Topological **M**ethods in **N**onlinear **A**nalysis
Volume 50, No. 1, 2017, 9–25
DOI: 10.12775/TMNA.2016.089

CONVEX HULL DEVIATION AND CONTRACTIBILITY

Grigory M. Ivanov

Abstract. We study the Hausdorff distance between a set and its convex hull. Let X be a Banach space, define the CHD-constant of the space X as the supremum of this distance over all subsets of the unit ball in X. In the case of finite dimensional Banach spaces we obtain the exact upper bound of the CHD-constant depending on the dimension of the space. We give an upper bound for the CHD-constant in L_p spaces. We prove that the CHD-constant is not greater than the maximum of Lipschitz constants of metric projection operators onto hyperplanes. This implies that for a Hilbert space the CHD-constant equals 1. We prove a characterization of Hilbert spaces and study the contractibility of proximally smooth sets in a uniformly convex and uniformly smooth Banach space.

1. Introduction

Let X be a Banach space. For a set $A \subset X$, we denote by $\partial A, \operatorname{int} A$ and $\operatorname{co} A$ the boundary, interior and convex hull of A, respectively. We use $\langle p, x\rangle$ to denote the value of the functional $p \in X^*$ at the vector $x \in X$. For $R > 0$ and $c \in X$ we denote by $B_R(c)$ a closed ball with center c and radius R. We denote the origin by 0.

2010 *Mathematics Subject Classification.* 52A07, 52A27, 52A35.

Key words and phrases. Hausdorff distance; characterization of inner product spaces; contractibility of a union of balls.

Research was partially supported by the Swiss National Science Foundation grants 200020-165977 and 200021-162884, and supported by the Russian Foundation for Basic Research, project 16-01-00259.

By $\rho(x, A)$ we denote the distance between a point $x \in X$ and a set A. We define the deviation from a set A to a set B as follows:

$$h^+(A, B) = \sup_{x \in A} \rho(x, B). \tag{1.1}$$

For the case $B \subset A$, which takes place below, the deviation $h^+(A, B)$ coincides with the Hausdorff distance between the sets A and B.

Given $D \subset X$, the deviation $h^+(\operatorname{co} D, D)$ is called the *convex hull deviation* (CHD) of D. We define the *CHD-constant* ζ_X of X as

$$\zeta_X = \sup_{D \subset B_1(0)} h^+(\operatorname{co} D, D).$$

REMARK 1.1. Directly from our definition it follows that for any normed linear space X we have $1 \leq \zeta_X \leq 2$.

We denote by ℓ_p^n the n-dimensional real vector space endowed with p-norm.

This article presents estimates for the CHD-constant for different spaces and some of its geometrical applications. In particular, for finite-dimensional spaces we obtain the exact upper bound of the CHD-constant depending on the dimension of the space:

THEOREM 1.2. *Let X_n be a normed linear space,* $\dim X_n = n \geq 2$, *then* $\zeta_{X_n} \leq 2(n-1)/n$. *If $X_n = \ell_1^n$ or $X_n = \ell_\infty^n$, then this bound is tight.*

Let the sets P and Q be the intersections of the unit ball with two parallel affine hyperplanes of dimension k, where P is a central section. In Corollary 2.3 we obtain the exact upper bound of the homothety coefficient, that provides covering of Q by P.

The next theorem gives an estimate for the CHD-constant in the L_p spaces, $1 \leq p \leq +\infty$:

THEOREM 1.3. *For any* $p \in [1, +\infty]$

$$\zeta_{L_p} \leq 2^{|1/p - 1/p'|}, \quad \textit{where } \frac{1}{p} + \frac{1}{p'} = 1. \tag{1.2}$$

Theorem 3.3 shows that the CHD-constant is not greater than the maximum of the Lipschitz constants of metric projection operators onto hyperplanes. This implies that for a Hilbert space the CHD-constant equals 1. Besides that, we prove a characterization of a Hilbert space in terms of the CHD-constant. The idea of the proof is analogous to the idea used by A.L. Garkavi in [9].

THEOREM 1.4. *The equation $\zeta_X = 1$ holds for a Banach space X if and only if X is a Euclidian space or* $\dim X = 2$.

In addition we study the contractibility of a covering of a convex set with balls.

Definition 1.5. A covering of a convex set with balls is called *admissible* if it consists of a finite number of balls with centres in this set and with the same radii.

Definition 1.6. A family of balls is called *admissible* when it is an admissible covering of the convex hull of its centres.

We say that a covering of a set by balls is contractible when the union of these balls is contractible. It is easy to show that in two-dimensional and Hilbert spaces any admissible covering is contractible (see Lemmas 2.4 and 2.5). On the other hand, using Theorem 1.4, we prove the following statement.

Theorem 1.7. *In a three dimensional Banach space X every admissible covering is contractible if and only if X is a Hilbert space.*

For 3-dimensional spaces we consider an example of an admissible covering of a convex set with four balls that is not contractible. To demonstrate the usefulness of this technique in Theorem 5.1 we obtain a sufficient condition for the contractibility of proximally smooth sets in a uniformly convex and uniformly smooth Banach space.

2. Proof of Theorem 1.2 and some other results

Lemma 2.1. *Suppose the set $B_1(o)\setminus \operatorname{int} B_r(o_1)$ is non-empty in an arbitrary linear normed space. Then it is arcwise connected.*

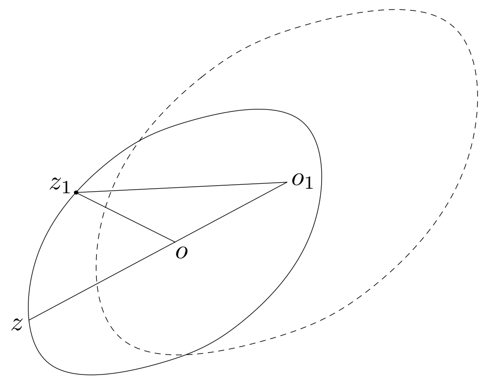

Figure 1. Illustration for Lemma 2.1. In the notations of the lemma (point z_1 is an arbitrary point of the sphere $\partial B_1(o)$), we have $\|z_1 - o_1\| \le \|z_1 - o\| + \|o - o_1\| = \|z - o_1\|$.

Proof. We suppose that $o \neq o_1$, otherwise the statement is trivial. Let z be the point of intersection of ray $o_1 o$ and the boundary of the closed ball $B_1(o)$ such that $o \in [z, o_1]$. The triangle inequality tells us that $B_1(o)\setminus \operatorname{int} B_r(o_1)$ contains z

(see Figure 1). We claim that $\partial B_1(o) \setminus \operatorname{int} B_r(o_1)$ is arcwise connected and thus prove the lemma. It suffices to show that in the two-dimensional case every point of the set $S = \partial B_1(o) \setminus \operatorname{int} B_r(o_1)$ is connected with z. Suppose on the contrary that it is not true. Hence, there exists a point $d \in S$ unconnected to z. By the triangle inequality we have that d does not lie on the line oo_1. Therefore on both arcs dz of the unit circle $\partial B_1(o)$ we can find points from $\operatorname{int} B_r(o_1)$. One of the arcs dz lies in the half-plane defined by the line oo_1, denote this arc as ω. Let c be a point from $\omega \cap \operatorname{int} B_r(o_1)$. Then $\|c - o_1\| < r$.

