k \in \mathbb{Z}_+$, $E \subset AP_H(\mathbb{R}, \mathbb{C})$ be a set bounded by a constant $M > 0$. From Theorem 2.7 it follows that the set $\mathcal{I}(E)$ is bounded by $\beta M/\delta$. By the property of integrals one gets

$$|\mathcal{I}(f)(t) - \mathcal{I}(f)(s)| = \left| \int_0^t f(\tau)\, d\tau - \int_0^s f(\tau)\, d\tau \right| \le M|t-s|$$

for all $t, s \in \mathbb{R}$ and $f \in E$. Since $a(f, \lambda) = i\lambda a(F, \lambda)$ for all $\lambda \in \mathbb{R}$, it follows that $\Lambda_{\mathcal{I}(f)} = \Lambda_f \subset H$ for all $f \in E$. Thus $\mathcal{I}(E)$ is u.a.p. by Theorem 2.5 and relatively compact by Remark 2.4.

Necessity. Let the set H have a limit point μ and $\{\mu_k\}_{k=1}^{\infty}$ be a sequence of different numbers in H which converges to μ. Then the set $\{i\mu_k \cdot e^{i\mu_k t}\}_{k=1}^{\infty}$ is bounded and its image $\mathcal{I}(\{i\mu_k \cdot e^{i\mu_k t}\}_{k=1}^{\infty}) = \{e^{i\mu_k t}\}_{k=1}^{\infty}$ is not relatively compact by Lemma 3.2 (a). Hence $\mathcal{I}$ is not compact on $AP_H(\mathbb{R}, \mathbb{C})$. □

Remark 3.3. The following are another three different proofs for the necessity of Theorem 1.1. Let μ be a limit point of H and $\{\mu_k\}_{k=1}^{\infty}$ be a sequence of different numbers in $H_\varepsilon := (\mu - \varepsilon, \mu + \varepsilon) \cap H$ converging to μ, where $0 < \varepsilon < |\mu|$.

(a) Since $\mathcal{I}(e^{i\lambda t}) = e^{i\lambda t}/i\lambda$, $1/i\lambda$ is an eigenvalue of $\mathcal{I}$ for each $\lambda \in H_\varepsilon$. Thus the spectrum $\sigma(\mathcal{I})$ of $\mathcal{I}$ contains $\{1/i\mu_k\}_{k=1}^{\infty}$ and has $1/i\mu$ as a limit point. Since the spectrum of a compact operator can have at most 0 as its limit point [30, p. 108], $\mathcal{I}$ is not compact on $AP_H(\mathbb{R},\mathbb{C})$.

(b) If $AP^1_{H_\varepsilon}(\mathbb{R},\mathbb{C}) = \{f \in AP(\mathbb{R},\mathbb{C}) : f'$ exists and is almost periodic and $\Lambda_f \subset H_\varepsilon\}$, then $f = \mathcal{I}(f')$ for all $f \in AP^1_{H_\varepsilon}(\mathbb{R},\mathbb{C})$. Furthermore, a theorem of Bochner asserts that $AP_{H_\varepsilon}(\mathbb{R},\mathbb{C}) = AP^1_{H_\varepsilon}(\mathbb{R},\mathbb{C})$ and there is a constant $C(\varepsilon) > 0$ such that $\|f'\| \leq C(\varepsilon)\|f\|$ for all $f \in AP_{H_\varepsilon}(\mathbb{R},\mathbb{C})$ [13, p. 67]. Therefore, the inverse image $\mathcal{I}^{-1}(B_1)$ of the unit ball B_1 in the infinite dimensional space $AP_{H_\varepsilon}(\mathbb{R},\mathbb{C})$ is a set bounded by $C(\varepsilon)$. The non-compactness of B_1 (by Riesz's theorem) shows that $\mathcal{I}$ is not compact on $AP_H(\mathbb{R},\mathbb{C})$.

(c) Notice that the set $\{i\mu_k \cdot e^{i\mu_k t}\}_{k=1}^{\infty}$ is bounded and its image $\mathcal{I}(\{i\mu_k \cdot e^{i\mu_k t}\}_{k=1}^{\infty}) = \{e^{i\mu_k t}\}_{k=1}^{\infty}$ is not equi-almost periodic by Lemma 3.2 (c). So $\mathcal{I}$ is not compact on $AP_H(\mathbb{R},\mathbb{C})$.

Corollary 3.4. *Let $\mathcal{N}\colon AP_H(\mathbb{R},\mathbb{C}) \to AP_H(\mathbb{R},\mathbb{C})$ be a nonlinear, continuous and bounded operator with H having no limit point. Then the integral operator $\mathcal{I}_{\mathcal{N}}$ defined by*

$$\begin{aligned} &\mathcal{I}_{\mathcal{N}}\colon AP_H(\mathbb{R},\mathbb{C}) \to AP_H(\mathbb{R},\mathbb{C}),\\ &f(t) \mapsto \int_0^t \mathcal{N}(f)(s)\,ds - \mathfrak{M}\bigg\{\int_0^t \mathcal{N}(f)(s)\,ds\bigg\}, \end{aligned} \tag{3.3}$$

is compact on $AP_H(\mathbb{R},\mathbb{C})$.

One may ask whether Corollary 3.4 is true for those nonlinear operators $\mathcal{N}$ defined by basic elementary functions, such as $f(t) \mapsto a(t)[f(t)]^2 + b(t)f(t) + c(t)$ and $f(t) \mapsto e^{f(t)}$ et al. Thus we need to consider the definition of a nonlinear operator on $AP_H(\mathbb{R},\mathbb{C})$. We are especially interested in the case when $AP_H(\mathbb{R},\mathbb{C})$ is a Banach algebra. The closedness of the space $AP_H(\mathbb{R},\mathbb{C})$ with respect to the operation of (pointwise) multiplication will be discussed in Section 4.

3.2. Integral operators on the spaces of almost periodic functions with absolutely convergent Fourier series. Theorem 1.1 remains true for integral operators on both the space $AP_1(\mathbb{R},\mathbb{C})$ and the space $B^2(\mathbb{R},\mathbb{C})$. The proofs are based on the natural isometric isomorphisms from the almost periodic function spaces to the Banach spaces $l^p(\mathbb{C})$ of complex sequences. So, it is of interest to provide detailed proofs.

Let $H = \{\lambda_k\}_{k=1}^{\infty} \subset \mathbb{R}$ be a set of different numbers and

$$AP_{1,H}(\mathbb{R},\mathbb{C}) = \{f \in AP_1(\mathbb{R},\mathbb{C}) : \Lambda_f \subset H\}.$$

Each $f \in AP_{1,H}(\mathbb{R},\mathbb{C})$ can be written as

$$f(t) = \sum_{k=1}^{\infty} a_k e^{i\lambda_k t}, \tag{3.4}$$

where all terms with exponents $\lambda_k \in H$ are written out for convenience such that a_k may vanish for some $k \in \mathbb{Z}_+$. Define an operator

$$\mathcal{S}_1 \colon AP_{1,H}(\mathbb{R},\mathbb{C}) \to l^1(\mathbb{C}), \qquad f \mapsto (a_1, a_2, \dots),$$

where $l^1(\mathbb{C})$ is the Banach space of complex sequences equipped with the norm

$$\|(a_1, a_2, \dots)\|_{l^1} = \sum_{k=1}^{\infty} |a_k| < \infty.$$

The following result is obvious by definition.

LEMMA 3.5. *The operator $\mathcal{S}_1 \colon AP_{1,H}(\mathbb{R},\mathbb{C}) \to l^1(\mathbb{C})$ is an isometric isomorphism.*

The following result is obvious by the completeness of $(l^1(\mathbb{C}), \|\cdot\|_{l^1})$ and Lemma 3.5.

LEMMA 3.6. *$(AP_{1,H}(\mathbb{R},\mathbb{C}), \|\cdot\|_1)$ is a Banach space.*

One can also obtain the completeness of $(AP_{1,H}(\mathbb{R},\mathbb{C}), \|\cdot\|_1)$ by showing that it is a closed subspace of $(AP_1(\mathbb{R},\mathbb{C}), \|\cdot\|_1)$. By Lemma 3.5, a set $E \subset AP_{1,H}(\mathbb{R},\mathbb{C})$ is relatively compact if and only if its image $\mathcal{S}_1(E) \subset l^1(\mathbb{C})$ is relatively compact. It follows from Lemma 2.12 that E is relatively compact if and only if E is bounded and the set of Fourier series of functions from E is equi-convergent.

As shown in Lemma 3.1, it is necessary to have $0 \notin \overline{H}$, the closure of H, for an integral operator mapping $AP_{1,H}(\mathbb{R},\mathbb{C})$ to $AP_{1,H}(\mathbb{R},\mathbb{C})$. In this case, let $\delta > 0$ be the number satisfying $|\lambda_k| \ge \delta$ for all $k \in \mathbb{Z}_+$. Given $f \in AP_{1,H}(\mathbb{R},\mathbb{C})$ by (3.4), a primitive of f is given by

$$F(t) := \int_0^t f(s)\,ds - \mathfrak{M}\biggl\{\int_0^t f(s)\,ds\biggr\} = \sum_{k=1}^{\infty} \frac{a_k}{i\lambda_k}\, e^{i\lambda_k t} \tag{3.5}$$

with norm

$$\|F\|_1 = \sum_{k=1}^{\infty} \left|\frac{a_k}{\lambda_k}\right| \le \frac{\|f\|_1}{\delta}. \tag{3.6}$$

It is obvious that $F \in AP_{1,H}(\mathbb{R},\mathbb{C})$ and the following integral operator:

$$\begin{gathered} \mathcal{I}_1 \colon AP_{1,H}(\mathbb{R},\mathbb{C}) \to AP_{1,H}(\mathbb{R},\mathbb{C}), \\ \sum_{k=1}^{\infty} a_k e^{i\lambda_k t} \mapsto \sum_{k=1}^{\infty} \frac{a_k}{i\lambda_k}\, e^{i\lambda_k t}, \end{gathered} \tag{3.7}$$

is well defined, linear and bounded by (3.6).