There exist points $a_1, b_1 \in \omega \cap \partial B_r(o_1)$ such that c lies on the arc a_1b_1 of the unit circle $\partial B_1(o)$. Consider two additional rays oa and ob co-directional with o_1a_1 and o_1b_1 respectively, where $a, b \in \partial B_1(o)$. Since balls $B_1(o)$ and $B_r(o_1)$ are similar, we have $a_1b_1 \parallel ab$. So, the facts that points a, b, a_1, b_1 lie on the same side of the line oo_1, $oa \cap o_1a_1 = \emptyset, ob \cap o_1b_1 = \emptyset$ and that a unit ball is convex, imply that segments ab and a_1b_1 lie on the same line. This means that segments ab and a_1b_1 belong to the circles $\partial B_1(o)$ and $\partial B_r(o_1)$ respectively. And what is more, the segment a_1b_1 belongs to the circle $\partial B_1(o)$, hence the point c belongs to the segment a_1b_1 and $\|c - o_1\| = r$. This contradicts $\|c - o_1\| < r$. □

PROOF OF THEOREM 1.2. Denote $r_n = 2(n-1)/n$. Suppose the inequality does not hold. It means that there exists a Banach space X_n with dimension $n \geq 2$, a set $D \subset B_1(0) \subset X_n$ and a point $o_1 \in \operatorname{co} D$ such that $B_{r_n}(o_1) \cap D = \emptyset$. But if $o_1 \in \operatorname{co} D$, then $o_1 \in \operatorname{co}(B_1(0) \setminus \operatorname{int} B_{r_n}(o_1))$. According to Lemma 2.1, the set $B = B_1(0) \setminus \operatorname{int} B_{r_n}(o_1)$ is connected. So, taking into consideration the generalized Carathéodory theorem ([16], Theorem 2.29), we see that the point o_1 is a convex combination of not more than n points from B. These points, denoted as $a_1, \dots, a_k$, $k \leq n$, may be regarded as vertices of a $(k-1)$-dimensional simplex A and the point $o_1 = \alpha_1 a_1 + \ldots + \alpha_k a_k$ lies in its relative interior ($\alpha_i > 0$, $\alpha_1 + \ldots + \alpha_k = 1$).

Let c_l be the point of intersection of the ray $a_l o_1$ with the opposite facet of the simplex A. So, $o_1 = \alpha_l a_l + (1 - \alpha_l) c_l$. Then

$$\|o_1 - a_l\| = (1 - \alpha_l)\|c_l - a_l\|.$$

And $[c_l, a_l] \subset A \subset B_1(0)$ implies that $\|a_l - c_l\| \leq 2$, for all $l \in \overline{1,k}$. Therefore $r_n < \|o_1 - a_l\| \leq 2(1-\alpha_l)$. Thus $\alpha_l < 1 - r_n/2 < 1/n$, and finally $\alpha_1 + \ldots + \alpha_k < k/n \leq 1$. We reach a contradiction.

Now let us show that the bound is tight for spaces the ℓ_1^n, ℓ_∞^n. Consider ℓ_1^n. Let $A = \{e_i\}_{i=1}^n$ be a standard basis for the space ℓ_1^n and $b = (e_1 + \ldots + e_n)/n \in \operatorname{co}\{e_1, \dots, e_n\}$. The distance between the point b and an arbitrary point from A is $\|a_i - b\| = 2(n-1)/n$.

Consider ℓ_∞^n. Let $a_{ij} = (-1)^{\delta_{ij}}$, where δ_{ij} is the Kronecker symbol, $a_i = (a_{i1}, \dots, a_{in})$ and $A = \{a_i\}_{i=1}^n$. Now let $b = (a_1 + \ldots + a_n)/n = ((n-2)/n, \dots,$

$(n-2)/n) \in \operatorname{co}\{a_1, \dots, a_n\}$. And the distance from point b to an arbitrary point from A is $\|a_i - b\| = 2(n-1)/n$. □

So, Theorem 1.2 and the inequality $\zeta_X \geq 1$ imply that the CHD-constant of any 2-dimensional normed space equals 1. Obviously, the CHD-constant of the infinite dimensional ℓ_1 space equals 2.

Clearly, in the definition of the CHD-constant we can consider only sets like $B_1(0) \setminus B_r(a)$. According to Lemma 2.1, such sets are arcwise connected. So, due to the generalized Carathéodory theorem and the Blashke selection theorem [15, Theorem 1.3.3], we have:

REMARK 2.2. Let X be a Banach space, $\dim X = n$. Then for every $d < \zeta_X$ there exists a set A that consists of not more than n points and meets the condition $h^+(\operatorname{co} A, A) = d$.

The following is a generalization of a result due to K. Leichtweiss [12].

COROLLARY 2.3. *Let sets P and Q be intersections of the unit ball with two parallel affine hyperplanes of dimension k, and let the hyperplane containing P contain* 0 *as well. Then it is possible to cover Q with the set* $\min\{2k/(k+1) : \zeta_X\}P$ *using a parallel translation.*

PROOF. Define $\eta = \min\{2k/(k+1) : \zeta_X\}$. Due to the Helly theorem it suffices to prove that we can cover any k-simplex $\Delta \subset Q$ with the set ηP.

Let us consider the k-simplex $\Delta \subset Q$ with vertices $\{x_1, \dots, x_{k+1}\}$. By the definition of ζ_X and by Theorem 1.2, for any set of indices $I \subset \overline{1,(k+1)}$, we have $\operatorname*{co}_{i\in I}\{x_i\} \subset \bigcup_{i\in I}(B_\eta(x_i) \cap \Delta)$. Using the KKM theorem [11], we obtain that $S = \bigcap_{i\in\overline{1,(k+1)}}(B_\eta(x_i) \cap \Delta) \neq \emptyset$. Then $\Delta \subset B_\eta(s)$, where $s \in S \subset \Delta$. □

Let us show that Hilbert and 2-dimensional Banach spaces satisfy the assumptions of Theorem 1.7. We consider the area covered with balls to be shaded. The balls' radii may be taken equal to 1. We will complete the proof of Theorem 1.7 in Section 4.

LEMMA 2.4. *Let X be a Banach space,* $\dim X = 2$, *then any admissible covering is contractible.*

PROOF. Without loss of generality, let us consider an admissible covering of a convex set V by balls $B_1(a_i)$, $i = \overline{1,n}$. Let us set $S = \bigcup_{i\in\overline{1,n}} B_1(a_i)$. Since the unit ball is a convex closed body, the set S is homotopically equivalent to its nerve [1], in our case it is a finite CW complex. Therefore, S is contractible if and only if it is connected, simply connected and its homology groups $H_k(S)$ are trivial for $k \geq 2$. Obviously, S is a connected set.