THEOREM 3.7. *Suppose that $0 \notin \overline{H}$, then the integral operator $\mathcal{I}_1$ defined by (3.7) is compact on $AP_{1,H}(\mathbb{R},\mathbb{C})$ if and only if H has no limit point.*

Proof. Sufficiency. Let H have no limit point, $\delta > 0$ be a number satisfying $|\lambda_k| \geq \delta$ for all $k \in \mathbb{Z}_+$, and $E \subset AP_{1,H}(\mathbb{R},\mathbb{C})$ be an arbitrary set bounded by a constant $M > 0$. Given $f \in E$, by (3.4), $\mathcal{I}_1(f)$ takes the form of (3.5) and $\|\mathcal{I}_1(f)\|_1 \leq M/\delta$ by (3.6). Hence both $\mathcal{I}_1(E)$ and $\mathcal{S}_1(\mathcal{I}_1(E))$ are bounded by Lemma 3.5.

Since H has no (finite) limit point if and only if $\lim_{k\to\infty} |\lambda_k| = \infty$, for each $\varepsilon > 0$ there exists $N > 0$ such that $|\lambda_k| > M/\varepsilon$ for every $k > N$. Therefore,

$$\sum_{k=N+1}^{\infty} \left|\frac{a_k}{i\lambda_k}\right| < \frac{\varepsilon}{M} \sum_{k=N+1}^{\infty} |a_k| \leq \varepsilon,$$

which implies that the set $\mathcal{S}_1(\mathcal{I}_1(E))$ is equi-convergent. So $\mathcal{S}_1(\mathcal{I}_1(E))$ is relatively compact by Lemma 2.12 and $\mathcal{I}_1(E)$ is relatively compact by Lemma 3.5.

Necessity. Assume the contrary that H has a limit point μ. Given $\varepsilon' > 0$, $\varepsilon' < 1/|\mu|$, there is a strictly increasing sequence $\{k_m\}_{m=1}^{\infty} \subset \mathbb{Z}_+$ such that $|1/\lambda_{k_m} - 1/\mu| < \varepsilon'$ for every $m \in \mathbb{Z}_+$. Let $f_m(t) = e^{i\lambda_{k_m}t}$, it follows that $\|f_m\|_1 = 1$ and $\mathcal{I}_1(f_m)(t) = e^{i\lambda_{k_m}t}/i\lambda_{k_m}$. The j-th component of $\mathcal{S}_1(\mathcal{I}_1(f_m))$ is

$$[\mathcal{S}_1(\mathcal{I}_1(f_m))]_j = \begin{cases} 0 & \text{if } j \neq k_m, \\ \dfrac{1}{i\lambda_{k(m)}} & \text{if } j = k_m. \end{cases}$$

Since $k_m > N$ whenever m is sufficiently large, it follows that

$$\sum_{j=N+1}^{\infty} \big|[\mathcal{S}_1(\mathcal{I}_1(f_m))]_j\big| = \frac{1}{|\lambda_{k_m}|} > \frac{1}{|\mu|} - \varepsilon' > 0,$$

which implies that the sequence $\{\mathcal{S}_1(\mathcal{I}_1(f_m))\}_{m=1}^{\infty}$ is not equi-convergent. Therefore, $\{\mathcal{S}_1(\mathcal{I}_1(f_m))\}_{m=1}^{\infty}$ is not relatively compact by Lemma 2.12. Hence the set $\{\mathcal{I}_1(f_m)\}_{m=1}^{\infty}$ is not relatively compact by Lemma 3.5. $\square$

Remark 3.8. One can also prove the necessity of Theorem 3.7 by an eigenvalue argument as in Remark 3.3 (i). Since $\|f\| \leq \|f\|_1$ for each $f \in AP_1(\mathbb{R},\mathbb{C}) \subset AP(\mathbb{R},\mathbb{C})$, a set which is bounded/compact in $(AP_1(\mathbb{R},\mathbb{C}), \|\cdot\|_1)$ must be bounded/compact in $(AP(\mathbb{R},\mathbb{C}), \|\cdot\|)$. However, a set $E \subset AP_1(\mathbb{R},\mathbb{C})$ which is bounded in $(AP(\mathbb{R},\mathbb{C}), \|\cdot\|)$ needs not to be bounded in $(AP_1(\mathbb{R},\mathbb{C}), \|\cdot\|_1)$. So an operator with domain $AP_H(\mathbb{R},\mathbb{C})$ which is compact on $(AP_{1,H}(\mathbb{R},\mathbb{C}), \|\cdot\|_1)$ may not be compact on $(AP_H(\mathbb{R},\mathbb{C}), \|\cdot\|)$. Theorem 3.7 is not a direct consequence of Theorem 1.1.

3.3. Integral operators on Besicovitch spaces. There are two different directions to generalize almost periodic functions. One is further structural generalizations of pure periodicity, by V.V. Stepanov, N. Wiener, H. Weyl, etc. One is to consider the class of limit functions of trigonometric polynomials in a more general sense than uniform convergence by A.S. Besicovitch [3, p. 67].

The Fourier series of Besicovitch almost periodic functions is just the series $\sum_k a_k e^{i\lambda_k t}$ for which $\sum_k |a_k|^2 < \infty$ (see [7]). Due to the similarity of the space $B^2(\mathbb{R},\mathbb{C})$ to $AP_1(\mathbb{R},\mathbb{C})$, we will state conclusions for $B^2(\mathbb{R},\mathbb{C})$ without proof.

Let $H = \{\lambda_k\}_{k=1}^{\infty} \subset \mathbb{R}$ be a set of different numbers, and define

$$B^2_H(\mathbb{R},\mathbb{C}) = \{f \in B^2(\mathbb{R},\mathbb{C}) : \Lambda_f \subset H\}.$$

For each $f \in B^2_H(\mathbb{R},\mathbb{C})$ given by

$$f(t) \sim \sum_{k=1}^{\infty} a_k e^{i\lambda_k t},$$

define an operator

$$\mathcal{S}_2\colon B^2_H(\mathbb{R},\mathbb{C}) \to l^2(\mathbb{C}), \qquad f \mapsto (a_1, a_2, \ldots),$$

where $l^2(\mathbb{C})$ is the Banach space of complex sequences equipped with the norm

$$\|(a_1, a_2, \ldots)\|_{l^2} = \sqrt{\sum_{k=1}^{\infty} |a_k|^2} < \infty.$$

From the Parseval equality (2.4) and the Riesz–Fisher–Besicovitch Theorem, Theorem 2.11, it follows that $\mathcal{S}_2\colon B^2_H(\mathbb{R},\mathbb{C}) \to l^2$ is an isometric isomorphism. Therefore, $B^2_H(\mathbb{R},\mathbb{C})$ is a Banach space and Lemma 2.12 can be used to determine compact sets in $B^2_H(\mathbb{R},\mathbb{C})$.

If $0 \notin \overline{H}$, the closure of H, an inverse operator to

$$\nabla\colon B^2_H(\mathbb{R},\mathbb{C}) \cap \operatorname{dom} \nabla \to B^2_H(\mathbb{R},\mathbb{C}), \qquad f \mapsto \nabla f$$

is given by

$$\begin{gathered} \mathcal{I}_2\colon B^2_H(\mathbb{R},\mathbb{C}) \to B^2_H(\mathbb{R},\mathbb{C}), \\ \sum_{k=1}^{\infty} a_k e^{i\lambda_k t} \mapsto \sum_{k=1}^{\infty} \frac{a_k}{i\lambda_k}\, e^{i\lambda_k t}. \end{gathered} \tag{3.8}$$

The proof of the following theorem is similar to that of Theorem 3.7.

THEOREM 3.9. *Suppose that $0 \notin \overline{H}$, then the operator $\mathcal{I}_2$ defined by* (3.8) *is compact on $B^2_H(\mathbb{R},\mathbb{C})$ if and only if H has no limit point.*

Since a primitive of a function from $\mathcal{M}$ may fail to be in $\mathcal{M}$, we present a new generalization of differential and integral operators to the space $B^2_H(\mathbb{R},\mathbb{C})$. It will be seen that they coincide with ∇ and $\mathcal{I}_2$ on $B^2_H(\mathbb{R},\mathbb{C})$ (Lemma 3.10).

For the set H let $\delta > 0$ be a number satisfying $|\lambda_k| \geq \delta$ for all $k \in \mathbb{Z}_+$. Recall that $\|\cdot\|_{\mathcal{M}}$ is a norm on $AP(\mathbb{R},\mathbb{C})$ [13, p. 36], it follows that the integral operator $\mathcal{I}$ defined by (1.1) is bounded with respect to the norm $\|\cdot\|_{\mathcal{M}}$, and the operator norm of $\mathcal{I}$ satisfies $\|\mathcal{I}\| \leq 1/\delta$. Since $AP_H(\mathbb{R},\mathbb{C}) \cap K = \{0\}$, extend $\mathcal{I}$ to the linear manifold $AP_H(\mathbb{R},\mathbb{C}) + K$ by

$$\begin{aligned}\mathcal{I}_K\colon AP_H(\mathbb{R},\mathbb{C}) + K &\to AP_H(\mathbb{R},\mathbb{C}) + K,\\ f + K &\mapsto \mathcal{I}(f) + K.\end{aligned}$$

It is easy to see that $\mathcal{I}_K$ is a linear and bounded operator on $(AP_H(\mathbb{R},\mathbb{C}) + K, \|\cdot\|_{\mathcal{M}})$. By the denseness of $AP_H(\mathbb{R},\mathbb{C}) + K$ in $B^2_H(\mathbb{R},\mathbb{C})$, there exists a unique continuous extension $\mathcal{I}_B\colon B^2_H(\mathbb{R},\mathbb{C}) \to B^2_H(\mathbb{R},\mathbb{C})$ of $\mathcal{I}_K$ such that $\mathcal{I}_B|_{AP_H(\mathbb{R},\mathbb{C})+K} = \mathcal{I}_K$ and $\|\mathcal{I}_B\| = \|\mathcal{I}_K\|$.