Let us show that the set S is simply connected and $H_k(S) = 0$ for $k \geq 2$. The unit circle is a continuous closed line without self-intersections, since the unit ball is convex. It divides a plane into two parts. A finite set of circles divides a plane in a finite number of connected components. Let us now shade the unit balls.

We prove now that the problem is stable with respect to small perturbations of the norm. To be more precise, if a norm does not meet the conclusions of the theorem, then there exists a polygon norm, which does not meet them either.

Let us choose a bounded uncovered area U with shaded boundary (area U may be non-convex). It is possible to put a ball of radius $3\varepsilon_1$, $\varepsilon_1 > 0$, inside this area. There exists ε_2, $\varepsilon_2 > 0$, such that if $B_1(a_{i_1}) \cap B_1(a_{i_2}) = \emptyset$ for $i_1, i_2 \in \overline{1,n}$, then $B_{1+\varepsilon_2}(a_{i_1}) \cap B_{1+\varepsilon_2}(a_{i_2}) = \emptyset$. Denote $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$.

Consider the following set:

$$B_1^c(0) = \bigcap_{p \in C} \{x : \langle p, x \rangle \leq 1\},$$

where C is a finite set of unit vectors from the space X^* such that $C = -C$. So, $B_1^c(0)$ is the unit ball for some norm. For an arbitrary unit functional p we have that $B_1(0) \subset \{x : \langle p, x \rangle \leq 1\}$, then $B_1(0) \subset B_1^c(0)$. According to [15, Corollary 2.6.1], it is possible to pick a set C such that $h^+(B_1^c(0), B_1(0)) \leq \varepsilon$. Then the set of balls $B_1^c(a_i)$, $i = \overline{1,n}$, is an admissible covering, contains the boundary of U and it does not cover U entirely. Furthermore the nerve and, consequently, the homology group of the sets $\bigcup\limits_{i \in \overline{1,n}} B_1^c(a_i)$ and S coincide.

Now it suffices to show that the statement of the lemma is true in the case of a polygon norm. In this case the set S and the unit ball are polygons, then S is a neighbourhood retract in $\mathbb{R}^2$ (see [17], Chapter 3, §Regular Neighbourhoods), therefore directly from Alexander duality (see [7], Chapter 4, § 6) we obtain that $H_k(S) = 0$ for $k \geq 2$.

Now we shall prove that S is simply connected. Assume the contrary, that is, there exist a norm, an admissible covering of a convex set V by balls $B_1(a_i)$, $i = \overline{1,n}$, and a non-shaded bounded set U with a shaded boundary. Note that its boundary appears to be a closed polygonal line without self intersections. Let $A = \operatorname{co}\{a_i : i = \overline{1,n}\}$.

Let x be an arbitrary point of the set U. The union of the balls $B_1(a_i)$ is an admissible covering of the set A, thus $x \notin A$. Then there exists a line l_a that separates x from the set A. This line may serve as a supporting line of the set A ([1]). Let $l \parallel l_a$ be a supporting line of U in a point v such that sets U and A lie on the same side of the line l (see Figure 2).

([1]) By a supporting line of a compact (not necessary convex) set we mean a line that intersects the set and for which the entire set is contained in one of closed half-spaces defined by the line.

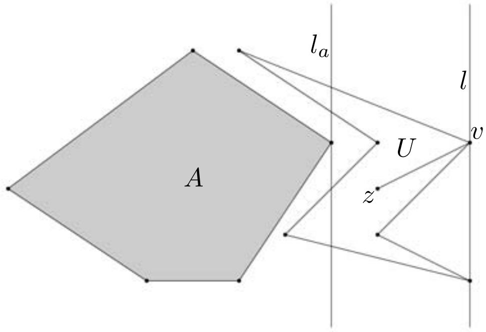

Figure 2. Illustration for Lemma 2.4.

The line l divides the plane into two half-planes. Let H_+ be the half-plane that does not contain A, we denote the other half-plane as H_- (i.e. $A \subset H_-$ and $U \subset H_-$). Let points $p, q \in l$ lie on different sides from v. We want to choose all the edges of the polygonal curve ∂U, that contain the point v. We will call them vb_i, $i \in \overline{1,k}$: $\cos\angle pvb_i > \cos\angle pvb_j$, $i > j$.

Since $v \in \partial U$, it follows that there exists a point z such that the interior of the segment vz lies in U and the ray vz lies between vb_1 and vb_k. Then, since the ball is convex, there is no ball $B_1(a_i)$ that simultaneously covers segments $[v, b_1^{'}]$ and $[v, b_k^{'}]$, where $b_1^{'}, b_k^{'}$ are arbitrary interior points of the segments vb_1, vb_k, respectively. Therefore the point v is covered by at least two balls, and the centres a_i, a_j of these balls are separated by the ray vz in the half-plane H_-. Again, since the ball is convex, the point $y = vz \cap a_ia_j$ is not covered by balls $B_1(a_i), B_1(a_j)$, thus $\|a_i - a_j\| = \|y - a_i\| + \|y - a_j\| > 2$, which contradicts the fact that a_i and a_j are contained in the ball $B_1(v)$. □

Lemma 2.5. *Let X be a Euclidean space. Then any admissible covering is contractible.*

Proof. Recall that a closed convex set is contractible and in a Hilbert space the projection onto a closed convex set is unique. Since a projection onto a convex set is a continuous function of the projected point, it is enough to prove that a line segment, which connects a shaded point with its projection onto a convex hull of centres of an admissible covering, is shaded. Suppose that we have an admissible set of balls. The convex hull of its center is a polygon. Let us call it C. If a shaded point a is projected onto the v-vertex of the polygon, then the segment av is shaded as well. Let a shaded point a, lying in the ball $B_1(v)$ from a set of balls, be projected onto the point $b \neq v$. Let L be a hyperplane passing through the point b and perpendicular to the line segment $[a, b]$. It divides the space into two half-spaces. The one with the point a we call H_a. C is convex, thus it contains the segment $[v, b]$. Then it is impossible for the point v to lie in H_a, so $\angle abv \geq \pi/2$, i.e. $\|v-a\| \geq \|v-b\|$. Thus, $b \in B_1(v)$ and, consequently, $ab \subset B_1(v)$. □

3. Upper bound for the CHD-constant in a Banach space

Let $J_1(x) = \{p \in X^* : \langle p, x\rangle = \|p\| \cdot \|x\| = \|x\|\}$. Let us introduce the following characteristic of a space:

$$\xi_X = \sup_{\substack{\|x\|=1,\\ \|y\|=1}} \sup_{p \in J_1(y)} \|x - \langle p, x\rangle y\|.$$

Note that if $y \in \partial B_1(0)$, $p \in J_1(y)$, then the vector $(x - \langle p, x\rangle y)$ is a metric projection of x onto the hyperplane $H_p = \{x \in X : \langle p, x\rangle = 0\}$. Denote by ξ_X^p the norm of the linear operator $x \mapsto (x - \langle p, x\rangle y)$, i.e. $\xi_X^p = \sup\limits_{x \neq 0} \|x - \langle p, x\rangle y\|/\|x\|$. For arbitrary vectors $a, b \in X$, we have

$$\|(a - \langle p, a\rangle y) - (b - \langle p, b\rangle y)\| = \|(a-b) - \langle p, (a-b)\rangle y\| \leq \xi_X^p \|a - b\|$$

and by the definition of ξ_X^p this inequality is tight, hence ξ_X^p is the Lipschitz constant for the metric projection operator onto H_p (here we project along the vector y). Since the unit ball and its metric projection onto any hyperplane are convex and centrally symmetric, and $(x - \langle p, x\rangle y) \in H_p$, we have that this Lipschitz constant equals to half of the diameter of the unit ball's projection onto the hyperplane H_p. Clearly, $\xi_X = \sup\limits_{\|y\|=1} \sup\limits_{p \in J_1(y)} \xi_X^p$. Therefore, ξ_X is the maximal value of the Lipschitz constant for metric projection operators onto a hyperplane. Obviously, $1 \leq \xi_X \leq 2$ and $\xi_H = 1$ for a Hilbert space H.

Let us use ξ_X to estimate the CHD-constant of X:

Lemma 3.1. *Let $y \in \operatorname{co}[B_1(0) \setminus \operatorname{int} B_r(y_1)]$ and let $p \in J_1(y)$. Then there exists $x \in B_1(0) \setminus \operatorname{int} B_r(y_1)$ such that $\langle p, x\rangle = \langle p, y\rangle$.*

Proof. Let $B = B_1(0) \setminus \operatorname{int} B_r(y)$. Since $y \in \operatorname{co} B$, there exist points $a_1, \ldots, a_n \in B$ and a set of positive coefficients $\lambda_1, \ldots, \lambda_n$, $\lambda_1 + \ldots + \lambda_n = 1$, such that

$$y = \lambda_1 a_1 + \ldots + \lambda_n a_n. \tag{3.1}$$

Let $H_p^+ = \{x \in X : \langle p, x\rangle \geq \langle p, y\rangle\}$. According to Lemma 2.1, the set B is connected, thus, since $B \setminus H_p^+$ is not empty, if the statement of the lemma is not true, we arrive at $B \cap H_p^+ = \emptyset$. Then $\langle p, a_i\rangle < \langle p, y\rangle$ and formula (3.1) implies

$$\langle p, y\rangle = \lambda_1\langle p, a_1\rangle + \ldots + \lambda_n\langle p, a_n\rangle < \langle p, y\rangle.$$

This is a contradiction. □

Lemma 3.2. $\zeta_X \leq \sup\limits_{\|y\| \in B_1(0)} \inf\limits_{p \in J_1(y)} \sup\limits_{\substack{x \in B_1(0):\\ \langle p, x-y\rangle = 0}} \|x - y\|.$

Proof. Let ε be a positive real number. Then, according to the definition of the CHD-constant, there exists a set $D \subset B_1(0)$ such that $h^+(\operatorname{co} D, D) \geq \zeta_X - \varepsilon$. It means that there exists a point $y \in \operatorname{co} D \setminus \{0\}$ such that $\rho(y, D) \geq \zeta_X - 2\varepsilon$. Let $r = \rho(y, D)$. So, $D \subset B_1(0) \setminus \operatorname{int} B_r(y)$. Hence, $y \in \operatorname{co}[B_1(0) \setminus \operatorname{int} B_r(y)]$.

Now let $p \in J_1(y)$. According to Lemma 3.1, there exists a vector $x \in B_1(0) \setminus \operatorname{int} B_r(y)$ such that $\langle p, x - y\rangle = 0$ and $r \leq \|x - y\|$. Therefore,

$$\zeta_X \leq \rho(y, D) + 2\varepsilon = r + 2\varepsilon \leq \|x - y\| + 2\varepsilon.$$

Now let ε tend to zero. The lemma is proved. □

Theorem 3.3. $\zeta_X \leq \xi_X$.

Proof. By Lemma 3.2, it is enough to show that

$$\xi_X = \sup_{\substack{\widehat{y} \in B_1(0),\\ p \in J_1(\widehat{y})}} \ \sup_{\substack{x \in B_1(0):\\ \langle p, x - \widehat{y}\rangle = 0}} \|x - \widehat{y}\|. \tag{3.2}$$

Denote the right-hand side of equality (3.2) as ξ. Note that in the definition of ξ we can assume that $\widehat{y} \neq 0$ and $\|x\| = 1$. Fix vectors $\widehat{y} \in B_1(0) \setminus \{0\}$ and $x \in \partial B_1(0)$ such that for some $p \in J_1(\widehat{y})$ we have $\langle p, x\rangle = \langle p, \widehat{y}\rangle$. Let $y = \widehat{y}/\|\widehat{y}\|$. Then $\|y\| = 1$, $p \in J_1(y)$ and $\widehat{y} = \langle p, \widehat{y}\rangle y = \langle p, x\rangle y$. Therefore, $x - \langle p, x\rangle y = x - \widehat{y}$. So, by the definition of ξ_X, we get $\xi_X \geq \xi$.

Let us show that $\xi_X \leq \xi$. In case $\xi_X = 1$ this inequality is trivial. Let $\xi_X > 1$. Fix $x, y \in \partial B_1(0)$ such that for some $p \in J_1(y)$ we have $\|x - \langle p, x\rangle y\| > 1$. Note that $\langle p, x\rangle \neq 0$ and $|\langle p, x\rangle| \leq 1$. Let $\widehat{y} = \langle p, x\rangle y$. Then $x - \langle p, x\rangle y = x - \widehat{y}$ and $\widehat{y} \in B_1(0)$. Hence, we get $\xi_X \leq \xi$. Equality (3.2) is proven. □

Using Remark 1.1 and Theorem 3.3 we get

Corollary 3.4. *If H is a Hilbert space, then $\zeta_H = 1$.*

With the following lemma we can pass to finite subspace limit in the CHD-constant calculations.

Lemma 3.5. *Let X be a separable Banach space and $\{x_1, x_2, \ldots\}$ be a vector system in it such that the subspace $\check{X} = \operatorname{Lin}\{x_1, x_2, \ldots\}$ is dense in X. Then*

$$\zeta_X = \lim_{n\to\infty} \zeta_{X_n}, \qquad \text{where } X_n = \operatorname{Lin}\{x_1, \ldots, x_n\}. \tag{3.3}$$

Proof. Let $\zeta = \zeta_X$, and fix a real number $\varepsilon > 0$. Since $X_n \subset X_{n+1} \subset X$, the sequence ζ_{X_n} is monotone and bounded and, consequently, convergent. Let $\zeta_2 = \lim\limits_{n\to\infty} \zeta_{X_n}$. Since $X_n \subset X$ it follows that $\zeta_2 \leq \zeta$. According to the CHD-constant definition, there exist a set $A \subset B_1(0)$ and a point $d \in \operatorname{co} A$ such that $\rho(d, A) > \zeta - \varepsilon/2$. Since $d \in \operatorname{co} A$, there exist a natural number N, points $a_i \in A$, and numbers $\alpha_i \geq 0$, $i \in \overline{1, N}$, $\alpha_1 + \ldots + \alpha_N = 1$, such that $d = \alpha_1 a_1 + \ldots + \alpha_N a_N$. Then $\|d - a_i\| > \zeta - \varepsilon/2$, $i \in \overline{1, N}$.