Lemma 3.10. *There holds $\mathcal{I}_B = \mathcal{I}_2$ on $B^2_H(\mathbb{R},\mathbb{C})$, by which the operator $\mathcal{I}_B\colon B^2_H(\mathbb{R},\mathbb{C}) \to \operatorname{ran}\mathcal{I}_B$ is invertible.*

Proof. Since $a(f,\lambda) = i\lambda a(\mathcal{I}(f),\lambda)$ for all $f \in AP_H(\mathbb{R},\mathbb{C})$ and $\lambda \in \mathbb{R}$, one has $a(\widetilde{f},\lambda) = i\lambda a(\mathcal{I}_K(\widetilde{f}),\lambda)$ for all $\widetilde{f} \in AP_H(\mathbb{R},\mathbb{C}) + K$ and $\lambda \in \mathbb{R}$. It is easy to check that the functionals $\{a(\,\cdot\,,\lambda)\colon B^2_H(\mathbb{R},\mathbb{C}) \to \mathbb{C}\}_{\lambda\in\mathbb{R}}$ are linear and uniformly bounded by 1. From the denseness of $AP_H(\mathbb{R},\mathbb{C}) + K$ in $B^2_H(\mathbb{R},\mathbb{C})$ it follows that $a(f,\lambda) = i\lambda a(\mathcal{I}_B(f),\lambda)$ for all $f \in B^2_H(\mathbb{R},\mathbb{C})$ and $\lambda \in \mathbb{R}$. Consequently,

$$\mathcal{I}_B = \mathcal{I}_2 : \sum_{k=1}^{\infty} a_k e^{i\lambda_k t} \mapsto \sum_{k=1}^{\infty} \frac{a_k}{i\lambda_k}\, e^{i\lambda_k t}$$

on $B^2_H(\mathbb{R},\mathbb{C})$. □

Denote by $\mathcal{D}_B\colon \operatorname{ran}\mathcal{I}_B \to B^2_H(\mathbb{R},\mathbb{C})$ the inverse to $\mathcal{I}_B\colon B^2_H(\mathbb{R},\mathbb{C}) \to \operatorname{ran}\mathcal{I}_B$. Define function spaces

$$\begin{aligned}V_{\mathbb{C}} &= \{f\colon \mathbb{R} \to \{c\} : c \in \mathbb{C}\},\\ V_K &= V_{\mathbb{C}} + K,\\ AP^1_H(\mathbb{R},\mathbb{C}) &= \{f \in AP_H(\mathbb{R},\mathbb{C}) : \text{there exists } f' \in AP_H(\mathbb{R},\mathbb{C})\}.\end{aligned}$$

It follows that $B^2_{H_0}(\mathbb{R},\mathbb{C}) = B^2_H(\mathbb{R},\mathbb{C}) \oplus V_K$, where $H_0 = H \cup \{0\}$. Define operators

$$\begin{aligned}\mathcal{D}_0\colon AP^1_H(\mathbb{R},\mathbb{C}) \oplus V_{\mathbb{C}} &\to AP_{H_0}(\mathbb{R},\mathbb{C}), \qquad f + c \mapsto f',\\ \mathcal{D} = \mathcal{D}_0|_{AP^1_H(\mathbb{R},\mathbb{C})}\colon AP^1_H(\mathbb{R},\mathbb{C}) &\to AP_{H_0}(\mathbb{R},\mathbb{C}), \qquad f \mapsto f',\\ \mathcal{D}_{0,K}\colon [AP^1_H(\mathbb{R},\mathbb{C}) + K] \oplus V_K &\to AP_{H_0}(\mathbb{R},\mathbb{C}) + K,\\ f + c + K &\mapsto f' + K,\\ \mathcal{D}_K = \mathcal{D}_{0,K}|_{AP^1_H(\mathbb{R},\mathbb{C})+K}\colon AP^1_H(\mathbb{R},\mathbb{C}) + K &\to AP_{H_0}(\mathbb{R},\mathbb{C}) + K,\\ f + K &\mapsto f' + K,\\ \mathcal{D}_{0,B}\colon \operatorname{ran}\mathcal{I}_B \oplus V_K &\to B^2_{H_0}(\mathbb{R},\mathbb{C}), \qquad \widetilde{f} + \widetilde{c} \mapsto \mathcal{D}_B\widetilde{f}.\end{aligned}$$

It is easy to prove the following result.

Lemma 3.11. *The following statements are true.*

(a) $\mathcal{D}\colon AP^1_H(\mathbb{R},\mathbb{C}) \to AP_H(\mathbb{R},\mathbb{C})$ *is the inverse to* $\mathcal{I}\colon AP_H(\mathbb{R},\mathbb{C}) \to \operatorname{ran}\mathcal{I}$.

(b) $\mathcal{D}_K\colon AP^1_H(\mathbb{R},\mathbb{C})+K \to AP_H(\mathbb{R},\mathbb{C})+K$ *is the inverse to* $\mathcal{I}_K\colon AP_H(\mathbb{R},\mathbb{C}) +K \to \operatorname{ran}\mathcal{I}_K$.

(c) $\mathcal{D}_K = \mathcal{D}_B|_{AP^1_H(\mathbb{R},\mathbb{C})+K} = \mathcal{D}_B|_{\operatorname{ran}\mathcal{I}_K}$.

(d) $\ker\mathcal{D}_{0,B} = V_K$, $\operatorname{ran}\mathcal{D}_{0,B} = \operatorname{ran}\mathcal{D}_B = B^2_H(\mathbb{R},\mathbb{C})$.

(e) $\mathcal{D}_{0,B}\colon \operatorname{ran}\mathcal{I}_B \oplus V_K \to B^2_{H_0}(\mathbb{R},\mathbb{C})$ *is a Fredholm operator of index* 0.

4. Semigroups and nonlinear operators

From now on we will only consider Bohr almost periodic function. Additive semigroups turns out to be the most suitable algebraic object for H when considering the compactness of integral operators on $AP_H(\mathbb{R},\mathbb{C})$ and the closedness of $AP_H(\mathbb{R},\mathbb{C})$ with respect to the operation of multiplication simultaneously.

Lemma 4.1. *The space* $AP_H(\mathbb{R},\mathbb{C})$ *is a Banach subalgebra of* $AP(\mathbb{R},\mathbb{C})$ *if and only if* H *is a semigroup.*

Proof. Sufficiency. Let H be a semigroup, and $f,g \in AP_H(\mathbb{R},\mathbb{C})$ be such that

$$f(t) \sim \sum_{k=1}^{\infty} a_k e^{i\lambda_k t}, \quad g(t) \sim \sum_{k=1}^{\infty} b_k e^{i\lambda_k t}.$$

Theorem 2.2 implies that there exist two sequences of trigonometric polynomials $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ such that

$$p_n(t) = \sum_{k=1}^{N_n} a_{n,k} e^{i\lambda_k t}, \quad \|p_n - f\| < \frac{1}{n},$$

$$q_n(t) = \sum_{k=1}^{N_n} b_{n,k} e^{i\lambda_k t}, \quad \|q_n - g\| < \frac{1}{n}.$$

Consequently, $\Lambda_{p_n q_n} \subset H$ by the definition of a semigroup, and

$$\lim_{n\to\infty} \|p_n q_n - fg\| = 0.$$

Since $a(p_n q_n, \lambda) = 0$ for all $\lambda \notin H$, it follows that $a(fg,\lambda) = 0$ for all $\lambda \notin H$. Therefore, $\Lambda_{fg} \subset H$ and $fg \in AP_H(\mathbb{R},\mathbb{C})$.

On the other hand, it is easy to check that $\|fg\| \le \|f\|\cdot\|g\|$. The completeness of $AP_H(\mathbb{R},\mathbb{C})$ implies that $AP_H(\mathbb{R},\mathbb{C})$ is a Banach subalgebra of $AP(\mathbb{R},\mathbb{C})$.

Necessity. Let $AP_H(\mathbb{R},\mathbb{C})$ be a Banach subalgebra of $AP(\mathbb{R},\mathbb{C})$. Since $e^{i\lambda t}$, $e^{i\mu t}$, $e^{i(\lambda+\mu)t} \in AP_H(\mathbb{R},\mathbb{C})$ for any $\lambda,\mu \in H$, it follows that $\lambda+\mu \in H$. Thus H is a semigroup. $\square$

From Lemma 4.1 we can show that there exists a large number of nonlinear, continuous and bounded operators on $AP_H(\mathbb{R},\mathbb{C})$ defined by power series.

Lemma 4.2. *Suppose that H is a semigroup and $h(z) = \sum_{n=1}^{\infty} C_n z^n$ is a complex power series such that $\sum_{n=1}^{\infty} |C_n| r^n < \infty$ for some $r > 0$. Then $\sum_{n=1}^{\infty} c_n(t)[f(t)]^n$ in $AP_H(\mathbb{R}, \mathbb{C})$ for every f and every sequence $\{c_n(t)\}_{n=1}^{\infty}$ in $AP_H(\mathbb{R}, \mathbb{C})$ satisfying $\|f\| \le r$ and $\|c_n\| \le |C_n|$, where $n \in \mathbb{Z}_+$.*

Proof. Since

$$\left| \sum_{n=1}^{\infty} c_n(t)[f(t)]^n \right| \le \sum_{n=1}^{\infty} |c_n(t)| \cdot |f(t)|^n \le \sum_{n=1}^{\infty} |C_n| r^n < \infty,$$

the function $\sum_{n=1}^{\infty} c_n(t)[f(t)]^n$ is well defined and uniformly convergent on $\mathbb{R}$. By Lemma 4.1 one has $\sum_{n=1}^{m} c_n(t)[f(t)]^n \in AP_H(\mathbb{R}, \mathbb{C})$ for all $m \in \mathbb{Z}_+$. It follows that $\sum_{n=1}^{\infty} c_n(t)[f(t)]^n \in AP_H(\mathbb{R}, \mathbb{C})$. □

Next we use three properties of additive semigroups to illustrate the conditions imposed on the set H for the space $AP_H(\mathbb{R}, \mathbb{C})$.

Lemma 4.3. *An additive subgroup $(G, +) < (\mathbb{R}, +)$ is isomorphic to $(\mathbb{Z}, +)$ if and only if $\gamma := \inf_{x \in G/\{0\}} |x| > 0$. In that case, γ is a generator of G.*

Proof. Necessity. Let $(G, +)$ be isomorphic to $(\mathbb{Z}, +)$. Since $\mathbb{Z} = \langle 1 \rangle$, the cyclic group generated by 1, there is $e \in G$, $e \ne 0$, such that $G = \langle e \rangle$. Thus $\gamma = \inf_{x \in G/\{0\}} |x| = |e| > 0$.

Sufficiency. Let $\gamma = \inf_{x \in G/\{0\}} |x| > 0$. If $|x| \ne \gamma$ for every $x \in G$, there would be a strictly decreasing sequence $\{x_n\}_{n=1}^{\infty} \subset G \cap (0, \infty)$ which converges to γ. Hence there exists $N > 0$ such that $0 < |x_m - x_n| < \gamma$ for all $m, n > N$. This contradicts the definition of γ since G is a group. Therefore, $\gamma \in G$ and $\langle \gamma \rangle \subset G$.

If there exists $y \in G \backslash \langle \gamma \rangle$, there is a unique $n \in \mathbb{Z}$ such that $n\gamma < y < (n+1)\gamma$. However, the inequality

$$0 < \min\{y - n\gamma, (n+1)\gamma - y\} \le \frac{\gamma}{2}$$

contradicts the definition of γ. So $G = \langle \gamma \rangle$. □

Example 4.4. Put

$$AP_{G,\delta}(\mathbb{R}, \mathbb{C}) := \{f \in AP(\mathbb{R}, \mathbb{C}) : \mathrm{mod}(f) \subset G,\ |\lambda| \ge \delta \text{ for all } \lambda \in \Lambda_f\},$$

where G is an additive subgroup of $\mathbb{R}$, and $\delta > 0$ is a constant. By Theorem 1.1 the integral operator $\mathcal{I}$ defined by (1.1) is compact on $AP_{G,\delta}(\mathbb{R}, \mathbb{C})$ if and only if G has no limit point. This leads G to be the cyclic group $\langle \gamma \rangle$ by Lemma 4.3. Therefore, $\mathcal{I}$ is compact on $AP_{G,\delta}(\mathbb{R}, \mathbb{C})$ if and only if $AP_{G,\delta}(\mathbb{R}, \mathbb{C})$ is a class of $2\pi/\gamma$-periodic functions. In this case, $AP_{G,\delta}(\mathbb{R}, \mathbb{C})$ can be viewed as a family of functions defined on the compact interval $[0, 2\pi/\gamma]$ and the classical Arzela–Ascoli theorem is applicable.

The following lemma shows the existence of a semigroup with no limit point.

Lemma 4.5. *Let $B \subset [0,\infty)$ be a set, then B has no limit point if and only if the semigroup generated by B, which is denoted by* $\mathrm{smod}(B)$*, has no limit point.*

Proof. Let $\mathrm{smod}(B)$ have no limit point. It is obvious that B has no limit point since $B \subset \mathrm{smod}(B)$.

To prove the necessity, we use a simple fact that a set $S \subset \mathbb{R}$ has no limit point if and only if for every $L > 0$ the interval $[-L, L]$ contains only finitely many numbers in S. Now suppose that B has no limit point. For every $L > 0$ the set $[-L, L] \cap B = [0, L] \cap B$ consists of only finitely many numbers, say

$$[0, L] \cap B = \{\beta_1 < \ldots < \beta_{m_L}\}.$$

If $\beta_1 > 0$, from the definition of a semigroup it follows that

$$[-L, L] \cap \mathrm{smod}(B) = \left\{ \sum_{k=1}^{m_L} n_k\beta_k : n_k \in \mathbb{N}, \sum_{k=1}^{m_L} n_k\beta_k \le L, \sum_{k=1}^{m_L} n_k \ne 0 \right\}.$$

Notice that the number of vectors $(n_1, \ldots, n_{m_L}) \in \mathbb{N}^{m_L}$ satisfying the inequalities

$$\beta_1 \cdot \sum_{k=1}^{m_L} n_k \le \sum_{k=1}^{m_L} n_k\beta_k \le L$$

is finite. Therefore, $[-L, L] \cap \mathrm{smod}(B)$ contains only finitely many numbers. If $\beta_1 = 0$, it follows that

$$[-L, L] \cap \mathrm{smod}(B) = \left\{ \sum_{k=2}^{m_L} n_k\beta_k : n_k \in \mathbb{N}, \sum_{k=2}^{m_L} n_k\beta_k \le L,\ k = 2, \ldots, m_L \right\}.$$

So, $[-L, L] \cap \mathrm{smod}(B)$ contains only finite numbers, which implies that the semigroup $\mathrm{smod}(B)$ has no limit point. $\square$

The following lemma is somewhat an inverse to Lemma 4.5 in the sense that a semigroup with particular properties must lie entirely on half of the real line.

Lemma 4.6. *A nonzero additive semigroup $G \subset \mathbb{R}$ which has no limit point is not a cyclic group if and only if $\lambda \cdot \mu \ge 0$ for all $\lambda, \mu \in G$.*

Proof. The sufficiency is obvious since a nonzero cyclic group contains numbers of both signs. For the necessity, assume the contrary that there exist $\lambda, \mu \in G$ satisfying $\lambda < 0 < \mu$. Since G has no limit point and $\lambda < 0 < \mu$, there exist two numbers λ_+ and λ_- such that

$$\lambda_+ = \min\{x \in G : x > 0\} > 0, \qquad \lambda_- = \max\{x \in G : x < 0\} < 0.$$

Then every $x \in G$, $x > 0$, is an integral multiple of λ_-. Otherwise, there exists $x \in G$, $x > 0$, and a unique $n \in \mathbb{N}$ such that

$$-n\lambda_- < x < -(n+1)\lambda_-.$$

It follows that $\lambda_- < x + (n+1)\lambda_- < 0$, which contradicts the definition of λ_-. Similarly one can show that every $x \in G$, $x < 0$, is an integral multiple of λ_+. Consequently, $\lambda_+ + \lambda_- = 0$ and G is the cyclic group generated by λ_+, which is absurd. $\square$

REMARK 4.7. [8] shows that for an additive semigroup $G \subset \mathbb{R}$ if $\inf_{x \in G} |x| > 0$, then all members of G are of the same sign. [12] proves a similar result to Lemma 4.6 but does not state it clearly. For the convenience of the readers, we provide a full statement and detailed proof.

Now it is clear about the conditions on the set H for the space $AP_H(\mathbb{R}, \mathbb{C})$. Theorem 1.1 shows when the integral operator $\mathcal{I}$ is compact on $AP_H(\mathbb{R}, \mathbb{C})$. Theorem 2.8 and Lemma 4.1 give a condition under which $AP_H(\mathbb{R}, \mathbb{C})$ is closed with respect to the operations of composition and multiplication. This makes it possible to define nonlinear and continuous operators on $AP_H(\mathbb{R}, \mathbb{C})$. Lemma 4.3 shows that it is not appropriate to make the set H an additive group with no limit point. Lemma 4.5 guarantees the existence of a nonzero semigroup with no limit point. Lemma 4.6 claims that a nonzero additive semigroup with no limit point must lie entirely on one half of the real line, either $[0, \infty)$ or $(-\infty, 0]$, if it is not a cyclic group. Therefore, a suitable set H for the space $AP_H(\mathbb{R}, \mathbb{C})$ turns out to be a nonzero additive semigroup such that H has no limit point, and either $H \subset [0, \infty)$ or $H \subset (-\infty, 0]$.

It is easy to give an example of noncompact and nonlinear operators on the space $AP_H(\mathbb{R}, \mathbb{C})$.

EXAMPLE 4.8. Let $H = \{\lambda_k\}_{k=1}^{\infty} \subset (0, \infty)$ be an additive semigroup with no limit point and $AP_H(\mathbb{R}, \mathbb{C})$ be the almost periodic function space defined by (3.1). The nonlinear operator

$$\mathcal{N} \colon AP_H(\mathbb{R}, \mathbb{C}) \to AP_H(\mathbb{R}, \mathbb{C}), \quad f \mapsto f^2,$$

is not compact. To prove this, note that the set $\{e^{i\lambda_k t}\}_{k=1}^{\infty}$ is bounded and its image $\mathcal{N}(\{e^{i\lambda_k t}\}_{k=1}^{\infty}) = \{e^{2i\lambda_k t}\}_{k=1}^{\infty}$ has no Cauchy subsequences by Lemma 3.2 (a).

5. Real function spaces

Coincidence degree theory works on real Banach spaces. However, the spectrum of a nonconstant real almost periodic function can never be contained in a half-line. In general, it is impossible for an integral operator $\mathcal{I}_{\mathcal{N}}$ defined by (3.3) to be compact on a real almost periodic function space. Thus in view of the spectral conditions, the present coincidence degree theory seems to be only applicable to complex almost periodic differential equations except for the linear case. In this section, we develop appropriate settings to make use of coincidence degree theory on a complex almost periodic function space.

Let $H = \{\lambda_k\}_{k=1}^{\infty} \subset (0,\infty)$ be an additive semigroup with no limit point and $AP_H(\mathbb{R},\mathbb{C})$ be the almost periodic function space defined by (3.1). Put $-H = \{-\lambda_k\}_{k=1}^{\infty}$ and define the following almost periodic function spaces

$$AP_H(\mathbb{R},\mathbb{R}) = \{f \in AP(\mathbb{R},\mathbb{R}) : \Lambda_f \subset H \cup (-H)\},$$
$$Z_H = \{(f,g) \in AP_H(\mathbb{R},\mathbb{R}) \times AP_H(\mathbb{R},\mathbb{R}) : \Lambda_{f+ig} \subset H\}.$$

Define

$$\|(f,g)\|_{Z_H} = \|f+ig\|$$

for every $(f,g) \in Z_H$. Then $\|\cdot\|_{Z_H}$ is a norm on Z_H.