Since $\overline{\check{X}} = X$, it is possible to pick points $b_i \in B_1(0) \cap \check{X}$, $i \in \overline{1,N}$, so that $\|a_i - b_i\| \le \varepsilon/4$. By the definition of a linear span, for some natural n_i we have $b_i \in X_{n_i}$. Let $M = \max n_i$, $i \in \overline{1,N}$. Consider the set $B = \{b_1, \ldots, b_N\}$ in the space X_M. Let $d_\varepsilon = \alpha_1 b_1 + \ldots + \alpha_N b_N \in \operatorname{co} B$, then

$$\|d_\varepsilon - d\| = \left\| \sum_{j=1}^{N} \alpha_j (b_j - a_j) \right\| \le \sum_{j=1}^{N} \alpha_j \|b_j - a_j\| \le \frac{\varepsilon}{4},$$

so, for every $i \in \overline{1,N}$, we have

$$\|d_\varepsilon - b_i\| = \|(d_\varepsilon - d) + (d - a_i) + (a_i - b_i)\| \ge \|d - a_i\| - \|d_\varepsilon - d\| - \|a_i - b_i\| \ge \zeta - \varepsilon.$$

Thus $\zeta - \varepsilon \le h^+(\operatorname{co} B, B) \le \zeta_{X_M} \le \zeta_2 \le \zeta$, and since $\varepsilon > 0$ was chosen arbitrarily, $\zeta = \zeta_2$. □

Let $p' \in [1, +\infty]$ be such that $1/p + 1/p' = 1$, $r = \min\{p, p'\}$, $r' = \max\{p, p'\}$.

LEMMA 3.6. *Given $p \in [1, +\infty]$. Let $x_i \in L_p$, $i = 1, \ldots, k$, be such that*

$$\sum_{i=1}^{k} \alpha_i = 1, \quad \alpha_i \ge 0 \quad (i = 1, \ldots, k), \quad x_0 = \sum_{i=1}^{k} \alpha_i x_i.$$

Then

$$\left(\sum_{i=1}^{k} \alpha_i \|x_i - x_0\|_p^r \right)^{1/r} \le 2^{-1/r'} \left(\sum_{i=1,\, j=1}^{k} \alpha_i \alpha_j \|x_i - x_j\|_p^r \right)^{1/r} \tag{3.4}$$

and

$$\left(\sum_{i=1,\, j=1}^{k} \alpha_i \alpha_j \|x_i - x_j\|_p^r \right)^{1/r} \le 2^{1/r} \max_{1 \le i \le k} \|x_i\|_p. \tag{3.5}$$

If $1 \le p \le 2$, then the latter inequality can be strengthened to

$$\left(\sum_{i=1,\, j=1}^{k} \alpha_i \alpha_j \|x_i - x_j\|_p^r \right)^{1/r} \le 2^{1/r} \left(\frac{k-1}{k} \right)^{2/p-1} \max_{1 \le i \le k} \|x_i\|_p. \tag{3.6}$$

PROOF. Inequality (3.5) follows from Schoenberg's inequalities [18, Theorem 15.1]:

$$\left(\sum_{i=1,\, j=1}^{k} \alpha_i \alpha_j \|x_i - x_j\|_p^r \right)^{1/r} \le 2^{1/r} \left(\max_{1 \le i \le k} \{1 - \alpha_i\} \right)^{2/r-1} \left(\sum_{i=1}^{k} \alpha_i \|x_i\|_p^r \right)^{1/r}.$$

Inequality (3.6) was deduced by S.A. Pichugov and V.I. Ivanov in [14, Assertion 1].

Using the Riesz–Thorin theorem for spaces with a mixed L_p-norm [18, § 14], S.A. Pichugov proved the following inequality [13, Theorem 1]:

$$\left(\sum_{i=1}^{k}\sum_{j=1}^{l}\alpha_i\beta_j\|(x_i-x_0)-(y_j-y_0)\|_p^r\right)^{1/r} \tag{3.7}$$
$$\leq 2^{-1/r'}\left(\sum_{i_1=1,\,i_2=1}^{k}\alpha_{i_1}\alpha_{i_2}\|x_{i_1}-x_{i_2}\|_p^r+\sum_{j_1=1,\,j_2=1}^{l}\beta_{j_1}\beta_{j_2}\|y_{j_1}-y_{j_2}\|_p^r\right)^{1/r},$$

where $\sum\limits_{i=1}^{k}\alpha_i=\sum\limits_{j=1}^{l}\beta_j=1$, $\alpha_i\geq 0$, $i=1,\dots,k$, $\beta_j\geq 0$, $j=1,\dots,l$, $x_0=\sum\limits_{i=1}^{k}\alpha_i x_i$, $y_0=\sum\limits_{j=1}^{l}\beta_j y_j$. Letting $y_j=0$ and $\beta_j=1/l$ in (3.7), we obtain inequality (3.4). □

Proof of Theorem 1.3. Consider the case $p\in(1;+\infty)$. For spaces L_p and an arbitrary set of vectors $A=\{x_0,\dots,x_k\}$ such that $x_0=\sum\limits_{i=1}^{k}\alpha_i x_i$, $\sum\limits_{i=1}^{k}\alpha_i=1$, $\alpha_i\geq 0$, $i\in\overline{1,k}$, $A\subset B_1(0)$ we have

$$\left(\min_{i\in\overline{1,k}}\|x_0-x_i\|_p^r\right)^{1/r}\leq\left(\sum_{i=1}^{k}\alpha_i\|x_i-x_0\|_p^r\right)^{1/r}.$$

Using (3.4) and (3.5), since the set of vectors A was chosen arbitrarily, we get $\zeta_{L_p}\leq 2^{(1/r-1/r')}=2^{|1/p-1/p'|}$. And it was shown in the proof of Theorem 1.2 that $\zeta_{\ell_1^n}=\zeta_{\ell_\infty^n}=2(n-1)/n$. Thus, $\zeta_{L_1}=\zeta_{L_\infty}=2$. □

Remark 3.7. If $1\leq p\leq 2$, then, using in the proof of Theorem 1.3 inequality (3.6) instead of (3.5), we arrive at

$$\zeta_{\ell_p^n}\leq\left(2\frac{n-1}{n}\right)^{|1/p-1/p'|}. \tag{3.8}$$

The following questions remain unanswered:

Question 3.8. Is inequality (3.8) true if $p\in(2;\infty)$?

Question 3.9. Is the estimate in inequality (1.2) sharp in the case of $p\in(1;\infty)$, $p\neq 2$?

4. Characterization of a Hilbert space

In order to prove Theorem 1.4 we need the following lemma, which it a straightforward consequence of the KKM theorem [11].