LEMMA 5.1. *The space $(Z_H, \|\cdot\|_{Z_H})$ is isometrically isomorphic to*

$$(AP_H(\mathbb{R},\mathbb{C}), \|\cdot\|).$$

PROOF. Define a linear map

$$\begin{aligned} \mathcal{T}\colon AP_H(\mathbb{R},\mathbb{C}) &\to Z_H, \\ h &\mapsto (\operatorname{Re} h, \operatorname{Im} h) = \left(\frac{h+\overline{h}}{2}, \frac{h-\overline{h}}{2i}\right). \end{aligned} \tag{5.1}$$

It is obvious that $\mathcal{T}$ is injective, we next show that $\mathcal{T}$ is surjective. Since the components of every $(f,g) \in AP_H(\mathbb{R},\mathbb{R}) \times AP_H(\mathbb{R},\mathbb{R})$ have Fourier series of the form

$$\begin{cases} f(t) \sim \dfrac{1}{2}\displaystyle\sum_{k=1}^{\infty}(a_k e^{i\lambda_k t} + \overline{a_k} e^{-i\lambda_k t}), \\ g(t) \sim \dfrac{1}{2}\displaystyle\sum_{k=1}^{\infty}(b_k e^{i\lambda_k t} + \overline{b_k} e^{-i\lambda_k t}), \end{cases}$$

it follows that

$$f(t) + ig(t) \sim \frac{1}{2}\sum_{k=1}^{\infty}[(a_k + ib_k)e^{i\lambda_k t} + (\overline{a_k} + i\overline{b_k})e^{-i\lambda_k t}].$$

If $(f,g) \in Z_H$, $\Lambda_{f+ig} \subset H$ implies that $\overline{a_k} + i\overline{b_k} = 0$ and $a_k = ib_k$ for all $k \in \mathbb{Z}_+$. If $h = f + ig$, it follows that

$$\begin{cases} f = \dfrac{h+\overline{h}}{2} = \operatorname{Re} h, \\ g = \dfrac{h-\overline{h}}{2i} = \operatorname{Im} h. \end{cases}$$

So $\mathcal{T}$ is surjective. At last, one has $\|(f,g)\|_{Z_H} = \|f+ig\| = \|h\|$ by definition. Hence $\mathcal{T}$ is an isometry. $\square$

The following simple equivalence relation is useful and can be proved easily by Lemma 5.1.

Lemma 5.2. *An operator* $\mathcal{A}\colon Z_H \to Z_H$ *is compact if and only if its conjugate* $\mathcal{T}^{-1}\mathcal{A}\mathcal{T}\colon AP_H(\mathbb{R},\mathbb{C}) \to AP_H(\mathbb{R},\mathbb{C})$ *is compact.*

We also need almost periodic functions with 0 as an exponent. Let $H_0 = H \cup \{0\}$ and define the following spaces:

$$\begin{aligned} V_{\mathbb{C}} &= \{f\colon \mathbb{R} \to \{c\} : c \in \mathbb{C}\}, \\ V_{\mathbb{R}} &= \{f\colon \mathbb{R} \to \{c\} : c \in \mathbb{R}\}, \\ AP_{H_0}(\mathbb{R},\mathbb{R}) &= \{f \in AP(\mathbb{R},\mathbb{R}) : \Lambda_f \subset H_0 \cup (-H)\}, \\ Z_{H_0} &= \{(f,g) \in AP_{H_0}(\mathbb{R},\mathbb{R}) \times AP_{H_0}(\mathbb{R},\mathbb{R}) : \Lambda_{f+ig} \subset H_0\}. \end{aligned}$$

It is easy to see that

$$\begin{aligned} AP_{H_0}(\mathbb{R},\mathbb{C}) &= AP_H(\mathbb{R},\mathbb{C}) \oplus V_{\mathbb{C}}, \\ AP_{H_0}(\mathbb{R},\mathbb{R}) &= AP_H(\mathbb{R},\mathbb{R}) \oplus V_{\mathbb{R}}, \\ Z_{H_0} &= Z_H \oplus V_{\mathbb{R}}^2, \end{aligned}$$

where $V_{\mathbb{R}}^2 = V_{\mathbb{R}} \times V_{\mathbb{R}}$. Define

$$\|(f,g)\|_{Z_{H_0}} = \|f + ig\|$$

for every $(f,g) \in Z_{H_0}$. Then $\|\cdot\|_{Z_{H_0}}$ is a norm on Z_{H_0}.

Lemma 5.3. *The space* $(Z_{H_0}, \|\cdot\|_{Z_{H_0}})$ *is isometrically isomorphic to*

$$(AP_{H_0}(\mathbb{R},\mathbb{C}), \|\cdot\|).$$

In this case, the components of each $(f,g) \in Z_{H_0}$ have Fourier series of the form

$$\begin{cases} f(t) \sim a_0 + \dfrac{1}{2}\displaystyle\sum_{k=1}^{\infty}(a_k e^{i\lambda_k t} + \overline{a_k} e^{-i\lambda_k t}), \\ g(t) \sim b_0 + \dfrac{1}{2}\displaystyle\sum_{k=1}^{\infty}(b_k e^{i\lambda_k t} + \overline{b_k} e^{-i\lambda_k t}), \end{cases}$$

where a_0, $b_0 \in \mathbb{R}$ and $a_k = ib_k \in \mathbb{C}$ for every $k \in \mathbb{Z}_+$. An isometric isomorphism from $AP_{H_0}(\mathbb{R},\mathbb{C})$ to Z_{H_0} is given by

$$\begin{gathered} \mathcal{T}_0\colon AP_{H_0}(\mathbb{R},\mathbb{C}) \to Z_{H_0}, \\ h \mapsto (\operatorname{Re} h, \operatorname{Im} h) = \left(\frac{h+\overline{h}}{2}, \frac{h-\overline{h}}{2i}\right). \end{gathered} \tag{5.2}$$

Lemma 5.4. *An operator* $\mathcal{A}\colon Z_{H_0} \to Z_{H_0}$ *is compact if and only if its conjugate* $\mathcal{T}_0^{-1}\mathcal{A}\mathcal{T}_0\colon AP_{H_0}(\mathbb{R},\mathbb{C}) \to AP_{H_0}(\mathbb{R},\mathbb{C})$ *is compact.*

Define an integral operator on Z_H by

$$\mathcal{I}_Z\colon Z_H \to Z_H,$$

$$(f(t), g(t)) \mapsto \left(\int_0^t f(s)\,ds - \mathfrak{M}\left\{ \int_0^t f(s)\,ds \right\}, \int_0^t g(s)\,ds - \mathfrak{M}\left\{ \int_0^t g(s)\,ds \right\} \right).$$

It is easy to check that $\mathcal{T}^{-1}\mathcal{I}_Z\mathcal{T}\colon AP_H(\mathbb{R},\mathbb{C}) \to AP_H(\mathbb{R},\mathbb{C})$ is the integral operator $\mathcal{I}$ defined by (1.1). The following criterion is useful for determining the $\mathcal{L}$-compactness of an operator $\mathcal{N}$ on Z_{H_0} and is a direct consequence of Lemma 5.2 and Theorem 1.1.

Lemma 5.5. *Let $\mathcal{N}\colon Z_{H_0} \to Z_{H_0}$ be a continuous and bounded operator, then the composite operator $\mathcal{I}_Z \circ (\mathrm{id}_{Z_{H_0}} - \mathfrak{M}) \circ \mathcal{N}\colon Z_{H_0} \to Z_H$ is compact, where $\mathrm{id}_{Z_{H_0}}$ and $\mathfrak{M}$ are the identity operator on Z_{H_0} and the mean value operator, respectively.*

One can check by Theorem 2.8, Lemmas 4.1 and 4.2 to see that Lemma 5.5 holds for those nonlinear operators $\mathcal{N}$ defined by basic elementary functions, such as $f(t) \mapsto a(t)[f(t)]^2 + b(t)f(t) + c(t)$ and $f(t) \mapsto e^{f(t)}$.

6. Applications

In this section, we apply coincidence degree theory to show the existence of almost periodic solutions to differential equations with an appropriate priori estimate structure.

6.1. Complex differential equations with analyticity in a bounded domain. Consider equation (1.2) with assumptions (A1)–(A3). Notice that (1.2) does not possess an exponential dichotomy if $\operatorname{Re}\alpha = 0$, in which case the non-resonance condition for those fixed point methods in [6], [8], [12], [14] fails. Moreover, (1.3) indeed can be satisfied in a number of situations.

Let $H = \{\lambda_k\}_{k=1}^{\infty} \subset (0,\infty)$ be the semigroup generated by $(\Lambda_\psi \cup \Lambda_\varphi) \setminus \{0\}$. Then H has no limit point by Lemma 4.5, and $H_0 = H \cup \{0\}$ is the semigroup generated by $\Lambda_\psi \cup \Lambda_\varphi \cup \{0\}$. Let $\delta > 0$ be a number such that $\lambda_k \geq \delta$ for all $k \in \mathbb{Z}_+$. Denote by $V_{\mathbb{C}}$, $V_{\mathbb{R}}$, $AP_H(\mathbb{R},\mathbb{C})$, $AP_H(\mathbb{R},\mathbb{R})$, $AP_{H_0}(\mathbb{R},\mathbb{C})$, $AP_{H_0}(\mathbb{R},\mathbb{R})$, Z_H and Z_{H_0} the function spaces as in Section 5. Define the space $AP_H^1(\mathbb{R},\mathbb{C})$ as in Subsection 3.3 and let

$$\begin{aligned} AP^1_{H_0}(\mathbb{R},\mathbb{C}) &= AP^1_H(\mathbb{R},\mathbb{C}) \oplus V_{\mathbb{C}}, \\ Z^1_H &= \{(f,g) \in Z_H : \text{there exists } (f',g') \in Z_H\}, \\ Z^1_{H_0} &= Z^1_H \oplus V^2_{\mathbb{R}}. \end{aligned}$$

Let $\mathcal{I}$ be the integral operators defined by (1.1). Define the following two operators

$$\mathcal{L}\colon AP^1_{H_0}(\mathbb{R},\mathbb{C}) \to AP_{H_0}(\mathbb{R},\mathbb{C}), \quad f \mapsto f', \tag{6.1}$$

$$\mathcal{N}\colon AP_{H_0}(\mathbb{R},\mathbb{C}) \to AP_{H_0}(\mathbb{R},\mathbb{C}), \quad z(t) \mapsto \alpha z(t) + \psi(z(t),t) + \varphi(t), \tag{6.2}$$

and two projectors

$$\mathcal{P} = \mathcal{Q}\colon AP_{H_0}(\mathbb{R},\mathbb{C}) \to AP_{H_0}(\mathbb{R},\mathbb{C}), \qquad f \mapsto \mathfrak{M}\{f\}.$$

Put $\mathcal{D} = \mathcal{L}|_{AP^1_H(\mathbb{R},\mathbb{C})}\colon AP^1_H(\mathbb{R},\mathbb{C}) \to AP_{H_0}(\mathbb{R},\mathbb{C})$. Let $\mathcal{T}$ and $\mathcal{T}_0$ be the isometric isomorphisms given by (5.1) and (5.2), respectively.