Lemma 4.1. *Let X be a Banach space. Suppose the triangle $a_1a_2a_3\subset X$ satisfies the inequality* $\operatorname{diam} a_1a_2a_3\leq 2R$ *and is covered by balls $B_R(a_i)$, $i=1,2,3$. Then these balls have a common point lying in the plane of the triangle.*

Taking into account Lemma 4.1, the proof of Theorem 1.4 is very similar to the one of Theorem 5 from [9].

Proof of Theorem 1.4. Using Theorem 1.2 and Corollary 3.4, it suffices to prove that a Banach space X, with $\dim X \geq 3$ and $\zeta_X = 1$, is a Hilbert space. According to the well-known results obtained by Fréchet and Blashke–Kakutani, it is enough to describe only the case when $\dim X = 3$. We need to show that if $\zeta_X = 1$, then for every 2-dimensional subspace there exists a unit-norm operator that projects X onto this particular subspace. Let $0 \in L$ be an arbitrary 2-dimensional subspace in X, and let c be a point not contained in L. We denote $B_n^2(0) = L \cap B_n(0)$ (it is a ball of radius $n \in \mathbb{N}$ in space L). For every $n \in \mathbb{N}$ let us introduce the following notations:

$$E_n = \{x \in L : \|c - x\| \leq n\}, \qquad F_n = \{x \in L : \|c - x\| = n\}.$$

If n is big enough, these sets are nonempty. Let x_1, x_2, x_3 be arbitrary points from E_n. The CHD-constant of space X equals 1, so the balls $B_n^2(x_i)$, $i = 1, 2, 3$, cover the triangle $x_1x_2x_3$. According to Lemma 4.1, their intersection is not empty. According to the Helly theorem, the set

$$S_n = \bigcap_{x \in E_n} B_n^2(x)$$

is non-empty as well.

Let us pick $a_n \in S_n$, then by construction we have for every $x \in F_n$

$$\|x - a_n\| \leq \|x - c\|. \tag{4.1}$$

Let us show that $\|x - a_n\| \leq \|x - c\|$ for every $x \in E_n$. Suppose that for some $x \in E_n$

$$\|x - a_n\| > \|x - c\|. \tag{4.2}$$

According to (4.1), we may assume that $x \in E_n \setminus F_n$. The set E_n is bounded and its boundary relatively to the subspace L coincides with F_n, thus there exists a point $b \in F_n$ such that x is contained in the interval (a_n, b). Then $a_n - x = \lambda(a_n - b)$, $0 < \lambda < 1$.

Note that $c - x = (c - a_n) + (a_n - x) = c - a_n + \lambda(a_n - b)$, then (4.2) may be reformulated as $\|c - a_n + \lambda(a_n - b)\| < \lambda\|a_n - b\|$. So,

$$\begin{aligned}\|c - b\| &= \|(c - a_n) + \lambda(a_n - b) + (1 - \lambda)(a_n - b)\| \\ &\leq \|(c - a_n) + \lambda(a_n - b)\| + (1 - \lambda)\|a_n - b\| \\ &< \lambda\|a_n - b\| + (1 - \lambda)\|a_n - b\| = \|a_n - b\|,\end{aligned}$$

and it contradicts (4.1).

Consider the sequence $\{a_n\}$. Note that $E_n \subset E_{n+1}$ and $\bigcup\limits_{i=1}^{\infty} E_i = L$. So, starting with a fixed natural k, the inclusion $0 \in E_n$, $n \geq k$, becomes true, thus when $x = 0$ inequality (4.1) implies $\|a_n\| \le \|c\|$, $n \ge k$, i.e. the sequence $\{a_n\}$ is bounded. It means that this sequence $\{a_n\}$ has a limit point a. Then every point $x \in L$ satisfies $\|x - a\| \le \|x - c\|$. Let us now represent every element $z \in X$ in the form

$$z = tc + x, \quad x \in L,\ t \in \mathbb{R}.$$

The operator $P(z) = P(tc + x) = ta + x$ projects X onto L. In addition,

$$\|P(z)\| = \|ta + x\| = |t| \left\| a + \frac{x}{t} \right\| \le |t| \left\| c + \frac{x}{t} \right\| = \|tc + x\| = \|z\|.$$

Hence, $\|P\| = 1$ and taking into consideration the theorem of Blashke and Kakutani we come to a conclusion that X is a Hilbert space. □

PROOF OF THEOREM 1.7. It remains to check that in every Banach space X that is not a Hilbert one, where $\dim X = 3$, there exist a convex set and an admissible and non-contractible covering.

To make the proof easier we first present a simple statement from geometry. Let a hyperplane H divide the space X in two half-spaces H_+, H_-. Let M be a bounded set in H. We want to cover the set M with balls

$$B = \left\{ \bigcup B_d(a_i) : i \in \overline{1, n},\ n \in \mathbb{N} \right\}.$$

Let us call such covering (ε, d, H_+)-good if $h^+(B, H_-) \le \varepsilon$.

LEMMA 4.2. *Let X be a Banach space, $3 \le \dim X < +\infty$. Let a hyperplane H divide X in two half-spaces H_+ and H_-. Let M be a bounded set in H. Then, for every $\varepsilon > 0$, $d > 0$, there exists an admissible set of balls $B_d(a_i) : i \in \overline{1, N}$, $N \in \mathbb{N}$ such that the set $B = \bigcup_{i \in \overline{1,N}} B_d(a_i)$ may be regarded as an (ε, d, H_+)-good covering of the set M and $\operatorname{co}(M \cup \{a_i\}) \subset B$, $i \in \overline{1, N}$.*

PROOF. Let $\dim X = n$. Without loss of generality we assume that $\varepsilon < d$ and H is the supporting hyperplane for the ball $B_d(0)$ and $B_d(0) \subset H_-$. For any $r > 0$ and $a \in X$ we use $C_r(a)$ to denote an $(n-1)$-dimensional hypercube centered at a that lies in the hyperplane parallel to H, where r is the length of its edges. Let $x \in H \cap B_d(0)$. Then $h^+(B_d(\varepsilon x/\|x\|), H_-) \le \varepsilon$. Let $D = B_d(\varepsilon x/\|x\|) \cap H$. Note that x is an inner point of the set D relatively to H. In a finite-dimensional linear space all norms are equivalent, so $C_r(x) \subset D$ for some $r > 0$. As the ball $B_d(\varepsilon x/\|x\|)$ is centrally symmetric, it contains the affine hypercube $\operatorname{co}(C_r(x) \cup C_r(\varepsilon x/\|x\|))$. Consider next an arbitrary bounded set $M \subset H$. Since it is bounded, $M \subset C_R(b)$, where $b \in H$, $R > 0$. We suppose that $R = kr$, $k \in \mathbb{N}$. Let us split the hypercube $C_R(b)$ into hypercubes with edges of length r and let b_i, $i \in \overline{1, N}$, be the centres of these hypercubes. Hence, from the above arguments, the balls $B_d(b_i - (d - \varepsilon)x/\|x\|)$ give us the necessary covering. □