Lemma 6.1. *The following statements are true for the above operators.*

(a) $\mathcal{I}\colon AP_H(\mathbb{R},\mathbb{C}) \to \operatorname{ran}\mathcal{I}$ *is the inverse to* $\mathcal{D}\colon AP^1_H(\mathbb{R},\mathbb{C}) \to AP_H(\mathbb{R},\mathbb{C})$.
(b) $\mathcal{T}\mathcal{I}\mathcal{T}^{-1}\colon Z_H \to \operatorname{ran}\mathcal{T}\mathcal{I}\mathcal{T}^{-1}$ *is the inverse to* $\mathcal{T}\mathcal{D}\mathcal{T}^{-1}\colon Z^1_H \to Z_H$.
(c) $\ker\mathcal{L} = V_{\mathbb{C}}$, $\operatorname{ran}\mathcal{L} = \operatorname{ran}\mathcal{D} = AP_H(\mathbb{R},\mathbb{C})$.
(d) $\ker\mathcal{T}_0\mathcal{L}\mathcal{T}_0^{-1} = V^2_{\mathbb{R}}$, $\operatorname{ran}\mathcal{T}_0\mathcal{L}\mathcal{T}_0^{-1} = \operatorname{ran}\mathcal{T}\mathcal{D}\mathcal{T}^{-1} = Z_H$.
(e) $\mathcal{T}_0\mathcal{L}\mathcal{T}_0^{-1}\colon Z^1_H \oplus V^2_{\mathbb{R}} \to Z_{H_0}$ *is a Fredholm operator of index* 0.

Proof. It is easy to check that $\mathcal{D}\circ\mathcal{I} = \mathrm{id}_{AP_H(\mathbb{R},\mathbb{C})}$ and $\mathcal{I}\circ\mathcal{D} = \mathrm{id}_{AP^1_H(\mathbb{R},\mathbb{C})}$. So (a) holds. (b) follows from (a) and Lemma 5.1. (c) is true by the fact that $f' = 0$ if and only if f is a constant and (a). (d) follows from (c) and Lemma 5.3. Therefore, (e) holds by (d). □

Let an isomorphism $\mathcal{J}\colon \operatorname{ran}\mathcal{Q} \to \ker\mathcal{L}$ be given by $\mathcal{J} = \mathrm{id}_{V_{\mathbb{C}}}$. We are in the position proving Theorem 1.2.

Proof of Theorem 1.2. Note that the following three equations

$$z' = \alpha z + \psi(z,t) + \varphi(t), \qquad \mathcal{L}z = \mathcal{N}z, \qquad \mathcal{T}_0\mathcal{L}\mathcal{T}_0^{-1}(\mathcal{T}_0 z) = \mathcal{T}_0\mathcal{N}\mathcal{T}_0^{-1}(\mathcal{T}_0 z),$$

are equivalent on $\overline{\Omega}\cap AP^1_{H_0}(\mathbb{R},\mathbb{C})$, where $\Omega = \{f \in AP_{H_0}(\mathbb{R},\mathbb{C}) : \|f\| \le R\}$ and the operators $\mathcal{L}$, $\mathcal{N}$ and $\mathcal{T}_0$ are defined by (6.1), (6.2) and (5.2), respectively. We will show that the pair $(\mathcal{T}_0\mathcal{L}\mathcal{T}_0^{-1}, \mathcal{T}_0\mathcal{N}\mathcal{T}_0^{-1})$ satisfies all the conditions (C1)–(C3) of the Continuation Theorem 2.13 on the open bounded set $\mathcal{T}_0(\Omega)$ in Z_{H_0}. For convenience we denote

$$K = \frac{\dfrac{1}{|\alpha|} + \dfrac{\beta}{\delta}}{1 - \dfrac{\beta}{\delta}\,|\alpha|}.$$

1. By (1.3), let ε be a number satisfying

$$0 < \varepsilon \le R - K\Big[\sup_{|z|\le R,\, t\in\mathbb{R}} |\psi(z,t)| + \|\varphi\|\Big],$$

and define a function on $\mathbb{C}\times\mathbb{R}$ by

$$\widetilde{\psi}(z,t) = \begin{cases} \psi(z,t) & K[|\psi(z,t)| + \|\varphi\|] \le R - \varepsilon, \\ \Big(\dfrac{R-\varepsilon}{K} - \|\varphi\|\Big)\cdot\dfrac{\psi(z,t)}{|\psi(z,t)|} & K[|\psi(z,t)| + \|\varphi\|] > R - \varepsilon. \end{cases}$$

It is easy to see that $\widetilde{\psi}\colon \mathbb{C}\times\mathbb{R}\to\mathbb{C}$ is continuous and

$$\|\widetilde{\psi}\| := \sup_{z\in\mathbb{C},\, t\in\mathbb{R}} |\widetilde{\psi}(z,t)| \le \frac{R-\varepsilon}{K} - \|\varphi\|,$$
$$\widetilde{\psi}(z,t) = \psi(z,t), \quad z\in\mathbb{C},\ |z|\le R, t\in\mathbb{R}.$$

Consider the auxiliary equation

$$z' = \alpha z + \widetilde{\psi}(z,t) + \varphi(t), \quad t\in\mathbb{R}. \tag{6.3}$$

Suppose that

$$z'(t) = \mu[\alpha z(t) + \widetilde{\psi}(z(t),t) + \varphi(t)], \quad t\in\mathbb{R}, \tag{6.4}$$

for some almost periodic z and $\mu\in(0,1)$. Therefore,

$$\mathfrak{M}\{z'\} = \mu\mathfrak{M}\{\alpha z + \psi(z(\,\cdot\,),\,\cdot\,) + \varphi\} = 0,$$

and

$$|\mathfrak{M}\{z\}| = \left|\frac{\mathfrak{M}\{\psi(z(\,\cdot\,),\,\cdot\,)+\varphi\}}{\alpha}\right| \le \frac{\|\widetilde{\psi}\| + \|\varphi\|}{|\alpha|}. \tag{6.5}$$

By integrating (6.4) one gets

$$\mathcal{I}(z') = z - \mathfrak{M}\{z\} = \mu\mathcal{I}(\alpha z + \psi(z(\,\cdot\,),\,\cdot\,) + \varphi),$$
$$\|z - \mathfrak{M}\{z\}\| \le \|\mathcal{I}(\alpha z + \psi(z(\,\cdot\,),\,\cdot\,) + \varphi)\|,$$

and

$$\begin{aligned} \|z\| - |\mathfrak{M}\{z\}| &\le \frac{\beta}{\delta}\,\|\alpha z + \widetilde{\psi}(z(\,\cdot\,),\,\cdot\,) + \varphi\| \\ &\le \frac{\beta}{\delta}\,\big[|\alpha|\cdot\|z\| + \|\widetilde{\psi}(z(\,\cdot\,),\,\cdot\,)\| + \|\varphi\|\big] \end{aligned} \tag{6.6}$$

by Theorem 2.7. From (6.5), and (6.6) it follows that

$$\left(1 - \frac{\beta}{\delta}\,|\alpha|\right)\cdot\|z\| \le \left(\frac{1}{|\alpha|} + \frac{\beta}{\delta}\right)\cdot(\|\widetilde{\psi}\| + \|\varphi\|),$$
$$\|z\| \le K(\|\widetilde{\psi}\| + \|\varphi\|) \le R-\varepsilon$$

If there exist $z\in\partial\Omega\cap\operatorname{dom}\mathcal{L}$ and $\mu\in(0,1)$ such that $\mathcal{L}z = \mu\mathcal{N}z$, then z is also a solution to (6.3). From the priori estimate above it follows that $\|z\|\le R-\varepsilon$, which is a contradiction since $z\in\partial\Omega$ and $\|z\| = R$. So $\mathcal{L}z\ne\mu\mathcal{N}z$ for each $z\in\partial\Omega\cap\operatorname{dom}\mathcal{L}$ and $\mu\in(0,1)$. It is easy to show that $\mathcal{T}_0(\Omega)$ is the open ball in Z_{H_0} centered at 0 with radius R and $\mathcal{T}_0(\partial\Omega) = \partial\mathcal{T}_0(\Omega)$. Hence condition (C1) of Theorem 2.13 is true for the pair $(\mathcal{T}_0\mathcal{L}\mathcal{T}_0^{-1}, \mathcal{T}_0\mathcal{N}\mathcal{T}_0^{-1})$ on $\mathcal{T}_0(\Omega)$.

2. It is easy to see that $\partial\Omega\cap\ker\mathcal{L} = \{w\in\mathbb{C} : |w| = R\}$ and

$$\mathcal{T}_0(\partial\Omega\cap\ker\mathcal{L}) = [\partial\mathcal{T}_0(\Omega)]\cap\ker\mathcal{T}_0\mathcal{L}\mathcal{T}_0^{-1} = \{(x,y)\in\mathbb{R}^2 : |x+iy| = R\}.$$

For any $z_0\in\partial\Omega\cap\ker\mathcal{L}$, from the definition of $\mathcal{N}$ and $\mathcal{Q}$ it follows that

$$(\mathcal{N}z_0)(t) = \alpha z_0 + \psi(z_0,t) + \varphi(t),$$

and

$$\mathcal{Q}\mathcal{N}z_0 = \mathfrak{M}\{\mathcal{N}z_0\} = \alpha z_0 + \mathfrak{M}\{\psi(z_0,\,\cdot\,) + \varphi\}.$$

By (1.3), if $\mathcal{Q}\mathcal{N}z_0 = 0$, then

$$|z_0| = R > K\bigg[\sup_{|z|\le R,\, t\in\mathbb{R}} |\psi(z,t)| + \|\varphi\|\bigg],$$

$$|z_0| \le \frac{1}{|\alpha|}\bigg[\sup_{|z|\le R,\, t\in\mathbb{R}} |\psi(z,t)| + \|\varphi\|\bigg],$$

which yield the contradiction

$$\bigg(K - \frac{1}{|\alpha|}\bigg)\bigg[\sup_{|z|\le R,\, t\in\mathbb{R}} |\psi(z,t)| + \|\varphi\|\bigg] < 0$$

since $K\cdot|\alpha| > 1$. Therefore, $\mathcal{Q}\mathcal{N}z \ne 0$ on $\partial\Omega \cap \ker\mathcal{L}$ and

$$(\mathcal{T}_0\mathcal{Q}\mathcal{T}_0^{-1})(\mathcal{T}_0\mathcal{N}\mathcal{T}_0^{-1})(x,y) \ne 0$$

on $[\partial\mathcal{T}_0(\Omega)] \cap \ker\mathcal{T}_0\mathcal{L}\mathcal{T}_0^{-1}$. Thus condition (C2) of Theorem 2.13 is true for the pair $(\mathcal{T}_0\mathcal{L}\mathcal{T}_0^{-1}, \mathcal{T}_0\mathcal{N}\mathcal{T}_0^{-1})$ on $\mathcal{T}_0(\Omega)$.

3. It is easy to check that $\Omega \cap \ker\mathcal{L} = \{w \in \mathbb{C} : |w| < R\}$ and

$$\mathcal{T}_0(\Omega \cap \ker\mathcal{L}) = \mathcal{T}_0(\Omega) \cap \ker\mathcal{T}_0\mathcal{L}\mathcal{T}_0^{-1} = \{(x,y) \in \mathbb{R}^2 : |x+iy| < R\}.$$

From the definition of $\mathcal{J}$, $\mathcal{Q}$ and $\mathcal{N}$ it follows that

$$\mathcal{J}\mathcal{Q}\mathcal{N}z = \mathfrak{M}\{\mathcal{N}z\} = \alpha z + \mathfrak{M}\{\psi(z,\,\cdot\,) + \varphi\}$$

for all $z \in \Omega \cap \ker\mathcal{L}$, and for all $(x,y) \in \mathcal{T}_0(\Omega \cap \ker\mathcal{L})$

$$\mathcal{T}_0\mathcal{J}\mathcal{Q}\mathcal{N}\mathcal{T}_0^{-1}(x,y) = \begin{pmatrix} \operatorname{Re}\mathcal{J}\mathcal{Q}\mathcal{N}(x+iy) \\ \operatorname{Im}\mathcal{J}\mathcal{Q}\mathcal{N}(x+iy) \end{pmatrix}.$$

We use the homotopy invariance property to calculate the Brouwer degree

$$\deg(\mathcal{T}_0\mathcal{J}\mathcal{Q}\mathcal{N}\mathcal{T}_0^{-1}, \mathcal{T}_0(\Omega \cap \ker\mathcal{L}), 0).$$

Define a function on $\mathcal{T}_0(\Omega \cap \ker\mathcal{L}) \times [0,1]$ by

$$(x,y,\mu) \mapsto F_\mu(x,y) = \begin{pmatrix} \operatorname{Re}[\alpha(x+iy) + \mu\mathfrak{M}\{\psi(x+iy,\,\cdot\,) + \varphi\}] \\ \operatorname{Im}[\alpha(x+iy) + \mu\mathfrak{M}\{\psi(x+iy,\,\cdot\,) + \varphi\}] \end{pmatrix}.$$

Suppose that $F_\mu(x,y) = 0$ for some $(x,y) \in \partial\mathcal{T}_0(\Omega \cap \ker\mathcal{L})$ and $\mu \in [0,1]$. If put $z = x+iy$, it follows that $|z| = R$ and $\alpha z + \mu\mathfrak{M}\{\psi(z,\,\cdot\,) + \varphi\} = 0$.

With a similar proof to that for $\mathcal{Q}\mathcal{N}z \ne 0$ on $\partial\Omega \cap \ker\mathcal{L}$, one can obtain the same contradiction:

$$|\alpha|\cdot R \le |\mathfrak{M}\{\psi(z,\cdot) + \varphi\}| \le \sup_{|z|\le R,\, t\in\mathbb{R}} |\psi(z,t)| + \|\varphi\|.$$

Consequently, the function $F_\mu(x,y)$ is a homotopy from $F_0(x,y)$ to $F_1(x,y)$ on $\mathcal{T}_0(\Omega \cap \ker\mathcal{L})$.

From the homotopy invariance property and the definition of degree for non-degenerate linear maps [11] it follows that

$$\begin{aligned}\deg(&\mathcal{T}_0\mathcal{J}\mathcal{Q}\mathcal{N}\mathcal{T}_0^{-1},\mathcal{T}_0(\Omega\cap\ker\mathcal{L}),0)\\ &= \deg(F_1,\mathcal{T}_0(\Omega\cap\ker\mathcal{L}),0) = \deg(F_0,\mathcal{T}_0(\Omega\cap\ker\mathcal{L}),0)\\ &= \operatorname{sgn}\det\begin{pmatrix}\operatorname{Re}\alpha & -\operatorname{Im}\alpha\\ \operatorname{Im}\alpha & \operatorname{Re}\alpha\end{pmatrix} = \operatorname{sgn}|\alpha|^2 = 1 \neq 0.\end{aligned}$$

Thus condition (C3) of Theorem 2.13 is true for the pair $(\mathcal{T}_0\mathcal{L}\mathcal{T}_0^{-1},\mathcal{T}_0\mathcal{N}\mathcal{T}_0^{-1})$ on $\mathcal{T}_0(\Omega)$. Therefore, the complex uniformly quasi-bounded differential equation (1.2) has at least one solution in $\overline{\Omega}\cap AP^1_{H_0}(\mathbb{R},\mathbb{C})$ by Theorem 2.13. □

6.2. Real linear differential equations with delays. In this subsection, we consider a class of real differential equations of the form

$$x'(t) = \sum_{j=1}^{m}\alpha_j x(t+\tau_j) + \varphi(t), \tag{6.7}$$

where $\alpha_j,\tau_j\in\mathbb{R}$ for $j=1,\dots,m$ and $\varphi\in AP(\mathbb{R},\mathbb{R})$, Λ_φ has no limit point.

The linear differential equation (6.7) is simpler than the complex one (1.2). Since a linear map always preserves the spectrum of an almost periodic function, there is no request for H to be a semigroup.

Let $H=\{\lambda_k\}_{k=1}^\infty := \Lambda_\varphi\cap(0,\infty)$, $\delta>0$ be a number satisfying $\lambda_k\ge\delta$ for all $k\in\mathbb{Z}_+$, and $H_0=H\cup\{0\}$. Denote by $V_\mathbb{R}$, $AP_H(\mathbb{R},\mathbb{R})$, $AP_{H_0}(\mathbb{R},\mathbb{R})$ the function spaces as before. Define spaces

$$\begin{aligned}AP^1_H(\mathbb{R},\mathbb{R}) &= \{f\in AP_H(\mathbb{R},\mathbb{R}) : \text{there exists } f'\in AP_H(\mathbb{R},\mathbb{R})\},\\ AP^1_{H_0}(\mathbb{R},\mathbb{R}) &= AP^1_H(\mathbb{R},\mathbb{R})\oplus V_\mathbb{R}.\end{aligned}$$

Theorem 6.2. *Let the following conditions hold*:

- (A4) $\varphi\in AP(\mathbb{R},\mathbb{R})$ *and* Λ_φ *has no limit point.*
- (A5) $H=\{\lambda_k\}_{k=1}^\infty=\Lambda_\varphi\cap(0,\infty)$, $H_0=H\cup\{0\}$, *and* $\delta>0$ *satisfies* $\lambda_k\ge\delta$ *for all* $k\in\mathbb{Z}_+$.
- (A6) $\alpha_j,\tau_j\in\mathbb{R}$ *for* $j=1,\dots,m$.
- (A7) $\sum_{j=1}^m\alpha_j\neq 0$ *and* $\sum_{j=1}^m|\alpha_j|<\delta/\beta$, *where* $\beta>0$ *is an absolute constant given by Theorem* 2.7.

Then there exists a unique solution $\phi\in AP^1_{H_0}(\mathbb{R},\mathbb{R})$ *to* (6.7) *with* $\Lambda_\phi=\Lambda_\varphi$.

Proof. Existence. The proof of the existence of a solution to (6.7) is similar to that of Theorem 1.2. For the reader's convenience, we provide a detailed one which may help in the understanding of the role of each assumption. Since $AP_H(\mathbb{R},\mathbb{R})$ is a closed subspace of

$$AP_{H\cup(-H)}(\mathbb{R},\mathbb{C}) = \{f\in AP(\mathbb{R},\mathbb{C}) : \Lambda_f\subset H\cup(-H)\},$$

from assumptions (A4) and (A5) and Theorem 1.1 for integral operators on the space $AP_{H\cup(-H)}(\mathbb{R},\mathbb{C})$ it follows that the following integral operator on a real almost periodic function space:

$$\begin{gathered}\mathcal{I}_R\colon AP_H(\mathbb{R},\mathbb{R})\to AP_H(\mathbb{R},\mathbb{R}),\\ f(t)\mapsto \int_0^t f(s)\,ds-\mathfrak{M}\bigg\{\int_0^t f(s)\,ds\bigg\},\end{gathered}$$

is compact. Define the following two operators:

$$\begin{aligned}\mathcal{L}\colon AP^1_{H_0}(\mathbb{R},\mathbb{R})&\to AP_{H_0}(\mathbb{R},\mathbb{R}), & f&\mapsto f',\\ \mathcal{N}\colon AP_{H_0}(\mathbb{R},\mathbb{R})&\to AP_{H_0}(\mathbb{R},\mathbb{R}), & x(t)&\mapsto \sum_{j=1}^m \alpha_j x(t+\tau_j)+\varphi(t),\end{aligned}$$

and two projectors

$$\mathcal{P}=\mathcal{Q}\colon AP_{H_0}(\mathbb{R},\mathbb{R})\to AP_{H_0}(\mathbb{R},\mathbb{R}),\qquad f\mapsto \mathfrak{M}\{f\}.$$

Put $\mathcal{D}_R=\mathcal{L}|_{AP^1_H(\mathbb{R},\mathbb{R})}\colon AP^1_H(\mathbb{R},\mathbb{R})\to AP_{H_0}(\mathbb{R},\mathbb{R})$. It is easy to check that the following statements hold for the operators defined above:

- (i) $\mathcal{I}_R\colon AP_H(\mathbb{R},\mathbb{R})\to\operatorname{ran}\mathcal{I}_R$ is the inverse to $\mathcal{D}_R\colon AP^1_H(\mathbb{R},\mathbb{R})\to AP_H(\mathbb{R},\mathbb{R})$.
- (ii) $\ker\mathcal{L}=V_{\mathbb{R}}$, $\operatorname{ran}\mathcal{L}=\operatorname{ran}\mathcal{D}_R=AP_H(\mathbb{R},\mathbb{R})$.
- (iii) $\mathcal{L}\colon AP^1_H(\mathbb{R},\mathbb{R})\oplus V_{\mathbb{R}}\to AP_{H_0}(\mathbb{R},\mathbb{R})$ is a Fredholm operator of index 0.