Let us consider an approach to constructing an admissible and non-contractible covering of a convex set. Let X be a non-Hilbert Banach space, with $\dim X = 3$. According to Theorem 1.4, $\zeta_X > 1$, and by Remark 2.2, there exist a set $A = \{a_1, a_2, a_3\} \subset B_1(0)$ and a point $b \in \operatorname{co} A$ such that $\rho(b, A) = 1+4\varepsilon > 1$. According to Theorem 1.2, $0 \notin H$. Consider the balls $B_{1+\varepsilon}(a_i)$, $i \in \overline{1,3}$, let $B_1 = B_{1+\varepsilon}(a_1) \cup B_{1+\varepsilon}(a_2) \cup B_{1+\varepsilon}(a_3)$. It is obvious that $b \notin B_1$. Since all the edges of the triangle $a_1a_2a_3$ lie in B_1, facets $0a_1a_2, 0a_1a_3, 0a_2a_3$ of the tetrahedron $0a_1a_2a_3$ lie in B_1. Let H be a plane passing through the points a_1, a_2, a_3.

Let H divide the space X in two half-spaces: H_+ and H_-. Let $0 \in H_+$. According to Lemma 4.2 there exists an $(\varepsilon, 1+\varepsilon, H_+)$-good covering of the triangle $a_1a_2a_3$ with an admissible set of balls that have centres lying in a set $C = \{c_i : i \in \overline{1,N}\}$, $N \in \mathbb{N}$. Let $B_2 = \bigcup\limits_{i\in\overline{1,N}} B_{1+\varepsilon}(c_i)$. Then the set $B = B_1 \cup B_2$ contains all the facets of the tetrahedron $oa_1a_2a_3$ and does not contain the interior of the ball $B_\varepsilon(b - 2\varepsilon b/\|b\|)$, i.e. the set B is non-contractible. However, $\operatorname{co}(A \cup C) \subset B_2 \subset B$, i.e. the union of balls $B_{1+\varepsilon}(x)$, $x \in A \cup C$, is an admissible covering for the set $\operatorname{co}(A \cup C)$ we were looking for. $\square$

There remain still some open questions:

QUESTION 4.3. What is the minimal number of balls in an admissible and non-contractible set of balls for a certain space X? How to express this number in terms of space characteristics, such as its dimension, modulus of smoothness and modulus of convexity?

QUESTION 4.4. How to estimate the minimal density (in terms of average distance between centres or in a some other way) of an admissible covering with balls for it to be contractible?

According to Lemma 4.1, it takes at least four balls to construct an admissible non-contractible set of balls in an arbitrary Banach space. The following example describes the case with precisely four balls.

EXAMPLE 4.5. Let $X = l_1^3$, $a_1 = (-2/3, 1/3, 1/3)$, $a_2 = (1/3, -2/3, 1/3)$, $a_3 = (1/3, 1/3, -2/3)$, $a_4 = (-1/6, -1/6, -1/6)$. The set of balls $B_1(a_i)$, $i = \overline{1,4}$, is admissible, however, the complement of the set $B = \bigcup\limits_{i=\overline{1,4}} B_1(a_i)$ has two connected components (see Figures 3–5).

PROOF. (1) Let us show that this set of balls is admissible. Every point x from the tetrahedron $A = a_1a_2a_3a_4$ may be represented in the form $x = \alpha_1 a_1 + \ldots + \alpha_4 a_4$, where $\alpha_1 + \ldots + \alpha_4 = 1$, $\alpha_i \geq 0$, $i \in \overline{1,4}$. Using the equation $\alpha_4 = 1 - \alpha_1 - \alpha_2 - \alpha_3$, we are going to prove an inequality which would detect that the point $x \in A$ is not contained in the ball $B_1(a_4)$:

$$1 < \|x - a_4\| = \left|\frac{-\alpha_1 + \alpha_2 + \alpha_3}{2}\right| + \left|\frac{\alpha_1 - \alpha_2 + \alpha_3}{2}\right| + \left|\frac{\alpha_1 + \alpha_2 - \alpha_3}{2}\right|. \tag{4.3}$$

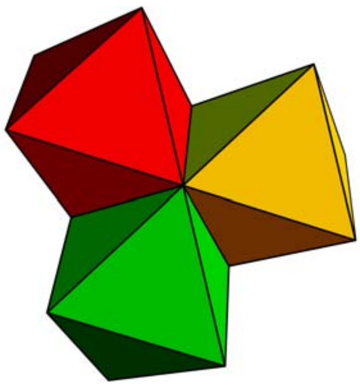

FIGURE 3. The balls $B_1(a_1)$, $B_1(a_2)$, $B_1(a_3)$.

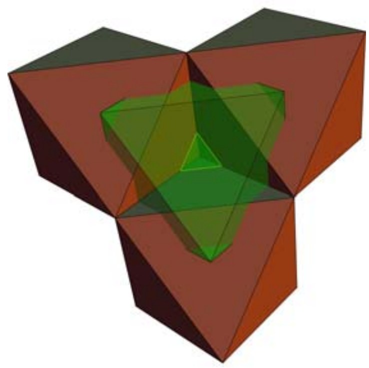

FIGURE 4. Balls $B_1(a_1)$, $B_1(a_2)$, $B_1(a_3)$ are brown. The ball $B_1(a_4)$ is green.

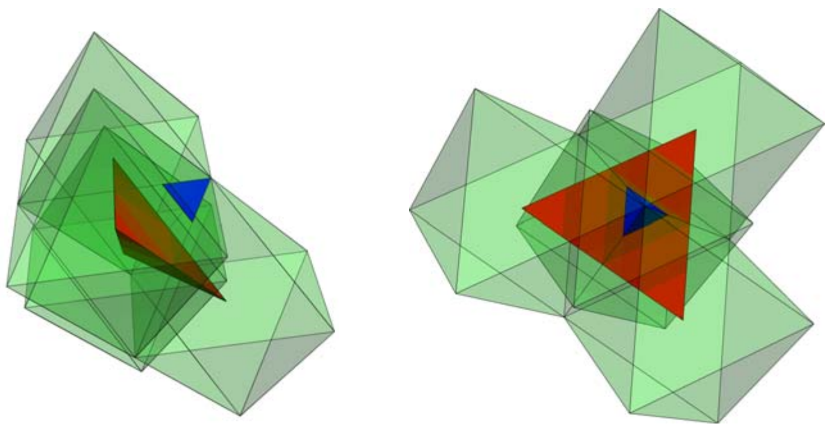

FIGURE 5. Balls $B_1(a_1)$, $B_1(a_2)$, $B_1(a_3)$, $B_1(a_4)$ are green. The tetrahedron $a_1a_2a_3a_4$ is red. The cavity is the blue tetrahedron.