Let an isomorphism $\mathcal{J}\colon\operatorname{ran}\mathcal{Q}\to\ker\mathcal{L}$ be given by $\mathcal{J}=\mathrm{id}_{V_{\mathbb{R}}}$. Since equation (6.7) is equivalent to $\mathcal{L}x=\mathcal{N}x$, we will show that the pair $(\mathcal{L},\mathcal{N})$ satisfies all the conditions (C1)–(C3) of Theorem 2.13 on an open bounded subset of $AP_{H_0}(\mathbb{R},\mathbb{R})$.

1. Suppose that $\mathcal{L}x=\mu\mathcal{N}x$ for some $x\in AP^1_{H_0}(\mathbb{R},\mathbb{R})$ and $\mu\in(0,1)$. Then

$$x'(t)=\mu\bigg[\sum_{j=1}^m\alpha_j x(t+\tau_j)+\varphi(t)\bigg]\tag{6.8}$$

for all $t\in\mathbb{R}$, which implies

$$\mathfrak{M}\{x'\}=\mu\bigg[\sum_{j=1}^m\alpha_j\mathfrak{M}\{x(\,\cdot\,+\tau_j)\}+\mathfrak{M}\{\varphi\}\bigg]=\mu\bigg[\bigg(\sum_{j=1}^m\alpha_j\bigg)\mathfrak{M}\{x\}+\mathfrak{M}\{\varphi\}\bigg]=0,$$

and

$$\mathfrak{M}\{x\}=-\mathfrak{M}\{\varphi\}\bigg/\sum_{j=1}^m\alpha_j.\tag{6.9}$$

By integrating (6.8) one gets

$$\begin{aligned}\mathcal{I}_R(x') = x - \mathfrak{M}\{x\} &= \mu\mathcal{I}_R\bigg(\sum_{j=1}^m \alpha_j\big[x(\,\cdot\,+\tau_j) - \mathfrak{M}\{x\}\big] + \varphi - \mathfrak{M}\{\varphi\}\bigg)\\ &= \mu\bigg[\sum_{j=1}^m \alpha_j\mathcal{I}_R\big(x(\,\cdot\,+\tau_j) - \mathfrak{M}\{x\}\big) + \mathcal{I}_R\big(\varphi - \mathfrak{M}\{\varphi\}\big)\bigg]\end{aligned}$$

and

$$\begin{aligned}\|x - \mathfrak{M}\{x\}\| &\le \sum_{j=1}^m |\alpha_j|\cdot\big\|\mathcal{I}_R\big(x(\,\cdot\,+\tau_j) - \mathfrak{M}\{x\}\big)\big\| + \big\|\mathcal{I}_R\big(\varphi - \mathfrak{M}\{\varphi\}\big)\big\|\\ &\le \frac{\beta}{\delta}\bigg[\sum_{j=1}^m |\alpha_j|\cdot\|x(\,\cdot\,+\tau_j) - \mathfrak{M}\{x\}\| + \|\varphi - \mathfrak{M}\{\varphi\}\|\bigg]\\ &= \frac{\beta}{\delta}\bigg[\bigg(\sum_{j=1}^m |\alpha_j|\bigg)\cdot\|x - \mathfrak{M}\{x\}\| + \|\varphi - \mathfrak{M}\{\varphi\}\|\bigg],\end{aligned}$$

where (6.9) and Theorem 2.7 are used to obtain the above inequalities. Consequently,

$$\bigg(\frac{\delta}{\beta} - \sum_{j=1}^m |\alpha_j|\bigg)\cdot\|x - \mathfrak{M}\{x\}\| \le \|\varphi - \mathfrak{M}\{\varphi\}\|,$$

$$\|x\| - |\mathfrak{M}\{x\}| \le \|x - \mathfrak{M}\{x\}\| \le \|\varphi - \mathfrak{M}\{\varphi\}\|\bigg/\bigg(\frac{\delta}{\beta} - \sum_{j=1}^m |\alpha_j|\bigg),$$

and by (6.9),

$$\|x\| \le \|\varphi - \mathfrak{M}\{\varphi\}\|\bigg/\bigg(\frac{\delta}{\beta} - \sum_{j=1}^m |\alpha_j|\bigg) + \bigg|\mathfrak{M}\{\varphi\}\bigg/\sum_{j=1}^m \alpha_j\bigg| =: R_0. \tag{6.10}$$

Choose $R > R_0$ and let $\Omega = B_R(0)$ be the open ball centered at 0 with radius R in $AP_{H_0}(\mathbb{R},\mathbb{R})$. Then $\mathcal{L}x \ne \mu\mathcal{N}x$ for each $x \in \partial\Omega \cap \operatorname{dom}\mathcal{L}$ and $\mu \in (0,1)$ by the above priori estimate. Hence condition (C1) of Theorem 2.13 is true for the pair $(\mathcal{L},\mathcal{N})$ on Ω.

2. It is easy to see that $\partial\Omega \cap \ker\mathcal{L} = \{R, -R\}$. For any $x_0 \in \partial\Omega \cap \ker\mathcal{L}$, from the definition of $\mathcal{N}$ and $\mathcal{Q}$ it follows that

$$(\mathcal{N}x_0)(t) = \bigg(\sum_{j=1}^m \alpha_j\bigg)x_0 + \varphi(t), \quad t \in \mathbb{R},$$

$$\mathcal{Q}\mathcal{N}x_0 = \bigg(\sum_{j=1}^m \alpha_j\bigg)x_0 + \mathfrak{M}\{\varphi\}.$$

If $\mathcal{Q}\mathcal{N}x_0 = 0$, then

$$|x_0| = R = \bigg|\mathfrak{M}\{\varphi\}\bigg/\sum_{j=1}^m \alpha_j\bigg|,$$

which contradicts (6.10). Therefore, $\mathcal{QN}x \neq 0$ on $\partial\Omega \cap \ker \mathcal{L}$ and condition (C2) of Theorem 2.13 is true for the pair $(\mathcal{L}, \mathcal{N})$ on Ω.

3. It is easy to check that $\Omega \cap \ker \mathcal{L} = (-R, R)$. The definitions of $\mathcal{J}$, $\mathcal{Q}$ and $\mathcal{N}$ imply

$$\mathcal{JQN}x = \mathfrak{M}\{\mathcal{N}x\} = \bigg(\sum_{j=1}^{m} \alpha_j\bigg) x + \mathfrak{M}\{\varphi\}$$

for all $x \in \Omega \cap \ker \mathcal{L}$. A direct calculation shows that

$$\mathcal{JQN}x = \mathfrak{M}\{\varphi\} \pm R \cdot \sum_{j=1}^{m} \alpha_j \neq 0$$

on $\partial\Omega \cap \ker \mathcal{L}$. From the definition of degree for non-degenerate linear maps [11] it follows that

$$\deg(\mathcal{JQN}, \Omega \cap \ker \mathcal{L}, 0) = \operatorname{sgn}\bigg(\sum_{j=1}^{m} \alpha_j\bigg) = \pm 1 \neq 0.$$

Thus condition (C3) of Theorem 2.13 is true for the pair $(\mathcal{L}, \mathcal{N})$ on Ω. Consequently, equation (6.7) has at least one solution in $\overline{\Omega} \cap AP^1_{H_0}(\mathbb{R}, \mathbb{R})$ by Theorem 2.13.

Uniqueness. Suppose that $\phi \in AP^1_{H_0}(\mathbb{R}, \mathbb{R})$ is a solution to equation (6.7), and

$$\varphi(t) \sim \varphi_0 + \sum_{k=1}^{\infty} (\varphi_k e^{i\lambda_k t} + \overline{\varphi_k} e^{-i\lambda_k t}) \in AP_{H_0}(\mathbb{R}, \mathbb{R}),$$

$$\phi(t) \sim \phi_0 + \sum_{k=1}^{\infty} (\phi_k e^{i\lambda_k t} + \overline{\phi_k} e^{-i\lambda_k t}) \in AP^1_{H_0}(\mathbb{R}, \mathbb{R}).$$

It follows that

$$\phi'(t) \sim \sum_{k=1}^{\infty} (i\lambda_k \phi_k e^{i\lambda_k t} - i\lambda_k \overline{\phi_k} e^{-i\lambda_k t}),$$

$$\phi(t + \tau_j) \sim \phi_0 + \sum_{k=1}^{\infty} (\phi_k e^{i\lambda_k \tau_j} e^{i\lambda_k t} + \overline{\phi_k} e^{-i\lambda_k \tau_j} e^{-i\lambda_k t})$$

for $j = 1, \ldots, m$. Since

$$a(\phi', \lambda) = \sum_{j=1}^{m} \alpha_j a(\phi(\tau_j + \cdot), \lambda) + a(\varphi, \lambda)$$

for all $\lambda \in \mathbb{R}$, there holds

$$\phi_k = \varphi_k \bigg/ \bigg(i\lambda_k - \sum_{j=1}^{m} \alpha_j e^{i\lambda_k \tau_j}\bigg)$$

for each $k \in \mathbb{Z}_+$. So the solution ϕ to (6.7) is uniquely determined. □

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