We use inequality (4.3) to estimate the distance between x and the vertex a_1:

$$\begin{aligned}\|x - a_1\| &= \|\alpha_2(a_2 - a_1) + \alpha_3(a_3 - a_1) + \alpha_4(a_4 - a_1)\| \\ &\le \alpha_2\|a_2 - a_1\| + \alpha_3\|a_3 - a_1\| + \alpha_4\|a_4 - a_1\| \\ &= 2(\alpha_2 + \alpha_3) + \frac{3}{2}\alpha_4 \le 2\left(\frac{1}{4} - \frac{3}{4}\alpha_4\right) + \frac{3}{2}\alpha_4 = \frac{1}{2}.\end{aligned}$$

Note that if every expression inside the absolute values is positive, then the right-hand side of (4.3) equals $(\alpha_1 + \alpha_2 + \alpha_3)/2 \le 1/2$. So, one of them has to be negative. Without loss of generality, let $\alpha_1 \geq \alpha_2+\alpha_3$. Then the other two expressions are positive and inequality (4.3) can be rewritten: $3\alpha_1 - \alpha_2 - \alpha_3/2 > 1$. Then $\alpha_1 > 2/3 + (\alpha_2 + \alpha_3)/3$. Using this relation, we arrive at

$$1 - \alpha_4 = \alpha_1 + \alpha_2 + \alpha_3 \geq \frac{2}{3} + \frac{4}{3}(\alpha_2 + \alpha_3).$$

Thus, $1/4 - 3\alpha_4/4 \geq \alpha_2 + \alpha_3$. So, we come to a conclusion that the set of balls is admissible.

(2) Let $b_1 = (1/3, 1/12, 1/12)$, $b_2 = (1/12, 1/3, 1/12)$, $b_3 = (1/12, 1/12, 1/3)$, $b_4 = (1/3, 1/3, 1/3)$, the tetrahedron $\Delta = b_1b_2b_3b_4$. It is easy enough to show that $\partial\Delta \subset B$, but $\operatorname{int}\Delta \cap B = \emptyset$. $\square$

5. About contractibility of proximally smooth sets

Clarke, Stern and Wolenski [5] introduced and studied proximally smooth sets in a Hilbert space H. A set $A \subset X$ is said to be *proximally smooth* with constant R if the distance function $x \mapsto \rho(x, A)$ is continuously differentiable on the set $U(R, A) = \{x \in X : 0 < \rho(x, A) < R\}$. Properties of proximally smooth sets in a Banach space and relations between such sets and akin classes of sets, including uniformly prox-regular sets, were investigated in [5], [4], [8], [6]. We study a sufficient condition for the contractibility of a proximal smooth set. G.E. Ivanov showed that if $A \subset H$ is proximally smooth (weakly convex in his terminology) with constant R and $A \subset B_r(o)$ with $r < R$, then A is contractible. The following theorem is a generalization of this result.

THEOREM 5.1. *Let X be a uniformly convex and uniformly smooth Banach space. Let A be a closed and proximally smooth with constant R, assume also that A is contained on a ball of radius $r < R/\zeta_X$. Then A is contractible.*

PROOF. Note that the set $\operatorname{co} A$ is contractible, so a continuous function $F\colon [0,1] \times \operatorname{co} A \to \operatorname{co} A$ such that $F(0, x) = x$, $F(1, x) = x_0$ for all $x \in \operatorname{co} A$ and some $x_0 \in A$ exists. Due to the CHD-constant definition and inequality $r < R/\zeta_X$, the set $\operatorname{co} A$ belongs to the R-neighbourhood of the set A. On the other hand, A is proximally smooth and in accordance with paper [2], the metric projection mapping $\pi\colon \operatorname{co} A \to A$ is single valued and continuous. Finally, we define the mapping $\widetilde{F}\colon [0,1] \times A \to A$ as follows $\widetilde{F}(t, x) = \pi(F(t, x))$ for all $t \in [0, 1]$, $x \in A$. The mapping F contracts the set A to the point x_0. $\square$

REFERENCES

[1] P. ALEXANDROFF, *Über den allgemeinen Dimensionsbegriff und seine Beziehungen zur elementaren geometrischen Anschauung*, Math. Ann. **98** (1928), 617–635.

[2] M.V. BALASHOV AND G.E. IVANOV, *Weakly convex and proximally smooth sets in Banach spaces*, Izv. Ross. Akad. Nauk Ser. Mat. **73** (2009), 23–66.

[3] A. Barvinok, *A Course in Convexity*, Graduate Studies in Mathematics, vol. 54, American Mathematical Society, Providence, 2002.

[4] F. Bernard, L. Thibault and N. Zlateva, *Prox-regular sets and epigraphs in uniformly convex Banach spaces: Various regularities and other properties*, Trans. Amer. Math. Soc. **363** (2011), 2211–2247.

[5] F.H. Clarke, R.J. Stern and P.R. Wolenski, *Proximal smoothness and lower-c^2 property*, J. Convex Analysis **2** (1995), 117–144.

[6] G. Colombo and L. Thibault, *Prox-regular Sets and Applications*, Handbook of Nonconvex Analysis, International Press, Somerville, 2010, pp. 99–182.

[7] A. Dold, *Lectures on Algebraic Topology*, Springer, New York, 1972.

[8] H. Federer, *Curvature measures*, Trans. Amer. Math. Soc. **93** (1959), 418–491.

[9] A.L. Garkavi, *On the Chebyshev center and convex hull of a set*, Uspekhi Mat. Nauk **19** (1964), 139–145.

[10] G.M. Ivanov, *Modulus of supporting convexity and supporting smoothness*, Eurasian Math. J. **6** (2015), 26–40.

[11] B. Knaster, C. Kuratowski and S. Mazurkiewicz, *Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe*, Fund. Math. **14** (1929), 132–137.

[12] K. Leichtweiss, *Zwei Extremalprobleme der Minkowski-Geometrie*, Math. Z. **62** (1955), 37–49.

[13] S.A. Pichugov, *On separability os sets by hyperplanes in l_p*, Anal. Math. **17** (1991), 21–33.

[14] S.A. Pichugov and V.I. Ivanov, *Jung constants of the l_p^n-spaces*, Math. Notes **48** (1991), 997–1004.

[15] E.S. Polovinkin and M.V. Balashov, *Elements of Convex and Strongly Convex Analysis*, Fizmatlit, Moscow, 2007 (in Russian).

[16] R.T. Rockafellar and R.J.-B. Wets, *Variational Analysis*, Grundlehren der Mathematischen Wissenschaften, vol. 317, Springer, Berlin, 2009.

[17] C.P. Rourke and B.J. Sanderson, *Introduction to Piecewise-Linear Topology*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 69, Springer, New York, 1972.

[18] J.H. Wells and L.R. Williams, *Embeddings and Extensions in Analysis*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 84, Springer, New York, 1975.

Manuscript received February 28, 2016

accepted June 8, 2016

Grigory M. Ivanov
Ecole Polytechnique Fédérale de Lausanne
Route Cantonale
1015 Lausanne, SWITZERLAND
and
Department of Higher Mathematics
Moscow Institute of Physics and Technology
Institutskiǐ pereulok 9
Dolgoprudny, Moscow region 141700, RUSSIA

E-mail address: grimivanov@gmail.com